Elements of Formal Semantics

An Introduction to Logic for Students of Language

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Alternative Semantics for Propositional Logic

7.1 THREE-VALUED LOGIC

7.1.1. The Third Value

One way that logicians often depart from classical semantics in the history of logic is to question the assumption that all sentences are either true or false. Such theories reject the fundamental principle of classical semantics, called variously *Tertium Non Datur*, the *Law of Excluded Middle*, or the *Principle of Bivalence*.

The Principle of Bivalence. Every sentence in every world is either true or false: for any $P \in \text{Sen}$, and any $R \in [R]$, $R(P) \in \{T, F\}$.

Such revisions face some major difficulties which we may group into three sorts. First, the theory must plausibly explain what the third kind of sentence is and how it differs from genuinely true and false sentences. Doing so adequately requires a close conceptual analysis of the key ideas—truth, falsity, and the concept represented by the third value. Providing this analysis is sometimes called the *problem of defining the third truth-value.* Second, given the new truth-value, the theory must reconstruct the truth-tables for the sentential connectives or otherwise explain how to assign truth-values to molecular sentences. Again, doing so is a matter of conceptual analysis, but this time the analysis concerns the meaning of the sentential connectives themselves. Defining the assignment of truth-values to the connectives in many-valued semantics is often called the *projection problem*. Last, given the revision of the idea of truth-value, the theory must redefine the notions of logical truth and valid argument. Again the definitions must conform to the conceptual content of these traditional ideas. Moreover, the classes delineated by the proposed definitions must have the right extensions as judged on the basis of logical intuition. We should not be able to find any intuitively valid arguments or logical truths excluded by the definitions, and all those embraced by the definition should be intuitively valid and logically true. Discussion of three-valued logic within the tradition of formal semantics centers on these three problems, and we shall discuss each in turn.

Claiming that there are three kinds of sentence—the true, the false, and something else—presupposes a fact about the meanings of words or, to put it another way, about the geography of concepts. Just as it is a linguistic fact about the meanings of the English words 'red', 'green', and 'blue' that they are mutually exclusive and that in a given world different things may be properly called by each, so it is a conceptual claim about the meanings of 'true', 'false', and the third category that they are exclusive and satisfiable. Obviously in order for these conceptual claims to be established, a meaning for the third category must be fixed. In the long tradition of the subject various different interpretations for the third value have been proposed. The important point theoretically is that once an interpretation is decided upon, an exercise in discussing the meanings of words is in order. Such a discussion usually makes two points. First it shows that given their ordinary meanings, the three categories divide up sentences much as color words divide up physical objects. Second, it shows that there are in fact important examples of the new category in ordinary speech—that there really is a difference in the linguistic data sufficient to justify the new distinction.

Before giving a list of the various meanings attached to the third value and examples of each, we must mention several challenges facing any such revision of classical semantics. Any such theory, first of all, must argue that given the meanings of basic semantic ideas, it is appropriate to divide up the category of the nontrue into two mutually exclusive subtypes. In addition, it must show that given the meaning of the term 'false', it is right to apply it to only one of these types. Last, it must establish that given the definition of the third value, however it is defined in that theory, it is correct to apply it to the remaining type.

Thus, the false sentences and those receiving the third value are viewed as proper subsets of the wider category of nontrue sentences. All such theories face a common objection to the effect that falsity really means nontruth, and that it does violence to the meaning of the word 'false' to distinguish a variety of nontrue sentences which are not false. In short, the objection claims

that any threefold classification of sentences into true, false, and other is incoherent. In this view 'falsity' is claimed to mean nontrue, and therefore a third category is inconsistent with the meanings of the theoretical terms used.

Defenders of the third value usually respond that the ordinary meaning of the term 'false' is somewhat vague and that it is only specialized theories in formal logic that rigorously divide all sentences into true and false. Ordinary speech, it is argued, allows for many unclear and borderline cases. Given this latitude in the ordinary language idea of falsity, the theorist of language may justifiably clarify the ordinary language idea if he or she has good theoretical reasons for doing so. These theoretical reasons usually consist of the claim that some phenomenon previously unremarked or unexplained in semantic theory can be adequately treated by clarifying the nontrue in such a way that it consists of two subspecies, one of which is the false.

Moreover, the point is also made that not only is ordinary language usage not clear enough in itself to resolve the issue of whether the nontrue is identical to the false, the more sophisticated usage of 'true', 'false', and 'nontrue' in works from the history of philosophy and grammar is no more definitive. We might call this usage *technical* or *theoretical* to distinguish it from that in ordinary language, and it might be thought that if ordinary language cannot fix the relative meaning of 'falsity' and 'nontruth' then technical usage might. The facts of intellectual history say otherwise. Though it is true that there have been many theorists who have held that the two are the same, there have also been numerous thinkers who have defended the opposing view that falsity is a proper subcategory of the nontrue. Among the latter are Aristotle, Ockham, Buridan, Hegel, Frege, and literally hundreds of twentieth-century logicians. Thus, the issue of the propriety of positing a third value turns not on general considerations about the meaning of the single term 'false', but on a detailed consideration of the meaning of the third value and on the cases that might be successfully explained by its means. We shall now review some of these cases and the category terms used to draw them together.

7.1.2 Category Mistakes

Some terms, it is held, are limited in the spheres of their meaningful usage. A given term can be correctly asserted or denied only for a restricted class of things. To attempt to apply it outside this class is nonsensical. In this view truth corresponds to correct assertion, falsity to correct denial, and the third value to meaningless uses.[1] For example, 'deductible' correctly applies to some expenses and not to others, but it is a conceptual mistake to think the question even arises whether anything other than an expense is deductible. Sentences that are grammatically correct but that link words together in a way that violates their restricted ranges of meaningfulness are called *category mistakes*. Here are some cases:

EXAMPLES Triangularity drinks procrastination. (Russell)
Colorless green ideas sleep furiously. (Chomsky)
Einstein's most important discovery supports combustion. (Thomason)

7.1.3 Vagueness

Words, especially those used to describe ordinary things, are *vague* in the sense that there is a large class for which it is unclear whether they fall under the term or not. When, it is argued, does it become true that I am in the corner as I advance toward it? There is a fuzzy border to the phrase 'in the corner'. Here are two cases that are often alleged to be so vague that they are not really either true or false.[2]

EXAMPLES France is hexagonal.
Italy is boot-shaped.

7.1.4 Presupposition Failure

Issues often presuppose facts, and often a question cannot arise without certain presuppositions being true. Both the meaningful assertion and denial of a sentence may presuppose the truth of another sentence, and if that presupposition is false, then any attempt to assert or deny the original

[1] In modern logic perhaps the first important statement of this idea was by Sorën Halldén, *The Logic of Nonsense* (1949). A much more developed treatment is found in L. Goddard and Richard Routley, *The Logic of Significance and Context* (1973). In both these works standard three-valued matrix theories (explained below) are used. For less standard many-valued semantics of category mistakes see Richmond Thomason, 'A Semantics of Sortal Incorrectness' (1972), John N. Martin, 'A Many-Valued Semantics for Category Mistakes' (1975), and Merrie Bergmann, 'Logic and Sortal Incorrectness' (1977).

[2] Perhaps the most interesting paper on a three-valued semantics of vagueness as distinct from nonsense is that by Hans Kamp, 'Two Theories about Adjectives' (1975), in which supervaluations are used. Vagueness is also the main inspiration for the movement known as 'fuzzy logic', which proposes a semantics similar to Łukasiewicz's infinitary-valued logic studied below. On fuzzy logic see L. Zadeh, 'Fuzzy Sets' (1965), and for a critical review of the movement see Charles Grady Morgan and Francis J. Pelletier, 'Some Notes Concerning Fuzzy Logics' (1977), and Alasdair Urquhart, 'Many-Valued Logic' (1986).

sentence, in this view, is meaningless.[3] The idea is important enough to merit a formal definition.

DEFINITION: *P presupposes Q* iff in any world *w*, if *P* is either T or F in *w*, then *Q* is T in *w*.

The definition assumes that assignments of T and F do not exhaust all the possible cases and that the law of bivalence is false. Here are two common examples. Each is presented as a set of three sentences. The first consists of an assertion and its denial, and both are said to presuppose the third sentence of the triple. Neither of the first two is true—indeed the question whether they are true does not arise—if the third is not true.

Existential Presupposition of Singular Terms
John is a bachelor.
John is not a bachelor.
John exists.

The king of France is bald.
The king of France is not bald.
The king of France exists.

Examples of this sort are the subject of a famous debate between Bertrand Russell and P. F. Strawson.[4] Russell argued that the affirmation 'The king of France is bald' may be false because it fails of presupposition but that its negation 'The king of France is not bald' would then be true or false depending on what one means by 'not'. If one means 'not' in the sense of 'it is not true that' then the negation is true. But if the negation means 'The king of France exists but is not bald' then this too is false because it fails of presupposition. In either case, claimed Russell, the negation does not violate the law of excluded middle. Strawson reserved the term 'false' for the subspecies of nontruth that satisfies its presuppositions. He held that in fact neither the affirmation nor the denial is true or false when the presupposition

[3] Some readers find this idea in Frege, 'On Sense and Reference' (1892), but it is first clearly distinguished in P. F. Strawson, *Introduction to Logical Theory* (1952), pp. 20–21, 175. The literature on presupposition is vast. The first clear use of presupposition as a motivation for a many-valued projection to molecular sentences was an application of the semantics of Bochvar to presupposition by T. J. Smiley, 'Sense Without Denotation' (1960). For some recent general discussions of semantic theories of presupposition using many-valued logic see John N. Martin, 'Some Misconceptions in the Critique of Semantic Presupposition' (1979), and William Lycan, *Logical Form in Natural Language* (1984).

[4] See Bertrand Russell, 'On Denoting' (1905), and 'Descriptions' (1919), and P. F. Strawson, 'On Referring' (1950) and 'Identifying Reference and Truth-Values' (1964). The logical literature on existential presupposition, both of singular and general terms, is described in the later sections on supervaluations and free logic.

fails. In such cases, to use his terminology, neither sentence can be used to make a meaningful statement. The debate is a good example of disagreement about the viability of three-valued logic in terms of a dispute about the scope of the term 'false'.

Another important kind of existential presupposition is that which is sometimes carried by common nouns. These cases will be discussed more fully in Chapter 8 under the topic of free logic. Here is a typical case:

Existential Presupposition of Class Terms

All the senators are crooks.
Not all the senators are crooks.
There are some senators.

One of the interesting discrepancies between Aristotelian and modern logic is that Aristotle captured this presuppositional inference in his logic in a way modern logic does not. In the syllogistic, in order for either an A or an O statement to be true, its subject term must stand for a nonempty set, and hence it would be true to say that there was something for which the subject term was true whenever either statement was true. But in modern logic predicates and class terms may have empty extensions, and the inference from $(\forall x)(Fx \rightarrow Gx)$ to $(\exists x)Fx$ is invalid.

