Pdf-version

The contribution of Leibniz for the quadrature of the circle

Bernard Aymon

Too much invested in the German policy to have the leisure to write long treaties of mathematics, Leibniz publishes its differential calculus in a fragmentary way. It is in a series of short articles, published since 1682 in the `` Acta eruditorum ", scientific newspaper founded in Leipzig with its support, which one finds the essence of his mathematical work. Numbers its results never were not published and are consigned in a newspaper. Leibniz noted its discoveries there as it did them. These incomplete and confused notes being, it is sometimes difficult to follow the evolution of its ideas on differential and integral calculus.

Leibniz openly recognized that it is only during its stay in Paris, of the years 1672 to 1676, that it started to study higher mathematics, encouraged and informed that it was by Christiaan Huygens. In its work `` Of will quadratura arithmetica Ciculi Ellipseos and Hyperbolae ", it establishes the first result of these studies. It succeeds in there expressing the surface of the quarter of disc of ray equal to 1 as equal to the infinite sum $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots$ This

sum bears its name still today, like Huygens, to which Leibniz communicated in first its discovery, had immediately predicted to him. It should however be specified that this formula to calculate π was also found by James Gregory (1638-1675) towards the end of its life and even before by the Indian mathematician Madhara about 1410. Many years after the publication of this work, Leibniz will claim to have drawn its first inspiration from the reading of a passage of the `` Traité sines of the quarter of the circle " of Blaise Pascal.

The establishment of this formula is the subject of this article. Let us start by giving extracts of a letter written in French that Leibniz probably addressed to the editor of the "Newspaper of Sçavans":

`` Sir,

Arithmetique squaring of the Circle and its segments or sectors, that I ay found and communicated to several excellens Geometres already a few years ago, their appeared rather extraordinary, and they exhorté me to make share with the public of it. But as I do not like to write a volume stuffed with a great number of proposals passed by again to give to only one which is new and considerable, I ay recourse to Vostre Journal which gives us the means of publishing a theoreme without making a book."

Further, it proposes its infinite sum there: $\frac{b}{1}-\frac{b^3}{3}+\frac{b^5}{5}-\frac{b^7}{7}+\ldots$ while specifying what

follows: `` the use of this squaring is which in addition to the beauty of a also simple theoreme and as surprising as celuy-Ci we have a means of finding the angles by costez and with wrong way; item spaces or portions of the Circles, Ellipses, Cones, Spheres, Spheroides and their surfaces, the whole by a simple addition of rationaux numbers or sizes commensurable with the defect even of tables all calculated, and without polygons, whose calculation requires a perpetuelle extraction of roots, in addition to thus one will approach vist well; because if b for example estoit $\frac{1}{10}$, b^{11} seroit $\frac{1}{100000000000}$ and by consequent all the higher powers will be able neglected boldly." What a beautiful example of the ill-considered love of German to make long sentences!

A last extract explains us `` the origins of its invention ": `` I ay thus consideré, that squarings which we found until icy by the ordinary analysis, dependent Arithmetiques rules to find the sums of the reglés rows, or progressions of rationaux numbers. But ordinates of the circle estant irrationelles, I ay stained to transform the circle into another figure, of the number of those that I call rationelles, i.e. whose ordinates are commensurable with their X-coordinates. For this effect I ay makes the enumeration of quantity of Metamorphoses, and having tested them by a very easy combination (because I pourrois by this means of writing in one hour of time a list of more than 50 plane or solid figures, different, and neantmoins dependantes of the circular) I ay found bientost the means that I of vays to be explained."

Thus let us follow the reasoning of Leibniz while being based on results which are in several of its articles or letters.

Let us start with a lemma (fig. 1): `` Three paralleles BC, GE, HF passing by the three angles of a triangle BEF and one of costez EF estant prolonged until the meeting of one of the paralleles in C, the rectangle under the interval BC between the point of meeting C and the angle B, by which passes this parallele, and under GH, the distance from two others paralleles GE, HF, i.e. the rectangle PGH (while supposing BGH normal with BC, and CP equal and parallele to BG) will be double Triangle BEF."

Angle fixed in B one traces a perpendicular (BJ) at the opposite side (EF). One obtains two similar triangles thus BJC and EFK. One thus has: $\sigma(BEF) = \frac{1}{2}EF \cdot BJ$ and like $\frac{EF}{KF} = \frac{BC}{BJ}$, one concludes that $\sigma(BEF) = \frac{1}{2}BC \cdot KF = \frac{1}{2}PG \cdot GH$.

Let us pass now to the situation represented by figure 2:

The surface of the `` zone " in staircases $B_1B_3(F_2)F_3(F_1)$ $F_1B_1$ is equal to the double of the `` segment " $C_1AC_3C_2C_1$. One uses the preceding lemma twice to show this result. The segments $[AE_i]$ or $[B_iF_i]$ are called the intercepted `` ''.