A third kind of presupposition requires that the sentence complement of some verbs, like 'discover', 'regret', and 'is surprised', must be true whether the sentence as a whole is an affirmation or a denial. Such verbs are called *factives*, and here is an instance:[5]

Factive Presupposition

He discovered she was there.
He did not discover she was there.
She was there.

Last some individual words (*lexical* items in linguistic jargon) carry their peculiar presuppositions. Here is a well-known case.

Presupposition of the word 'stop'

He stopped beating his wife.
He did not stop beating his wife.
He beat his wife.

[5] See C. A. S. and R. P. V. Kiparsky, 'Fact' (1970), and Lauri Karttunen, 'Some Observations on Factivity' (1971).

7.1.5 Future Contingents

Aristotle reasoned that if the law of excluded middle were true, then the future tense sentence 'There will be a sea battle tomorrow' is either true or false, and the fact it describes is therefore determined. But matters that depend on human decision are not determined. Hence, the law must be false. Medieval philosophers were greatly exercised over whether God's foreknowledge is compatible with human freedom. Like Aristotle's examples of future contingent sentences, these facts about the future were classified as undetermined rather than true or false. These cases are interesting because the need to find an alternative to truth and falsity arises from a philosophical conundrum rather than a straightforward attempt to classify examples of sentences on the fuzzy border between truth and falsity.[6] In classifying their values as indeterminate rather than meaningless, they also offer quite a different conceptual account of the meaning of the new category.

7.1.6 Paradoxes

If the sentence below is true, it is false, and if it is false, it is true:

This sentence is false.

This sort of absurdity led Jean Buridan in the fourteenth century and D. A. Bochvar in the twentieth century to classify this and similar paradoxes as neither true nor false.[7] In this way they attempted to explain away apparent exceptions to another traditional principle of logic, the law of noncontradiction (i.e., $\vDash \sim(P \wedge \sim P)$). Whether such attempts succeed is controversial, but they are interesting because in the concept of paradoxicalness they have quite a new meaning for the third value. Like the case of future contingents this departure from the law of excluded middle is motivated by logical puzzles.

[6] See the discussion of Aristotle in William and Martha Kneale, *The Development of Logic* (1962), and the application of Aristotle's ideas on nonbivalence to many-valued logic in Bas van Fraassen, 'Singular Terms, Truth-Value Gaps, and Free Logic' (1966).

[7] For discussion of Buridan's solutions to the 'sophismata' see James Cargile, *Paradoxes: A Study in Form and Predication* (1979), and Hans G. Herzberger, 'Dimensions of Truth' (1973). See also D. A. Bochvar, 'On a Three-Valued Logical Calculus and Its Application to the Analysis of the Paradoxes' (1937). There is quite a large literature on the application of many-valued logic to the paradoxes. Many of the most important papers may be found in Robert L. Martin, ed., *The Paradox of the Liar* (1970), and *Recent Essays on Truth and the Liar Paradox* (1984), as well as in the special issue on the paradoxes, Vol. 13 (1984), of the *Journal of Philosophical Logic*.

☐ EXERCISES

1. Show that if the law of bivalence were true, then presuppositions would all be trivial in the sense that only logical truths could be presuppositions. That is, given bivalence and the formal definition of 'presupposition', show that if P presupposes Q, then in all R, $R(Q) = \text{T}$.
2. Find some examples of the use of common nouns in which the sentences clearly presuppose that the extensions of the nouns are nonempty, and then other examples of the use of the same or other common nouns in which it clearly remains an open question whether the extensions of the terms are nonempty.
3. Find some examples using the verb 'to report' in which the complement is clearly presupposed to be true, and other examples in which it remains an open question whether the complement is true.

7.1.7 The Projection Problem

Once a meaning for the third value, which we shall call N, is decided upon and defended, both in its compatibility with truth and falsity and in its ability to cover important examples, one must then explain how the truth-values of molecular sentences are determined. One way to do so is to assume that essentially the same mechanisms will work in many-valued semantics that apply in classical semantics. In particular, Jan Łukasiewicz at the beginning of this century supposed that many-valued semantics would remain truth-functional. On this view there are rules for determining the truth-value of a whole sentence given the truth-values of its immediate parts, and these rules may be formulated in many-valued truth-tables. These tables must pass what is really a conceptual test. It must be convincingly argued that the connectives as interpreted by the tables are being used in a way consistent with their ordinary meanings. We have a rough idea of the meaning of the ordinary words 'not', 'and', 'or', 'if...then', etc. We also know roughly what we mean by 'true', 'false', and whatever we are choosing as our rendering of N. This linguistic knowledge must be consistent with the relations stated in a given truth-table. We must intuitively agree that the whole has the values assigned in the various cases. Such judgments depend on our intuitive understanding not only of the connective but of the concepts represented by the truth-values. Below are three of the best known three-valued tables for the connectives.

	~	∧	T	F	N	∨	T	F	N	→	T	F	N
T	F		T	F	N		T	T	N		T	F	N
F	T		F	F	N		T	F	N		T	T	N
N	N		N	N	N		N	N	N		N	N	N

Kleene's weak connectives
(Bochvar's internal connectives)

	~	∧	T	F	N	∨	T	F	N	→	T	F	N
T	F		T	F	N		T	T	T		T	F	N
F	T		F	F	F		T	F	N		T	T	T
N	N		N	F	N		T	N	N		T	N	N

Kleene's strong connectives

	~	∧	T	F	N	∨	T	F	N	→	T	F	N
T	F		T	F	N		T	T	T		T	F	N
F	T		F	F	F		T	F	N		T	T	T
N	N		N	F	N		T	N	N		T	N	T

Łukasiewicz's three-valued connectives

The weak connectives treat N as an infection that corrupts the whole if any part is affected, and they seem to conform best with the reading of N as meaningless. They are, therefore, often defended as the tables most suitable for theories for category mistakes and vagueness. Bochvar viewed paradoxicalness as a kind of infection and thus used the first tables in his explanation of the semantic paradoxes, and Buridan seems at times to have had something similar in mind.

Presupposition failure has sometimes been explained as a kind of corruption of this sort, but just as frequently it is explained by the second set of tables. These strong connectives are usually explained by reading N as 'unknown'. If one part of a conjunction is false, then regardless of what we know about the other part the whole must be false. Likewise a disjunction with a true disjunct must be T. Of the readings given to N earlier, this treatment conforms best to that taking N as marking indeterminateness because indeterminateness may be understood in the quasi-epistemic sense of undecided.

Łukasiewicz's tables are explained, with one exception, just like those of the strong connectives. In the case of the conditional, when both parts are N the whole is T. The motivation for this analysis seems to reside in intuitions about logical truth. Unless the truth-table for the conditional yields T for the arguments N and N, it looks like the obvious logical truth $P \rightarrow P$ would sometimes not be T, viz. when P is N. This bow in the direction of capturing an adequate logic raises the whole question of how logical truth and validity are to be defined in a three-valued logic. The adequacy of these definitions of logical ideas constitutes a major problem for a many-valued theory to which we shall return in a moment. First, however, we must discuss in some detail the use of epistemic ideas in truth-tables.[8]

7.1.8 Epistemic Readings of Truth-Values

Let us grant for the sake of argument that the strong tables correctly describe how to determine the epistemic status of a whole expression from the status of its sentential parts. Even if this supposition were true, there would remain the prior question of what these epistemic facts have to do with truth. Is it proper to use epistemic ideas in truth theory?

Explanations that define truth-values in terms of epistemic ideas have been open to a traditional challenge. Given the traditional definition of knowledge as justified true belief, a vast difference exists between a state of knowledge and a fact about the world. The truth or falsity of a sentence is a function of how it corresponds to the world. Knowledge, on the other hand, is a function of what reasons a person can advance for his or her beliefs. It is perfectly possible and indeed highly likely, in the traditional view, that something could be true but unknown. To translate the truth-value T as 'known', then, seems to collapse two quite different ideas. Epistemology, in this view, has nothing to do with the theory of truth, and it is a confusion to use a truth-value to represent an epistemic category. This traditional analysis of truth and knowledge is sometimes called *realism*.[9]

The reply to this objection consists of rejecting the traditional definition of truth as correspondence to the world, and with it the definition of knowledge as justified belief that corresponds to the world. The alternative conception of

[8] The projection problem is so named by Lauri Karttunen, 'Presuppositions of Compound Sentences' (1973). The original source of the weak connectives is S. C. Kleene, 'On a Notation for Ordinal Numbers' (1938). That of the strong connectives is S. C. Kleene, *Introduction to Metamathematics* (1959). For early statements of Łukasiewicz's three-valued matrix see Jan Łukasiewicz, 'On 3-Valued Logic' (1920), 'On Determinism' (1923), and 'Philosophical Remarks on Many-Valued Systems of Propositional Logic' (1930). Though we shall not study the proof theory corresponding to these matrices in this book, their characteristic logical truths and valid arguments can be completely specified both by axioms and in natural deduction systems. For a natural deduction account using much the same terminology as that of the classical proof theory given in Chapter 5 but applied to Łukasiewicz's matrix (which has Kleene's strong connectives as a part) and generalizations from it, see Richard B. White, 'Natural Deduction in the Łukasiewicz Logics' (1980).

[9] For references to the philosophical literature on coherence theories and antirealism see Chapter 3. References to the literature on intuitionistic logic and its semantics are given below. For varieties of many-valued logics with epistemic interpretations see the probabilistic semantics of Hartry Field, 'Logic, Meaning, and Conceptual Role' (1977), and an epistemic interpretation of supervaluations is developed in John N. Martin, 'Epistemic Semantics of Classical and Intuitionistic Logic' (1984).

truth is that of coherence, and holds that the truth of a sentence is a function of its place in a larger systematic body of sentences that, as a whole, has the property of coherence. Thus, there would be no need to posit a world beyond language to serve as the measure by which we evaluate sentences. Knowledge, then, collapses into truth. To know something is to believe it as a part of a larger coherent set. This view has come to be called *antirealism*.

It is hard at first glance to see the attraction of antirealism. It is open to two major objections that it has never satisfactorily answered. The theory should be able to define its central idea of coherence, but beyond the notion of consistency there is little agreement about what coherence amounts to, and clearly consistency itself is an insufficient explanation of a notion of coherence that aspires to be equivalent to truth. Second, given a coherence theory of truth, it appears that more than one set of sentences could be coherent and hence true. Moreover, nothing in the idea of coherence seems to preclude the possibility that these sets could be mutually contradictory. If so, their simultaneous truth would violate the law of noncontradiction, and a theory that violates this law clashes with very deep logical intuitions.

The turn to antirealism came as the outcome of long discussions in two related areas of philosophy, the philosophy of mathematics and epistemology. Plato held the rather implausible view that mathematical objects such as triangles exist on some special level of reality specific to them, and that it is the task of mathematics to describe this special world. This view that mathematical truth is a special case of correspondence to abstract mathematical objects is known in the philosophy of mathematics as *Platonism*. But these mathematical objects are not part of common sense, and it strains plausibility, to say the least, to say that mathematicians investigating objects such as imaginary numbers are *describing* the entities of a world. Thus Platonism in mathematics has never been a particularly convincing application of the correspondence theory of truth. What, then, is truth in mathematics? Coherence is an obvious candidate, and here the idea seems to make a good deal of sense. There is a tradition called *constructivism* in mathematics and *intuitionism* in logic that says that a sentence is true in mathematics if, and only if, it has been proven, and is false if, and only if, it has been refuted. Because being proven or refuted is a matter of justification and is epistemic, in this tradition the theory of truth and epistemology merge.

Difficulties in mainstream epistemology have also led philosophers to take the coherence theory seriously, not only as a model for knowledge in mathematics, but for all knowledge. The traditional view of knowledge as justified true belief has always had the problem of explaining what justification means. Classical philosophers thought justification makes use of some kind of special access to the truth that is so foolproof that it imparts certainty. Modern philosophers have generally backed away from the claim that all knowledge is certain, but most have held that what knowledge we do have is based in some remote way on a foundation of certainties. The rationalists of the Renaissance believed these foundations were truths of reason that we grasp by the intellect alone; more recently the general view has been that most knowledge is built upon sensation. In this view, our senses make reports that we are constrained to accept as 'givens', and we then make inferences or inductions on the basis of these reports. We construct a world view, a body of scientific knowledge and common sense, that is compatible with and ultimately verified by a foundation of sensation. Such is the empiricist view of knowledge accepted by most scientists and by the movement known as *logical positivism* earlier in this century. It is now generally admitted that this sort of *foundationalism* in epistemology has not been worked out. In the theory there is an important relation between the foundation of knowledge and the larger body of scientific theory and common sense based upon it. But the theory has not been able to explain what that relation is and how it works. In particular, many attempts to explain the relation seem to lead to skepticism, the denial that there is any knowledge at all. There is, then, the temptation to think that the whole idea of foundationalism is misconceived, that perhaps knowledge is not based on indubitable links to an external world. The alternative seems to be some kind of coherence theory.

In this book we cannot pursue epistemology nor the philosophy of mathematics, and these remarks will have to suffice as a sketch of what is at issue in using epistemic ideas to explain truth-values. Some such rejection of traditional realism underlies many of the developments of nonclassical semantics and many-valued logics. Intuitionism, which is one such view, is discussed in some detail shortly.

☐ **EXERCISES**

1. Using the syntactic definition of 'consistent' from Chapter 5, make up two sets of sentences from propositional logic that are each consistent, but which contradict each other in the sense that they could not be put together in a consistent set.
2. Describe a simple situation in which somebody has a very good justification for believing something, good enough to satisfy the ordinary requirements for expert knowledge of the subject, yet he is mistaken because what he is justified in believing happens to be false.

7.1.9 The Importance of Explaining Logical Intuitions

In three-valued logic we continue to use essentially Leibniz's definitions of logical truth and validity. There are two ways to read Leibniz's definitions once two kinds of nontruth have been distinguished. A logical truth may be either one that is always true or one that is never false, one that is always T, or always either T or N. Likewise a valid argument may be either one that whenever its premises are true so is its conclusion, or one in which whenever its premises are not false, its conclusion is not false. That is, it is one in which whenever the premises are T the conclusion is T, or one in which whenever the premises are either T or N, the conclusion is T or N. In what follows we allow for these various possibilities by singling out a subset of truth-values, called the *designated values*, as those relevant to defining logical truth and validity. A logical truth will be one that always has a designated value, and a valid argument will be one in which whenever the premises are designated so is the conclusion. Then we specify whether we mean $\{T\}$ or $\{T, N\}$ as the set of designated values.

Once the logical notions are defined, they are tested to see if they conform to our logical intuitions. In practice this testing amounts to seeing how the new notions correspond to their classical counterparts: does the new theory agree with classical logic? This comparison with classical logic may be done with mathematical rigor, and we now turn to the techniques used to make the comparison. As we have seen, the evaluation of many-valued semantics involves conceptual issues and the collection of different kinds of examples from usage that might reasonably be candidates for sentences having a value other than T or F. Many-valued truth-tables also are supposed to serve as explanations of the ordinary meaning of the connectives. Much more than we have space to discuss can be said about the strengths and weaknesses of many-valued semantics from the perspective of these criteria. Here, however, we shall focus most of our discussion on the appraisal of three-valued semantics from a logical point of view. We do so because the proper place for discussing the methodology for appraising the conceptual adequacy and accuracy of a semantic theory is theoretical linguistics and the philosophy of language. But one of the most important kinds of critique of many-valued semantics has focused on its logic. The concepts and techniques used in this logical appraisal are a special case of the general methods we shall see applied again and again in this book. They consist of studies in the way the structure of syntax relates to that of semantics, and the ideas used are drawn from abstract algebra. We pause in the next section to state in general terms the concepts needed to evaluate the logics of specific many-valued theories.

☐ **EXERCISE**

Assume that the relation of valid argument is defined by reference to a set D of designated values as follows: $P \vDash Q$ iff for all $R \in [R]$, if $R(P) \in D$ then $R(Q) \in D$. Show that the relation $\vDash$ is transitive.

7.2 COMPARING LOGICAL ENTAILMENT RELATIONS IN DIFFERENT MANY-VALUED LOGICS

7.2.1 The Idea of a Matrix Language

Many-valued logic studies the semantics and proof theory of the highly simplified and abstract syntax of the sentential logic, the formal language built up from atomic sentences by the sentential connectives.[10] For the purposes of this chapter we may use a rather abstract notion of sentential syntax:

DEFINITION: A *sentential syntax* Syn is any structure $\langle \text{Sen}, B, F\rangle$ such that

(1) B is an ordered set of basic sentences (called *atomic* and usually countably infinite in number);

(2) F is an ordered family of 1–1 syntactic functions of various finite degrees (called *formation rules*, usually finite in number);

(3) Sen is the inductive set formed by closing B under the operations in F.

In the examples we shall discuss we shall assume a single set of atomic sentences $B = \{P_1, \ldots, P_n, \ldots\}$, and frequently make use of the formation rules $f_\sim, f_\wedge, f_\vee, f_\rightarrow$ for negation, conjunction, disjunction, and the conditional as well as a few new rules. To simplify our presentation, we shall let the index of the function stand for the function as a whole and write $\wedge(x, y)$ for $f_\wedge(x, y)$, $\vee(x, y)$ for $f_\vee(x, y)$, $\rightarrow(x, y)$ for $f_\rightarrow(x, y)$, and $\sim(x)$ for $f_\sim(x)$.

Semantically a syntax will be interpreted by a structure of truth-values organized by truth-functions which are intended to interpret the various connectives. The structure specifies, in order, the truth-values used in the semantics, the set of truth-values used in the definition of logical entailment, and the truth-functions corresponding to the formation operations. Possible worlds are then defined as functions which map atomic sentences into truth-values and then project truth-values to complex sentences by applying to the truth-values of its immediate parts the truth-function corresponding to the connective. A language could be defined as we have in the past as a pair consisting of a syntax and its set of possible worlds, but the custom in many-valued semantics is to identify a language with the syntax paired with its semantic structure.

[10] The major extended studies of the semantic theory of many-valued logics are those by J. B. Rosser and A. R. Turquette, *Many-Valued Logics* (1952), Nicholas Rescher, *Many-Valued Logic* (1969), and George Epstein, ed., *Multiple-Valued Logic* (1976). A good review of the technical issues in many-valued logic, which is, however, somewhat too dismissive of its conceptual and linguistic motivation, is by Alasdair Urquhart, 'Many-Valued Logic' (1986). Another critical discussion from a logical perspective is that of Dana Scott, 'Does Many-Valued Logic Have Any Use?' (1976). The metatheory presented in this section is a development of that in John N. Martin, 'The Semantics of Frege's *Grundgesetze*' (1984).

As explained earlier, there are two possible ways to define valid argument once three values are allowed. Such an argument may be defined as one that always takes you from premises that are T to a conclusion that is T. That is, it may be defined as truth-preserving. Alternatively, it may be defined as non-falsity-preserving, as one in which whenever the premises are not F then the conclusion is not F. In classical semantics in which the only value other than F is T, the two definitions are equivalent, but they are not so when three or more values are allowed. To state the definition in general terms, we set aside a subset of truth-values as those preserved in logical inference, and define validity by reference to this set of 'designated' values.

DEFINITION: A *logical matrix* M is defined as any structure $\langle U, D, G\rangle$ such that

(1) U is a nonempty set (called the set of *truth-values*);

(2) D is a subset of U (the set of *designated values*);

(3) G is an ordered set of finitely valued functions on U.

DEFINITION: A *sentential matrix language* L is any structure $\langle \text{Syn}, M\rangle$ such that

(1) Syn is some sentential syntax $\langle \text{Sen}, B, F\rangle$;

(2) M is some logical matrix $\langle U, D, G\rangle$, such that

(a) F and G contain the same number of functions and

(b) for any f_i in F, if f_i is of degree n, so is g_i, and if f_i is nonempty, so is g_i.

DEFINITION: For a sentential matrix language $L = \langle \text{Syn}, M\rangle$ where Syn $= \langle \text{Sen}, B, F\rangle$ and $M = \langle U, D, G\rangle$, we define the set $[R]$ (called the *set of possible worlds for L*) as follows:

$$[R] = \{R \mid (\text{Sen} \xrightarrow[\text{into}]{} U) \text{ such that}$$

$$(1) \text{ if } P \in B, \text{ then } R(P) \in U;$$
$$(2) \text{ if } f_i \in F \text{ and } P_1, \ldots, P_n \in \text{Sen, then } R(f_i(P_1, \ldots, P_n)) = g_i(R(P_1), \ldots, R(P_n))\}.$$

In the following definition we introduce the abbreviation $R(X) \in D$ for the condition in which all the sentences in a set X are assigned designated elements in D by R.

DEFINITION: For any set X of sentences and any reference relation R, we define $R(X) \in D$ to mean that for any element P of X, $R(P)$ is in D.

DEFINITION: For a matrix language L and a subset X of Sen of L and a sentence P in Sen, we say

X logically entails P in L (briefly, $X \vDash_L P$) iff
for all R in $[R]$ of L, $R(X) \in D$ only if $R(P) \in D$.