Let us come to its significant result that it names `` its Characteristica ''. It takes again the preceding drawing, but it considers the case where the point C_1 is confused with the point A. Moreover, it brings the following precision: that are as many taken points as one will want on the curve proposed (of equation $y = f(x)$), that is to say $C_1C_2C_3C_4$..., and in

same number the correspondents $F_1F_2F_3F_4$..., being (CE) to them the touching `` "
(tangent) with the curve proposed. It shows then (fig. 3) that the surface of the `` zone " delimited by the axis $(AB_i)$, the vertical $x = B_i$ and the curve of the intercepted `` " is equal to the double of the curvilinear segment of `` bases " $[AC_i]$ and delimited by the curve given.

Here how Leibniz proceeds without indicating the indices of the points concerned (fig. 4).

Let us pose $AB = EF = x$, $BC = y$, $AE = BF = z$ and $FC = y - z$. In the same way, under the touching `` " EC, let us consider the differential elements dx and dy. One has then: $\frac{dx}{dy} = \frac{x}{y-z}$ or $ydx - zdx = xdy$, or $2ydx - zdx = xdy + ydx(1)$.

However $\int xdy + \int ydx = \int(xdy + ydx) = xy(2)$. The result is immediate while considering, on figure 5, the sums $\sum_i x_i dy_i + \sum_i y_i dx_i = \sum_i (x_i dy_i + y_i dx_i) \approx xy$.

From (1) and (2) one obtains: $2\int ydx - \int zdx = xy$ or $\int ydx - \frac{1}{2}xy = \frac{1}{2}\int zdx(3)$.

However $\frac{1}{2}\int zdx$ other than half of the surface of the `` zone " formed by the intercepted `` " is nothing, $\frac{1}{2}xy$ the surface of the triangle AB_iC_i and $\int ydx$ the surface of the `` zone " delimited by the curve. Equality (3), one obtains well that the surface of the curvilinear segment is equal to half of the surface of the `` zone " delimited by the curve of the intercepted `` ''.

It any more but does not remain to use this result for the quarter of disc.

The equation of the half-circle of ray $a = AI$ is given by $y = \sqrt{2ax - x^2}$. The usual methods of the calculation of the tangents make it possible Leibniz to deduce the expression from the X-coordinate x according to `` intercepted the " z:

$$x = \frac{2az^2}{a^2 + z^2}.$$

Here how.

Let us pose $x_1 = x$ abcisse it point C located on the circle and $z_1 = AE$ the value of intercepted for $x_1 = x$.

Of $y^2 = a^2 - (x-a)^2$ one finds $2yy' = -2(x-a)$ and $y' = -\frac{x-a}{y}$.

The equation of the tangent to the circle in C is thus: $y = -\frac{x_1-a}{y_1}x + k$ where k is equal to z_1. One deduces some $z_1 = y_1 + \frac{(x_1-a)x_1}{y_1} = \frac{2ax_1-x_1^2+x_1^2-ax_1}{y_1}$, i.e. $z_1 = \frac{ax_1}{y_1}$. One finds, in the general case: $z = \frac{ax}{y}$ (4).

Let us use the equality $z_1 = AE = \frac{ax_1}{\sqrt{a^2-(x_1-a)^2}}$ to express x_1 according to z_1 only.
Leibniz proposes this equality for x_1, y_1 and z_1 unspecified:

$$\frac{AE^2}{IE^2} = \frac{AE^2}{AE^2+a^2} = \frac{1}{1+\frac{a^2}{AE^2}} = \frac{1}{1+\frac{a^2}{z^2}}.$$

From $z^2 = \frac{a^2x^2}{a^2-(x-a)^2}$, one obtains $\frac{AE^2}{IE^2} = \frac{x}{2a}$. While posing in this equality $AE = z$ and $IE = \sqrt{a^2+z^2}$, one can write the sought equality: $\frac{z^2}{a^2+z^2} = \frac{x}{2a}$(5).

Let us return to `` its Characteristica " applied to the circle. For an unspecified $C(x;y)$ point of the quadrant, the surface of the circular sector ICA is equal to the surface of the triangle ICA added with the surface with the circular segment AC.

However this last surface is equal to half of the surface given by the curve of the intercepted `` ''. The surface of the circular sector is thus equal to $\frac{1}{2}ay + \frac{1}{2}\int_0^x zdx$. Gold $\int_0^x zdx = xz + \int_0^z xdz$. Let us calculate initially ay. From $z = \frac{ax}{y} = \frac{axy}{y^2}$, one also obtains $zy^2 = axy$, i.e. $z(2ax - x^2) = axy$ and finally $ay = z(2a-x)$.

Thus the surface of the circular sector is equal to
$\frac{1}{2}(z(2a-x)) + \frac{1}{2}(xz - \int_0^z xdz) = az - \int_0^z \frac{az^2}{a^2+z^2}dz$.