From this point on we shall assume that L ranges over sentential matrix languages, and that Syn $= \langle \text{Sen}, B, F \rangle$ is its syntax, $M = \langle U, D, G \rangle$ is its matrix, and $[R]$ is its world set. Distinct languages, their syntaxes, matrices, world sets, and entailment relations will be distinguished by subscripts and prime marks. As we have in the past we let X, Y, and Z range over the subsets of the set Sen under discussion, and P, Q, and S over its members. We let f range over the set F of operations, and g over the operations of the set G of a matrix.

Our goal in this section is to define the relevant concepts and prove the necessary theorems for the systematic comparison of the entailment relations of various many-valued languages. The need to make such comparisons is dictated by the theoretical goal of maintaining an entailment relation that approximates that recognized by our logical intuitions. One way to make such comparisons is to evaluate directly a proposed entailment relation by comparison with our raw intuitions. If some arguments which we intuitively accept fall outside the entailment relation as defined, the definition is too narrow, and if the definition sanctions some arguments rejected by intuition, it is too broad. In practice we often compare a new entailment relation with another that is already formally defined, especially that of classical two-valued semantics. We know the classical entailment relation very well. We know its strengths and weaknesses. Even though there are some intuitively strange arguments sanctioned by classical logic (e.g., the argument from P to $Q \vee \sim Q$, or that from P to $P \vee Q$), classical logic is an impressive theory. Any competitor would do well to equal classical logic at capturing logical intuitions, and it is usual practice to contrast new semantics with the standard classical account. We shall see that it is a far from trivial matter to introduce new truth-values and preserve a truth-functional logic that even approximates classical logic.

To carry out the comparison we need some theory. Our procedure will be to introduce a series of new concepts, some syntactic and some semantic, and along with each concept we shall prove the relevant metatheorem which explains how the concept is to be used in comparing entailment relations.

7.2.2 The Part-Of Relation among Syntaxes

A natural way to think of one syntax as being part of another is that the larger syntax contains all the atomic sentences of the smaller, and all the sentences built up from them. The larger syntax might also contain atomic sentences not present in the smaller, as well as the molecular sentences built up from these. The formation rules of the larger would accordingly be defined for the additional sentences not present in the smaller.

DEFINITION: A Syn is *part of* Syn′ iff $B \subseteq B'$, and for each i, $f_i \subseteq f'_i$.

The notion of part as defined here allows for the possibility that a given f_i might be empty, and indeed we shall consider one matrix to be a part of another if it can be made a part in the sense just defined by the addition of various empty formation rules and re-indexing the set F. Note also that a matrix for a syntax will also count as a matrix for all syntaxes of which it is a part.

THEOREM. Let Syn be a part of Syn′ and let M be a matrix for both. For any R in $[R]$, there is some R' in $[R']$ such that $R \subseteq R'$. Likewise, for any R' in $[R']$, there is an R in $[R]$ such that $R \subseteq R'$.

Proof. For the first part, we assume R, define R' in terms of R, and then show by induction that for any P in Sen, $R(P) = R'(P)$. We define R' as follows: for any atomic P in B of Syn, we require that $R'(P) = R(P)$ and that for any P in $B' - B$, $R'(P)$ is in U'. For any molecular $P = f_i(Q_1, \ldots, Q_n)$, we define $R'(P) = g_i(R'(Q_1), \ldots, R'(Q_n))$. We have defined R' so that it trivially meets the conditions for membership in $[R']$. For induction consider the following property of sentences in Sen:

(1) $R(P) = R'(P)$.

We show that it holds for all P in Sen by induction.

Atomic case. Let P be atomic. Then by definition of R', $R(P) = R'(P)$.

Molecular case. Let P be $f_i(Q_1, \ldots, Q_n)$, and we assume as our induction hypothesis that (1) holds for all sentences shorter than P. In particular, we assume that (1) holds for its immediate parts: for any j, $R(Q_j) = R'(Q_j)$. Now,

$R(P) = g_i(R(Q_1), \ldots, R(Q_n))$ [by the assumption that R is in $[R]$]

$= g_i(R'(Q_1), \ldots, R'(Q_n))$ [by the induction hypothesis and substitution of identities]

$= R'(P)$ [by the definition of R'].

Proof of the second part of the theorem is left for the reader.

□ **EXERCISE**

Prove the second part of the previous theorem by assuming R', defining R so as to be in $[R]$. Show by induction that for any P in Sen, $R(P) = R'(P)$.

We now state the results which show how to employ the notion of syntactic part in comparing entailment relations.

THEOREM. If Syn is a part of Syn′ and $M = M'$, then $X \vDash_L P$ only if $X \vDash_{L'} P$.

Proof. For some R' of $[R']$ of L' assume for a *reductio* proof that $R'(P) = \text{F}$ but $R'(X) = \text{T}$. Since $X \vDash_L P$, X is a subset and P an element of Sen of L. Then, by the previous metatheorem, there is some R of $[R]$ that agrees with

R' such that $R(X) = \mathrm{T}$ and $R(P) = \mathrm{F}$, contradicting our assumption that $X \vDash_L P$. End of Proof.

THEOREM. If Syn is a part of Syn$'$, $M = M'$, and P is in B of Syn, then $X \vDash_{L'} P$ only if $X \vDash_L P$.

☐ **EXERCISE**

Prove the previous theorem.

7.2.3 Sameness of Syntax

The next concept we shall use to compare logics is a variety of sameness of syntax. Clearly if all that differentiates two syntaxes is the shape of the connectives, they differ only in style, not substance. The formal idea of sameness in this sense is that of isomorphism. It is also possible to show rather easily that if one syntax differs from another only in having more atomic sentences but is otherwise the same, then the logic of the two will at least agree on the expressions they share. This variety of sameness is discussed under the concept of homomorphism.

DEFINITION: A function h is a *homomorphism* from a structure $\langle X, O_1, \ldots, O_n\rangle$ to a structure $\langle X', O'_1, \ldots, O'_n\rangle$, where each of O_i is an operation on X, each O'_i is an operation on X', and O_i is of the same degree as O'_i, iff

(1) h maps X into X', and

(2) for any i, $h(O_i(x_1, \ldots, x_n)) = O'_i(h(x_1), \ldots, h(x_n))$.

DEFINITION: A function h is an *isomorphism* iff h is a 1–1 'onto' homomorphism.

THEOREM. If h is a homomorphism from Syn to Syn$'$ and $M = M'$, then for any R' in $[R']$, there is an R in $[R]$ such that for any P in Sen, $R(P) = R'(h(P))$.

Proof. Assume h is such a homomorphism. We assume R' and define R: for any P in B, $R(P) = R'(h(P))$. We now show by induction that for any P in Sen, $R(P) = R'(h(P))$.

Atomic case. Trivially true by definition of R.

Molecular case.

$$\begin{aligned} R(f_i(P_1, \ldots, P_n)) &= g_i(R(P_1), \ldots, R(P_n)) \text{ [by definition of } R \text{ and } R'] \\ &= g_i(R'(h(P_1)), \ldots, R'(h(P_n))) \text{ [by induction hypothesis]} \\ &= R'(f'_i(h(P_1), \ldots, h(P_n))) \text{ [by definition of } R'] \\ &= R'(h(f_i(P_1, \ldots, P_n))) \text{ [since } h \text{ is a homomorphism].} \end{aligned}$$

End of Proof.

THEOREM. If h is a homomorphism from Syn to Syn$'$ and $M = M'$, then $X \vDash_L P$ only if $h(X) \vDash_{L'} h(P)$. (Here $h(X) = \{h(P) | P \in X\}$.)

THEOREM. If h is a homomorphism from Syn to Syn$'$ and $h(P)$ is in B' of Syn$'$, then P is in B of Syn.

☐ **EXERCISE**

Prove the previous two theorems. The proof strategy for the first is similar to that of previous results. The second can be shown by a straightforward *reductio* proof.

THEOREM. h is a homomorphism from Syn to Syn$'$ and for any atomic P in B, $h(B)$ is a unique element of B' (i.e., h restricted to B is 1–1 and onto B') iff h is an isomorphism from Syn to Syn$'$.

Proof. *'If' part.* We need only show that h is 1–1 and onto molecular sentences since it is so by definition for atomic sentences. We shall show by induction that h assigns a unique value to every element in Sen. Clearly the basis step is satisfied because by definition h assigns unique values to all atomic elements in B. For the inductive step we assume h assigns unique values to the parts of the whole sentence $f_i(P_1, \ldots, P_n)$ and then show it assigns a unique value to the sentence itself. Now, $h(f_i(P_1, \ldots, P_n))$ is the composition value $f'_i(h(P_1), \ldots, h(P_n))$. By the induction hypothesis these h values are unique (i.e., h restricted to the parts of the whole is 1–1) and f'_i is a 1–1 function by definition. Thus their composition function is unique.

'Only if' part. It follows trivially from the definition of isomorphism that h is a homomorphism from Syn to Syn$'$. We now show that h restricted to B is 1–1 and onto B'. It is 1–1 by definition. Since h is by assumption an 'onto' function, it assigns values to all elements in its domain Sen$'$, including the subset B' of Sen$'$. By a previous metatheorem, we also know that if $h(P)$ is in B', then P is in B. Thus for every value in B' there is a unique argument in B assigned to it by h. It remains to be shown that h restricted to B assigns values only in B'. Suppose otherwise, that for some f'_i, $P'_1, \ldots, P'_n$, and some P in B, $h(P) = f'_i(P'_1, \ldots, P'_n)$. Then for some $P_1, \ldots, P_n$ of Sen, $h(P_1) = P'_1, \ldots, h(P_n) = P'_n$ and then $h(P) = f'_i(h(P_1), \ldots, h(P_n))$ [by substitution of identities] $= h(f_i(P_1, \ldots, P_n))$ [by definition of homomorphism], contradicting our assumption that h is 1–1. End of Proof.

THEOREM. If h is an isomorphism from Syn to Syn$'$ and $M = M'$, then for any R in $[R]$, there is an R' in $[R']$ such that for any P in Sen, $R(P) = R'(h(P))$.

Proof. Assume h is such an isomorphism and that R is in $[R]$. We define R'. If P' is in B', there is a unique P in B such that $h(P) = P'$. We define $R'(P') = R'(h(P))$ to be $R(P)$. Likewise for the molecular case. For any molecular $f'_i(P'_1, \ldots, P'_n)$, there are unique $P_1, \ldots, P_n$, such that $h(P_j) = P_j$. We defne $R'(f'_i(P'_1, \ldots, P'_n)) = g'_i(P'_1, \ldots, P'_n))$. We show by induction that for all P in Sen, $R(P) = R'(h(P))$. The reader may fill out the details as an exercise.

THEOREM. If h is an isomorphism from Syn to Syn' and $M = M'$, then $X \vDash_L P$ iff $h(X) \vDash_{L'} h(P)$.

☐ EXERCISE

Prove the previous two theorems. Note that half of the latter result has in effect been proven earlier.

7.2.4 Part-Whole among Matrices

Thus far we have compared two languages by varying their syntaxes but keeping the matrix determining their semantics fixed. We can also compare languages by varying their matrices. We begin by defining a part-whole relation for matrices. It will turn out that possible worlds according to the smaller matrix also count as worlds according to the larger and that entailments valid under the larger hold also for the smaller.

DEFINITION: M is a *part of* M' iff $U \subseteq U'$, $D \subseteq D'$, and for each i, g_i is nonempty and a subset of g_i'.

THEOREM. If Syn = Syn' and M is a part of M', then $[R] \subseteq [R']$.

Proof. Assume the antecedent, and that R is in $[R]$. It is evident that R meets the defining conditions for membership in $[R']$. For any atomic P, $R(P)$ is in U' because the fact that R is in $[R]$ entails that $R(P)$ is in U and $U \subseteq U'$. Similarly in the molecular case $R(f_i(P_1, \ldots, P_n)) = g_i(R(P_1), \ldots, R(P_n))$ [since R is in $[R]$] $= g_i'(R(P), \ldots, R(P_n))$ [since g_i is contained in g_i']. Therefore R is in $[R']$. End of Proof.

THEOREM. If Syn = Syn' and M is a part of M', then $X \vDash_{L'} P$ only if $X \vDash_L P$.

☐ EXERCISE

Prove the last theorem.

7.2.5 Sameness of Matrix

Just as similarity of structure among syntaxes forces similarity of entailment, so too does structural sameness among matrices. The general concepts of morphism defined for arbitrary structures will serve as the relevant notions of sameness of structure but we must adjust them slightly since matrices have the extra set of designated elements not allowed for in the general case. We do so with the following definitions.

DEFINITION: h is a *morphism* from M to M' iff h is a morphism from

$$\langle U, g_1, \ldots, g_n \rangle \quad \text{to} \quad \langle U', g_1', \ldots, g_n' \rangle.$$

DEFINITION: A morphism h from M to M' is said to *preserve designation* iff for any x, $x \in D$ only if $h(x) \in D'$, and is said to *preserve nondesignation* iff for any x, $x \notin D$ only if $h(x) \notin D'$.

THEOREM. If Syn = Syn' and h is a homomorphism from M to M', then $\{h \cdot R | R \text{ is in } [R]\} \subseteq [R']$.

Proof. Here $h \cdot R$ is the composition function of R and h defined in the usual way: $h \cdot R(x) = h(R(X))$. Assume the antecedent of the theorem, and consider an arbitrary $h \cdot R$ such that R is in $[R]$. We show that it meets the conditions of membership in $[R']$. For an atomic P, we need note only that h is defined for $R(P)$ and that the range of h is included in U'. Hence $h(R(P))$ is in U'. For the molecular case we note that $h(R(f_i(P_1, \ldots, P_n))) = h(g_i(R(P_1), \ldots, R(P_n)))$ [since R is in $[R]$] $= g_i'(h(R(P_1), \ldots, h(R(P_n)))$ [since h is a homomorphism]. Thus $h \cdot R$ meets the conditions for membership in $[R']$. End of Proof.

THEOREM. If Syn = Syn' and h is a homomorphism from M to M' that preserves both designation and nondesignation, then $X \vDash_{L'} P$ only if $X \vDash_L P$.

Proof. Assume the condition that $X \vDash_{L'} P$, and assume for a *reductio* proof that for R in $[R]$, $R(X) \in D$ but that $R(P) \notin D$. We know from the last theorem that $h \cdot R$ is in $[R']$. Since h preserves designation, $h \cdot R(X) \in D'$, and since h preserves nondesignation, $h(P) \notin D'$. But this contradicts the assumption that $X \vDash_{L'} P$. Thus if $R(X)$ is in D, so is $R(P)$. End of Proof.

THEOREM. If Syn = Syn' and h is a homomorphism from M onto M', then $\{h \cdot R | R \text{ is in } [R]\} = [R']$.

THEOREM. If Syn = Syn' and h is a homomorphism from M onto M', then $X \vDash_L P$ iff $X \vDash_{L'} P$.

☐ EXERCISE

Prove the previous two theorems.

7.2.6 Sublanguages and Conservative Extensions of Entailment

Thus far we have allowed structures to be extensions of one another if their various features are subsets of one another. There is another sense of part-whole that is useful when considering matrices. It is frequently the case that we want to compare one language with another like it except that the second contains some additional connectives and corresponding truth-functions.

Normally we should not expect the addition of some new connectives to alter the meaning of the old. In particular, we would expect that the logical relations validated in the restricted language would remain valid in the new, and any argument in the new language formulated solely in terms of the old would be valid in the extended language only if it had been so in the original. To express these ideas, we first define the relevant notion of part-whole and then the idea of being faithful in the whole to the logic of the part.

DEFINITION: L is a *sublanguage* of L' iff Syn is a part of Syn$'$, $U = U'$, $D = D'$, and for each i, either g_i is empty or identical to g'_i.

DEFINITION: L' is a *conservative extension* of L iff Syn is a part of Syn$'$ and the logical entailment relation of L' restricted to Sen is identical to that of L (i.e., for any subset X of Sen and any P in Sen, $X \vDash_{L'} P$ iff $X \vDash_L P$).

Notice that for any language it is required that if f_i is nonempty, so is g_i. Thus, if g_i is empty, so is f_i. Intuitively, an empty formation operation is equivalent to no operation at all, and we shall in fact identify any structure with empty operations with the structure obtained by deleting the empty operations and renumbering. The idea is intuitive enough but its formal statement is a bit baroque.

DEFINITION: A *deflation* of a structure $\langle X, O_1, \ldots, O_m\rangle$ is the structure $\langle X, O'_1, \ldots, O'_n\rangle$ such that there is a function p on $\{1, \ldots, m\}$ such that $\{0, 1, \ldots, n\}$ is the range of p, $O'_{p(i)} = O_i$, and p is defined (recursively) as follows:

(i) $p(1) = 1$ if Q_1 is nonempty and $p(1) = 0$ otherwise, and
(ii) $p(i+1) = p(i) + 1$ if Q_{i+1} is nonempty and $p(i+1) = 0$ otherwise.

When convenient we shall assume without comment that a structure is identical with its deflation.

THEOREM. If L is a sublanguage of L', then

(1) for any R in $[R]$, there is an R' in $[R']$ such that for any P in Sen, $R(P) = R'(P)$, and

(2) for any R' in $[R']$, there is an R in $[R]$ such that for any P in Sen, $R(P) = R'(P)$.

Proof. We prove part (2). Assume the antecedent of the theorem and that R' is in $[R']$. We define R as the restriction of R' to Sen. Since Sen is a subset of Sen$'$, R' is defined for Sen, and since R is the restriction of R' to Sen, R and R' agree on all values assigned to elements in Sen. What remains to be shown is that R meets the conditions for membership in $[R]$. Clearly, R assigns atomic elements values in U because R' does. Moreover, for the molecular case consider a function f_i of F that is nonempty, $R(f_i(P_1, \ldots, P_n)) = R'(f_i(P_1, \ldots, P_n))$ [since R is the restriction of R' to Sen] $= g_i(R'(P_1), \ldots, R'(P_n))$ [since R' is in $[R']$] $= g_i(R(P_1), \ldots, R(P_n))$ [since R is the restriction of R' to Sen]. Hence R is in $[R]$. End of Proof.

THEOREM. If L is a sublanguage of L', then L' is a conservative extension of L.

☐ **EXERCISE**

Prove the last theorem and part (1) of the one that precedes it.

Yet another technique for constructing logically similar languages is to introduce into a new language explicit connectives for truth-functions expressible in the original but only by means of molecular sentences. For example, we know that it is sufficient for classical propositional logic to have only negation and the conditional as primitive connectives in the syntax. We could then 'introduce' connectives for conjunction and disjunction by using $P \wedge Q$ as an abbreviation of the longer expression $\sim(P \rightarrow \sim Q)$ and $P \vee Q$ as short for $\sim P \rightarrow Q$. In such a theory a distinction is made between, on the one hand, the primitive notation defined as any sentence of the syntax and, on the other, the abbreviated forms. An abbreviated expression is not really a sentence of the syntax, though a genuine sentence of the syntax that it abbreviates is in principle obtainable from it. Thus an expression employing the defined forms is translated into primitive notation in order to transform it into a genuinely well-formed expression. It follows that the nonprimitive forms receive no direct semantic interpretation because in the matrix for the syntax there is no semantic operation corresponding to or interpreting the defined connectives. The expressions introduced by definition must be interpreted indirectly by first transforming them into primitive notation. A language using primitive and defined notation may be contrasted with one in which the syntax has a separate formation operation for all the connectives. Each of these connectives has its own syntactic operation in the syntax that generates genuine sentences, and each of these syntactic operations is interpreted by a corresponding semantic operation in the matrix. Of course, both sorts of language are in a deep sense capable of saying the same thing. We now take up discussion of this equivalence and how to formulate it. The first idea we need is that of a truth-function definable in terms of the other truth-functions of a matrix. It is a property of the abbreviated forms that their truth-functions are definable in this way.

DEFINITION: We define the notion of a function $[k]$ *constructible in a structure* $S = \langle X, O_1, \ldots, O_n\rangle$ inductively.

(1) (*Basis clause*) Any of the functions O_i of S is constructible in S; and

(2) (*Inductive clause*) if $g, h_i, \ldots, h_j$ are functions constructible in S and $\langle x_1, \ldots, x_m\rangle = \langle x_q, \ldots, x_r, \ldots, x_s, \ldots, x_t, x_u, \ldots, x_v\rangle$, then the function $[k]$ defined on X as follows is constructible in S:

$$[k](x_1, \ldots, x_m) = g(h_1(x_q, \ldots, x_r), \ldots, h_j(x_s, \ldots, x_t), x_u, \ldots, x_v).$$

(In this case we refer to $[k]$ as $[g \cdot h_i \cdot \ldots \cdot h_j]$).

We shall apply this definition to matrices in the obvious way by calling a function constructible in M iff it is constructible in the structure obtained by deleting D.

DEFINITION: L' is a *definitional extension* of L iff L is a sublanguage of L', B of Syn is identical to B' of Syn', and for each i, if g_i is empty but g'_i is nonempty, then g'_i is constructible in M.

THEOREM. If L' is a definitional extension of L, then there is a translation function t from Sen' onto Sen such that $X \vDash_{L'} P$ iff $t(X) \vDash_L t(P)$.