To calculate this integral, Leibniz calls upon the `` beautiful method " developed by Mercator (Nikolaus Kauffmann):

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + t^4 - \ldots$$

While posing $a = 1$ and $z = b$, it is necessary to calculate

$\int_0^b \frac{z^2}{1+z^2}dz = \int_0^b z^2(1 - z^2 + z^4 - z^6 + \ldots)dz.$

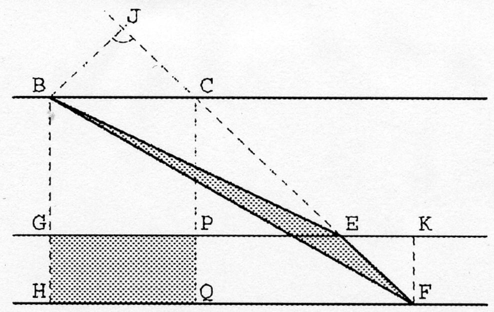
However Leibniz knows well squarings of the polynomial curves of equation $y = ax^n$ for which $\int x^n dx = \frac{1}{n+1}x^{n+1}$, expression that he writes: `` $\int x^n dx = \frac{diff(x)^{n+1}, x^{n+1}}{n+1}$ ''. It thus obtains: $\int_0^b \frac{z^2}{1+z^2}dz = \frac{b^3}{3} - \frac{b^5}{5} + \frac{b^7}{7} - \frac{b^9}{9} + \ldots$ and finally its significant result for the surface of the circular sector which is equal to $b - \frac{b^3}{3} + \frac{b^5}{5} - \frac{b^7}{7} + \ldots$

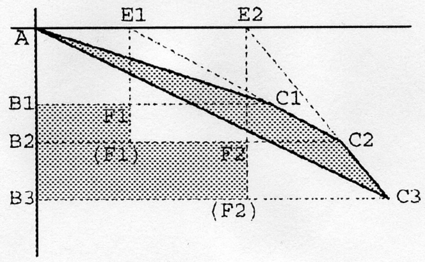
To obtain the quarter of the disc, one poses $b = 1$. Thus Leibniz finds its formula:

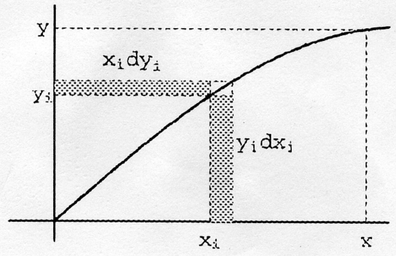
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

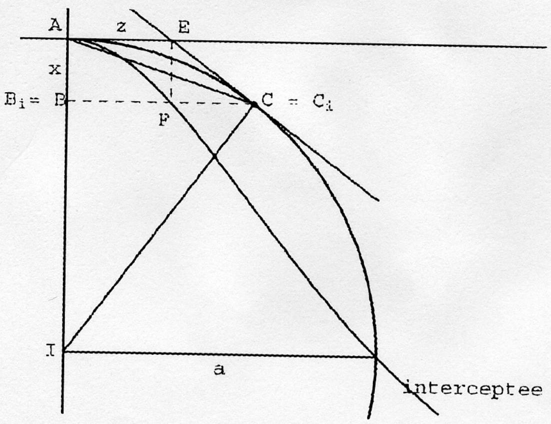
It writes: `` I acknowledge that this demonstration will not be able estre heard everyone because it supposes many things which are known only with those which are pour in the news decouvertes and which sçavent to handle the characteres or symbols... If it is necessary of esperer which one will be able to never arrive at an analytical reason, expressed in finished terms, Diameter with the circonference, I croy that it will be by this voye, because quoyque the expressions soyent infinite, we do not leave some times find the sums of them."

What does he say limit of this sum? In an article entitled `` Of proportione circuli AD quadratum circumscriptum in numeris rationalibus expressa. will vera '', appeared in the `` Acta eruditorum Lipsiensium " in 1682, one reads there: `` the whole series contains all the approximations at the same time or the larger and smaller values of the true value. It will have to be continued as far as possible so that the error is smaller than a given fraction, and, consequently, than any quantity given. This is why all the series expresses the exact value." Further, it continues as follows: `` If some circle is not commensurable by a square, it cannot be expressed by a number, but will have to be necessarily exhibé by a series of rational, and the same for the diagonal of a square, and the extreme and average section made in a report/ratio, that some call divine, and much other quantities which are irrational."


J
B
C
G
P
E
K
H
Q
F


E1
E2
A
B1
C1
F1
B2
C2
(F1)
F2
B3
C3
(F2)


y
$x_i dy_i$
y_i
$y_i dx_i$
x_i
x


A
z
E
x
B_i = B
C = C_i
F
I
a
interceptee