Proof. First, for a function $[k]$ constructible in M we define by recursion the function $*$ that assigns to $[k]$ a function constructible in Syn such that $[k]$ intuitively interprets the grammatical operation $[k]*$. If $[k]$ is constructible in M, we define $[k]*$ recursively.

(1) if for some i, $[k] = g_i$, then $[k]* = f_i$; and

(2) if $[k] = [g \cdot h_i \cdot \ldots \cdot h_j]$ for some functions g, h_i, h_j constructible in M, then $[k]* = [g* \cdot h_i * \cdot \ldots \cdot h_j *]$.

We make the claim that embodies the intuitive interpretation of $*$:

(I) for any R in $[R]$, $R([k]*(P_1, \ldots, P_n)) = [k](R(P_1), \ldots, R(P_n))$.

Proof of claim (I) is by induction on the inductive set of sentences in Syn and is left as an exercise. We define by recursion the relevant translation function t from Syn' onto Syn:

(1) if P is in B', $t(P) = P$;

(2) if P is some $f'_i(P_1, \ldots, P_n)$, then $t(P)$ is $[g'_i]*(t(P_1), \ldots, t(P_n))$.

It is clear that t is an 'into' function, but we also claim that

(II) t is an 'onto' function from Sen' to Sen.

Proof is again by induction on the set of sentences in Sen and is also left as an exercise. Now, let us use the usual notation $f \mid A$ for the restriction of the function f to A, i.e., $f \mid A = \{\langle x, y\rangle \mid \langle x, y\rangle \in f \wedge x \in A\}$. Note that below in the notation $f \mid A(x)$, $f \mid A$ is grouped together as a function name, and the whole $f \mid A(x)$ refers to the value of that function for the argument x. We are now ready to make our final claim:

(III) for any R' in $[R']$, $R'(P) = R' \mid \mathrm{Sen}(t(P))$.

Proof is by induction on the set of sentences in Sen'. For brevity let $R' \mid S$ be $R' \mid$ Sen.

Atomic case. Since $B' = B$, if P is in B', $R'(P) = R' \mid S(P) = R' \mid S(t(P))$ [since $t(P) = P$].

Molecular case.

$$\begin{aligned} R'(f'_i(P_1, \ldots, P_n)) &= [g'_i](R'(P_1), \ldots, R'(P_n)) \text{ [since } R' \text{ is in } [R']] \\ &= g'_i(R' \mid S(t(P_1)), \ldots, R' \mid S(t(P_n))) \text{ [by induction hypothesis]} \\ &= R' \mid S([g_i]*(t(P_1), \ldots, t(P_n)) \text{ [by claim (I)]} \\ &= R' \mid \mathrm{Sen}(t(f'_i(P_1, \ldots, P_n)) \text{ [by definition of } t]. \end{aligned}$$

End of Proof.

We are now suitably armed to attempt a broad review of how the logical entailment relations of various many-valued logics compare to that of classical logic.

☐ **EXERCISE**

Prove claims (I) and (II) in the preceding proof.

7.3 EXAMPLES OF MANY-VALUED LOGICS

7.3.1 Kleene's Weak and Strong Connectives

S. C. Kleene is responsible for two quite interesting three-valued matrix semantics. In the first the classical truth-values are augmented by a third that may be read as representing meaninglessness. A sentence assigned T is true in the usual sense, and one assigned F is false in the usual sense, but a sentence assigned the third value (we shall use N), though grammatical and hence genuinely a member of Sen, is supposed to be semantically ineffectual. Questions of its truth or falsity do not arise. This idea alone does not tell us how to project truth-values to molecular sentences. We need a second idea. Kleene proposed that any sentential whole containing a meaningless part is itself meaningless. Thus the semantic imperfection of the part affects the whole, and this idea is sufficient for completely determining a set of three-valued truth-functions for the usual connectives:

	~
T	F
F	T
N	N

∧	T	F	N
T	T	F	N
F	F	F	N
N	N	N	N

∨	T	F	N
T	T	T	N
F	T	F	N
N	N	N	N

→	T	F	N
T	T	F	N
F	T	T	N
N	N	N	N

Kleene's weak connectives

In another context Kleene proposes another reading of the third value and a correspondingly different projection of it to molecular sentences. The theory is meant to apply to those sorts of sentences in mathematics that can in principle be decided by an algorithm. Kleene's idea is to divide the mathematically relevant sentences into three classes: those for which there is a mathematical algorithm showing that it is true, those for which there is one showing that it is false (i.e., that its negation is true), and those which are undecided in the sense that there is no algorithm establishing either it or its negation. Here an algorithm is meant to be what we have called an effective process in Chapter 6. In the vocabulary of that discussion, we may express

Kleene's idea as follows. There is, he suggests, a decidable subset A of the set U of all mathematical propositions. Thus, the characteristic function f of A is definable as an effective process. For any P in U, if f is defined for P and $f(P) = \mathrm{T}$, then we say that the sentence is 'true'; if f is defined for P and $f(P) = \mathrm{F}$, then we say P is 'false' and $\sim P$ is 'true'; and if P is in $U - A$ and f is undefined for P, we suspend judgment on P. A computer program that answers mathematical inquiries for defined inputs with either a 'yes' or 'no' in a finite period of time would be an example. Such a testing procedure for mathematical truths would be, if accurate, a justification for believing those sentences that pass the test and for disbelieving those that fail. Thus, classification of sentences according to their status as decidable by an algorithm may be viewed as an epistemic semantics.

A more general reading of the truth-values as recording epistemic status is a natural abstraction from Kleene's particular application to sentences in mathematics. By this three-valued generalization any sentence receives one of the three values according to whether it is justified, refuted, or neither. There are those sentences that are fully justified and receive T, those which are fully refuted (their negations are fully justified) and receive F, and the remainder about which we are still in epistemic doubt. We shall find similar ideas underlying the semantics for intuitionistic logic when we discuss it later in this chapter.

The projection to molecular sentences that Kleene proposed for such truth-values is one that preserves a feature of the classical truth-tables. If the truth-value of one part of a sentence is enough to determine that of the whole, then it should remain so even when an additional value is introduced. Thus a conjunction with a false conjunct should be false regardless of whether the other conjunct is true, false, or neither. Likewise, a disjunction with a true disjunct is true, and a conditional with a false antecedent or a true consequent is true. Using this principle, which seems to be quite reasonable under an epistemic reading of the truth-values, we arrive at the following operations:

	$\sim$	$\wedge$	T	F	N	$\vee$	T	F	N	$\rightarrow$	T	F	N
T	F		T	F	N		T	T	T		T	F	N
F	T		F	F	F		T	F	N		T	T	T
N	N		N	F	N		T	N	N		T	N	N

Kleene's strong connectives

In order to define the matrices in the customary way, we shall use numbers as truth-values. It is customary to regard 1 as T, 0 as F, and 1/2 as N. We will also follow the usual practice in mathematics of identifying a (natural) number with the set of all its predecessors. For example, 0 is the empty set, $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$, etc. We define KW, the matrix for the weak connectives, to be the matrix $\langle\{0, 1/2, 1\}, \{1\}, \sim, \wedge, \vee, \rightarrow\rangle$ such that the operations $\sim$, $\wedge$, $\vee$, and $\rightarrow$ are as defined in the tables for the weak connectives. Similarly, KS, the matrix for the strong connectives, is defined as $\langle\{0, 1/2, 1\}, \{1\}, \sim, \wedge, \vee, \rightarrow\rangle$ such that the operations are defined by the truth-tables for the strong connectives. We shall now compare the logics of these matrices with those of classical logic. Let C be the matrix $\langle\{0, 1\}, \{1\}, \sim, \wedge, \vee, \rightarrow\rangle$ such that the operations conform to the classical truth-tables. Let us further assume $\mathrm{Syn} = \langle \mathrm{Sen}, \sim, \wedge, \vee, \rightarrow\rangle$ is a syntax for propositional logic, and the following languages are defined: $\mathrm{LC} = \langle \mathrm{Syn}, C\rangle$, $\mathrm{LKW} = \langle \mathrm{Syn}, \mathrm{KW}\rangle$, and $\mathrm{LKS} = \langle \mathrm{Syn}, \mathrm{KS}\rangle$.

THEOREM. (1) If $X \vDash_{\mathrm{LKW}} P$ or $X \vDash_{\mathrm{LKS}} P$, then $X \vDash_{\mathrm{LC}} P$;
(2) there is an X and P such that $X \vDash_{\mathrm{LC}} P$ and not($X \vDash_{\mathrm{LKW}} P$); and
(3) there is an X and P such that $X \vDash_{\mathrm{LC}} P$ and not($X \vDash_{\mathrm{LKS}} P$).

Proof. Proof of clause (1) follows directly from the fact that there are homomorphisms from KW and KS into C that preserve designation and nondesignation. Clauses (2) and (3) are proven by finding the right examples. Consider the argument from P to $(P \wedge Q) \vee (P \wedge \sim Q)$. End of Proof.

Though there are interesting generalizations of these matrices to more than three values, we shall not consider them here but turn instead to some other three-valued theories that readily lend themselves to comparison with classical logic by the techniques we have developed.

☐ **EXERCISE**

By KW$*$ let us mean the matrix like KW except that both 1 and 1/2 are designated (i.e., $D = \{1/2, 1\}$). Likewise, let KS$*$ be like KS except that both 1 and 1/2 are designated. Let LKW$*$ be $\langle \mathrm{Syn}, \mathrm{KW}*\rangle$, and LKS$*$ be $\langle \mathrm{Syn}, \mathrm{KS}*\rangle$. Prove that a metatheorem just like that above continues to hold with the $*$ matrices replacing the original versions.

7.3.2 Lukasiewicz's Three-Valued Logic

One of the earlier versions of three-valued logic was developed by the Polish logician Jan Łukasiewicz to deal with problems like future contingents. Those sentences that are not determined receive the third value. The matrix he proposed is quite like that of the strong connectives except for the conditional which has the special feature that if both the antecedent and consequent are undetermined, the whole is true:

	$\sim$
T	F
F	T
N	N

$\wedge$	T	F	N
T	T	F	N
F	F	F	F
N	N	F	N

$\vee$	T	F	N
T	T	T	T
F	T	F	N
N	T	N	N

$\rightarrow$	T	F	N
T	T	F	N
F	T	T	T
N	T	N	T

Łukasiewicz's three-valued connectives

Again let us use numbers for truth-values and define the matrix L3, Łukasiewicz's three-valued matrix, to be $\langle\{0, 1/2, 1\}, \{1\}, \wedge, \vee, \rightarrow\rangle$, and set LL3 = $\langle$Syn, L3$\rangle$. Likewise, let L3* and LL3* be like their unstarred originals except that both 1 and 1/2 are designated.

THEOREM. The logical entailment relations $\vDash_{LL3}$ and $\vDash_{LL3*}$ are proper subsets of $\vDash_{LC}$.

Proof. The examples for LKS and LKS* will work for Łukasiewicz's matrix because they do not make use of the conditional.

It is possible to generalize the ideas in this matrix to arbitrarily many values. Let min be a function that pairs with any two arguments their minimum, and let max be the function that pairs with them their maximum.[11]

DEFINITION: By Ln we mean the matrix $\langle U_n, \{1\}, \sim, \wedge, \vee, \rightarrow\rangle$ such that

(1) $U_n = \{n/n, \ldots, 0/n\}$;
(2) $\sim(x) = 1 - x$;
(3) $\wedge(x, y) = \min(x, y)$;
(4) $\vee(x, y) = \max(x, y)$; and
(5) $\rightarrow(x, y) = \min(1, (1 - x) + y)$.

DEFINITION: By Lω we mean the matrix $\langle R, \{1\}, \sim, \wedge, \vee, \rightarrow\rangle$ such that R is the set of rational numbers (ratios of positive integers), and the operations are as defined in clauses (2)-(5) above. (Here ω is the limit ordinal representing the set of all natural numbers $\{0, 1, 2, \ldots\}$.) These matrices may be used to form a sequence of languages with increasingly strong entailment relations culminating in classical logic. Let LLω be $\langle$Syn, L$\omega\rangle$, LLn be $\langle$Syn, Ln$\rangle$, etc.

THEOREM. The logical entailment relation $\vDash_{LL\omega}$ is a proper subset of any $\vDash_{LLn}$; $\vDash_{LLn}$ is a proper subset of any $\vDash_{LLm}$ such that $m < n$; and L2 = C.

[11] The following results are due to techniques of Gödel and Dugundi. See Nicholas Rescher, *Many-Valued Logic* (1969), pp. 188–195.

Proof. Proof that the relations are subsets follows from the fact that there are relevant homomorphisms from the larger matrices into the smaller that preserve designation and nondesignation. However, proof that the subsets are proper is more difficult and will not be attempted here.

7.3.3 Product Logics

Product logics, first developed by the Polish logician Stanisław Jaskowski, are another kind of logic that illustrates the technique of comparing entailment relations through structural similarities among matrices.[12] First we define the general idea and then illustrate it by an example.

DEFINITION: M^n, relative to a matrix $M = \langle U, D, g_1, \ldots, g_m\rangle$, is that matrix $\langle U^n, D^n, h_1, \ldots, h_m\rangle$ such that

(1) U^n and D^n are the sets of n-tuples of U and D respectively; and
(2) for each i, $h_i(\langle x_{1,1}, \ldots, x_{1,n}\rangle, \ldots, \langle x_{m,1}, \ldots, x_{m,n}\rangle) = \langle g_i(x_{1,1}, \ldots, x_{m,1}), \ldots, g_i(x_{1,n}, \ldots, x_{m,n})\rangle$.

A special case is the matrix C^2, which interested Łukasiewicz a good deal because its logic is classical. Indeed it is possible to show by structural similarities among matrices that M and M^n have the same logic. Let LM^n = $\langle$Syn, $M^n\rangle$.

	$\sim$
11	00
10	01
01	10
00	11

$\wedge$	11	10	01	00
11	11	10	01	00
10	10	10	00	00
01	01	00	01	00
00	00	00	00	00

$\vee$	11	10	01	00
11	11	11	11	11
10	11	10	11	10
01	11	11	01	01
00	11	10	01	00

The four-valued tables for C^2

THEOREM. $X \vDash_{LM^n} P$ iff $X \vDash_{LM} P$.

Proof. The theorem holds because there are relevant 'onto' homomorphisms from M^n to M.

So far our comparison of languages has employed homomorphism among matrices only. For examples of the use of the other structural properties, we

[12] See S. Jaskowski, 'Investigations into the System of Intuitionist Logic' (1936). For more recent applications of product logics to problems in the philosophy of language, see Hans Herzberger, 'Dimensions of Truth' (1973), John N. Martin, 'A Many-Valued Semantics for Category Mistakes' (1975), and Merrie Bergmann, 'Presupposition in Two Dimensions' (1981).

now turn to an interesting attempt by the Russian logician D. A. Bochvar to introduce a third value and retain precisely classical logic.

7.3.4 Bochvar's Internal and External Connectives

Bochvar's idea is that sentences with meaningless parts are indeed meaningless, as Kleene suggested in his interpretation of the weak connectives. We can however render the parts of a sentence bivalent (i.e., either true or false) by first affixing a truth operator to them. The operator assigns any true sentence the value true, and any sentence that is not true, whether it be false or something else, the value false. Frege had used a similar operator in his formalizations of mathematics, and Bochvar showed that the portion of the language rectified by having its atomic sentences prefixed by the truth operator consitutes a perfectly classical fragment of the language. All that is necessary, then, for logic to be perfectly classical is that we ensure that our atomic sentences be bivalent by prefixing them with a truth operator meaning the same as the English phrase 'It is true that...'.

Historically, the sort of meaninglessness that Bochvar was concerned with is that characteristic of paradoxes. He marked with the third value the semantic deviance of sentences (like the liar paradox or the paradoxes of naive set theory) that are provably both true and false. Later, in an important paper, Timothy Smiley interpreted Bochvar's third value as marking the failure of presupposition, thus beginning a long discussion among logicians of the proper way, if any, of representing presuppositions within many-valued semantics.

Bochvar's result is developed in stages. We first introduce a language using essentially Kleene's weak connectives, though our version will generalize the idea to arbitrarily many values.[13] First, let SynC be the syntax for classical logic using only negation and conjunction formation operations, let MC be the classical matrix for these connectives, and let LC = ⟨SynC, MC⟩. We shall use the same names $\sim C$ and $\wedge C$ for the operations of the two structures. Though we shall suppress mention of the other connectives, all the results continue to hold if, relative to each syntax, they are introduced by the usual definitions in terms of negation and conjunction. We now give names to two important ways in which a many-valued matrix may resemble the classical two-valued matrix. First of all, it may treat the classical values in the same way that the classical matrix does, and second, it may assign classical values only to wholes which have classical values as parts.

DEFINITION: The operations $\sim$ and $\wedge$ of a matrix are called *normal* iff whenever x and y are in $\{0, 1\}$, then $\sim(x) = \sim C(x)$, and $\wedge(x, y) = \wedge C(x, y)$; and they are called *sensitive* iff whenever x and y are not in $\{0, 1\}$, then $\sim(x)$, and $\wedge(x, y)$ are not in $\{0, 1\}$.

[13] These generalizations are from John N. Martin, 'The Semantics of Frege's *Grundgesetze*' (1984).

DEFINITION: An internal language is any LI = ⟨SynI, MI⟩ such that

(1) SynI, which we shall assume is ⟨Sen, B, $\sim I$, $\wedge I$⟩, is any propositional logic syntax such that $\sim I$ is unary and $\wedge I$ is binary; and

(2) MI is ⟨U, $\{1\}$, $\sim I$, $\wedge I$⟩ (called an *internal matrix*) such that U is a set of values including 0 and 1, and the operations $\sim I$ and $\wedge I$ are normal and sensitive.

The reader can easily verify that if U is the three-valued set {T, F, N} then MI determines the same tables as Kleene's weak connectives:

	~
T	F
F	T
N	N

∧	T	F	N
	T	F	N
	F	F	N
	N	N	N

∨	T	F	N
	T	T	N
	T	F	N
	N	N	N

→	T	F	N
	T	F	N
	T	T	N
	N	N	N

Bochvar's internal connectives

DEFINITION: An *extension* LI+ of an internal language LI is any ⟨SynI+, MI+⟩ such that

(1) SynI+ = ⟨SenI+, B, $\sim I$, $\wedge I$, τ, $\sim E$, $\wedge E$⟩, τ is a one-place operator (the truth operator), and $\sim E$ and $\wedge E$ are new one-place and two-place operators, respectively.

(2) MI+ = ⟨U, $\{1\}$, $\sim I$, $\wedge I$, τ, $\sim E$, $\wedge E$⟩ such that

(a) for any x, if $x = 1$, $\tau(x) = 1$, and $\tau(x) = 0$ otherwise;

(b) for any x and y, $\sim \mathrm{E}(x) = \sim \mathrm{I}(\tau(x))$ and $\wedge E(x, y) = \wedge I(\tau(x), \tau(y))$.

DEFINITION: By the *external language* LE relative to an internal language LI and its extension LI+ we mean ⟨SynE, ME⟩ such that

(1) SynE is ⟨SenE, B, $\sim E$, $\wedge E$⟩; and

(2) ME is ⟨U, $\{1\}$, $\sim E$, $\wedge E$⟩.

In the three-valued case in which U is {T, F, N} it is straightforward to verify that the truth operator and the external connectives conform to the following tables:

	T
T	T
F	F
N	F

	~
T	F
F	T
N	T

∧	T	F	N
	T	F	F
	F	F	F
	F	F	F

∨	T	F	N
	T	T	T
	T	F	F
	T	F	F

→	T	F	N
	T	F	F
	T	T	T
	T	T	T

Bochvar's truth operator and external connectives

THEOREM. There is some isomorphism from SynE to SynC such that $X \vDash_{LC} P$ iff $t(X) \vDash_{LI} t(P)$.

THEOREM. ME is homomorphic onto MC in a way that preserves designation and nondesignation, and hence $X \vDash_{LE} P$ iff $X \vDash_{LI+} P$.

THEOREM. LE is a sublanguage of LI+ and hence $X \vDash_{LE} P$ iff $X \vDash_{LI+} P$.

☐ EXERCISE

Prove the previous three metatheorems by using the results of the previous section.

THEOREM. There is a translation function t from SenC to SenI such that $X \vDash_{LC} P$ iff $t(X) \vDash_{LI} t(P)$.

Proof. The previous results establish that there are homomorphisms h and h' such that for any X and P of SynE:

$$h(X) \vDash_{LC} h(P) \text{ iff } h'(X) \vDash_{LI} h'(P).$$

Assume for arbitrary Y and Q of SenC that $Y \vDash_{LC} Q$. Then since h is an isomorphism, its inverse h^{-1} is also, and $h(h^{-1}(Y)) \vDash_{LC} h(h^{-1}(Q))$. Then by the previous results $h'(h^{-1}(Y)) \vDash_{LI} h'(h^{-1}(Q))$ and the function t defined as $t(P) = h'(h^{-1}(P))$ is the translation desired. End of Proof.

7.4 NON-TRUTH-FUNCTIONAL SENTENTIAL LOGICS

7.4.1 The Issue of Truth-Functionality

The motivation for many-valued semantics lies in the desire to classify sentences into more categories than just the true and the false, but as the survey of the last section shows, introducing new truth-values as representatives of the additional classes has unwanted consequences for logical entailment. In most cases the entailment relations of the new languages reject some classically valid arguments. The rejection of classical validities can be responded to in various ways. One reply is to maintain that the rejected validities are nonintuitive anyway, and it is fair to say that there is at least some plausibility to this view inasmuch as arguments rejected by the strong connectives and Łukasiewicz' matrices often depend on unintuitive features of classical logic. Among these are the paradoxes of material implication (Chapter 1) and arguments that introduce in the conclusion new sentences that do not even appear as parts of those sentences in the premises. It is, moreover, possible to characterize exactly what the rejected arguments are, though we shall not do so here.[14] It is also possible to experiment with novel ways to define logical entailment while retaining a matrix semantics.[15] In this

[14] For example, Bas van Fraassen has shown that

$$P \vDash_{LKW} Q \leftrightarrow (\vDash_{LC} P, \text{ or all atomic sentences in } Q \text{ are in } P \text{ and } P \vDash_{LC} Q).$$

It is also fairly easy to show that $P \vDash_{LKW*} Q$ iff ($P \vDash_{LC} Q$ and (if not $P \vDash_{LC} Q$, then all the atomic sentences of P are in Q)). Less obvious results of a similar nature may be found in John N. Martin, 'A Syntactic Characterization of Kleene's Strong Connectives' (1975), and Merrie Bergmann, 'Logic and Sortal Incorrectness' (1977).

[15] For example, it can be shown that $\vDash_{LKW*} P$ iff $\vDash_{LKS*} P$ iff $\vDash_{LC} P$.

chapter the response we shall discuss in detail involves the rejection of the matrix format for semantical theory. If we abandon the assumption that semantics is truth-functional, we may be able to obtain a more adequate logic.

Before discussing some examples of such theories, we should pause first to explain the nature of the assumption that they sacrifice. What is truth-functionality and why is it important? Briefly put, truth-functionality is the property which reference relations possess when they assign truth-values to a whole in a manner uniquely determined by the truth-values assigned to its immediate parts. The formal idea of a truth-functional semantics is captured in the concept of a matrix language. Intimately tied to truth-functionality is a logical property characteristic of classical logic and lost in non-truth-functional semantics. This is the property of the valid substitutability of one sentence for another of like truth-value in a larger sentence, or what is called *substitutability salva veritate*. This property fails if the reference relations are non-truth-functional.

THEOREM. *Substitutability of Material Equivalents.* Make the following assumptions: L is a matrix $\langle \text{Syn}, M \rangle$, S is some sentence of Syn that contains the sentence P, $[S]^P_Q$ is like S except for containing the sentence Q at one or more places where S contains P, and R is a member of the set $[R]$ of possible worlds for L. It follows then that if $R(P) = R(Q)$, then $R(S) = R[S]^P_Q$.

This theorem is really a special case of a more general property of abstract algebras proven in Chapter 2. If the matrix relations g_i are functions, then each reference relation R is a homomorphism from Syn to M, and the relation $P \equiv Q$ defined as $R(P) = R(Q)$ is a congruence relation admitting substitutability of coreferential parts without altering the reference of the whole.

When pressed, however, it is difficult to know how much importance to attach to this sort of substitutability, and individual cases of nonmatrix languages are appraised in terms of their various compensating virtues. This kind of substitutability is part of what is meant when classical logic is said to be *extensional*, and we shall find other examples, especially in modal and intensional logic where the property is abandoned in order to enrich semantic theory.

7.4.2 Intuitionistic Logic

Historically, intuitionistic logic was developed first as a particular theory of proof motivated by perceived weaknesses in classical proof theory. It was provided with a semantics only much later. Let us begin then by discussing intuitionistic ideas of proof.

Mathematically minded logicians concerned with codifying not reasoning in general but the particular sort of reasoning done in mathematics observed that mathematicians often encountered nonintuitive results when they used some questionable reasoning techniques of classical logic, and especially when they assumed the law of excluded middle, which in its syntactic form is $\vdash(P \vee \sim P)$. This group of logicians, whose views came to be known as *intuitionism*, observed that in mathematics it is inappropriate to assume that every sentence is either true or false. It may well be possible, they reasoned, that there may be some mathematical questions that are not resolvable in principle. Certainly, it is the case that there are still open questions, and it is only a matter of classical dogma, they suggested, to assume that these open questions can all ultimately be decided one way or the other.

These ideas were developed and explained by making them a part of a wider theory of mathematical truth. Truth, at least in mathematics, they claimed, may be analyzed through the concept of proof. A sentence is true if it is proven, false if it is refuted, and neither true nor false otherwise. Moreover, negation is interpreted as saying 'this sentence is refuted'. Hence, to prove $\sim P$ is the same as refuting P. The other connectives are explained in a similar way. A conjunction is proven iff both its conjuncts are, a disjunction is proven iff either of its disjuncts are, and a conditional is proven iff there is a proof that attaches to a proof of the antecendent so as to yield a proof of the consequent.

These semantic ideas defining truth and provability were only used informally to explain the more formal statement of the theory which was proof theoretic. The earliest formulations were in terms of axiom systems, but the same ideas are usually expressed today in the more general form of natural deduction systems.

Proof theoretically, the way to eliminate exluded middle must proceed indirectly by eliminating one or more of the classical assumptions used in proving it. As established in an earlier exercise, one classical rule used in its derivation, negation elimination (or more familiarly, double negation), is doubtful for the same reason excluded middle is. To have proven $\sim\sim P$ is to have refuted the proposition that P is false. But that does not entail that P is true because the possibility remains that P is neither. Thus the intuitionistic adjustment to classical proof theory is to replace $\sim$-elimination with

$$\frac{X \vdash P \qquad Y \vdash \sim P}{X, Y \vdash Q}$$

Intuitionistic $\sim$-elimination

We keep the other rules the same. This one change results in quite major changes in the resulting notion of $\vdash$. We now list some of the classical results that fail in the new theory and some that remain true. Let $\vdash$ remain as defined in classical proof theory as the closure of the basic deductions under the classical rules amended to include the intuitionistic version of $\sim$-elimination.[16]

THEOREM. The following are not true in intuitionistic natural deduction:

(1) $\vdash P \vee \sim P$

(2) $$\frac{\sim\sim P}{P}$$

(3) $\vdash \sim\sim P \to P$

(4) $$\frac{\sim P \vdash \bot}{\vdash P}$$

(5) $P \to Q, \sim P \to Q \vdash Q$

(6) $$\frac{\sim(\sim P \wedge \sim Q)}{P \vee Q} \qquad \frac{\sim(\sim P \vee \sim Q)}{P \wedge Q}$$

(7) $$\frac{\sim P \to \sim Q}{Q \to P} \qquad \frac{\sim P \to Q}{\sim Q \to P}$$

(8) $$\frac{\sim\sim P \to P}{P \vee \sim P} \qquad \frac{P \to Q}{\sim P \vee Q}$$

(9) $$\frac{P \to (Q \vee S)}{(P \to Q) \vee (P \to S)}$$

THEOREM. The following classical results continue to hold in intuitionistic natural deduction:

(1) $\vdash \sim(P \wedge \sim P)$ and $\vdash \sim\sim(P \vee \sim P)$

(2) $$\frac{P}{\sim\sim P}$$

(3) $\vdash(P \to \sim\sim P)$

[16] For an introduction to the ideas motivating intuitionistic logic see Chapter 1 of Michael Dummett, *Elements of Intuitionism* (1977), and Section 13 of S. C. Kleene, *Introduction to Metamathematics* (1971). For a fuller exposition of the natural deduction system see Dummett, Chapter 4, and for a detailed comparison to classical natural deduction see Section 4.5 of Neil Tennant, *Natural Logic* (1978), as well as Chapter VI of Kleene.

(4) $\dfrac{P \vdash \bot}{\vdash \sim P}$

(5) $\dfrac{\sim(P \vee Q)}{\sim P \wedge \sim Q}$ $\dfrac{\sim P \wedge \sim Q}{\sim(P \vee Q)}$

(6) $\dfrac{P \rightarrow Q}{\sim Q \rightarrow \sim P}$ $\dfrac{P \rightarrow \sim Q}{Q \rightarrow \sim P}$

(7) $\dfrac{P \vee \sim P}{\sim \sim P \rightarrow P}$

(8) $\dfrac{\sim P \vee Q}{P \rightarrow Q}$

THEOREM. If $\vdash \sim P$ in classical natural deduction, then $\vdash \sim P$ in intuitionistic natural deduction.

☐ EXERCISE

Prove the last metatheorem as well as results (1)–(3) of the one preceding it.

7.4.3 Beth's Semantics for Intuitionistic Logic

We can conclude from the foregoing discussion that there is a clear sense in which intuitionistic logic rejects some doubtful classical inferences, but what remains unclear is the semantics of the natural deduction proof theory. In particular, how is the semantic theory to be developed so that its valid inferences coincide exactly with the provable deductions of the system? It must be done, moreover, by means of a conceptually plausible analysis of truth in terms of a proof that rejects the law of excluded middle.

Conceptually, the whole idea of a semantics for intuitionistic logic is somewhat strange if not incoherent. The argument for this view runs as follows. What is characteristic of intuitionism is the rejection of a classical tenet of philosophy that there is a difference between truth, understood as sentences corresponding to an objective real world, and knowledge, consisting of true, well-justified beliefs about the world. In intuitionism truth and knowledge are collapsed inasmuch as a sentence being true is conceptually identical to the epistemic state of it being proven. Indeed, it is perfectly consistent with the intuitionistic explanation of mathematical truth that there is no external world for sentences to correspond to, but only a mental life with its various epistemic states of possession of proof, possession of refutation, and ignorance. Some philosophers like Michael Dummett have even taken the intuitionistic conception of mathematical truth as a key to the understanding of all truth, and have accordingly raised doubts about the need to posit a real world at all for the purposes of semantics and logical theory. A second reason for being skeptical of the project of developing a semantics for intuitionistic logic is a technical result by Gödel that shows that there is no finitely valued matrix language whose logical entailment relation is that of intuitionistic logic.[17] A straightforward two- or three-valued matrix semantics is therefore impossible.

Nevertheless, there are semantical accounts of the intuitionistic connectives, but they differ from traditional many-valued logic in being nontruth-functional. Here we shall present an interpretation due to E. W. Beth. The fundamental idea of this semantics is to evaluate sentences relative not to possible worlds in a robust realistic sense but rather to worlds understood epistemically as states of information. The supposition is that relative to these states it is possible to classify sentences according to whether they are provable or not. Those that are so are true (in the intuitionistic sense) and are assigned 1, those that are not are assigned 0. The semantics is epistemic in the sense that the intended readings of the truth-values are provided by concepts from traditional epistemology, but it is at the same time two-valued. The law of excluded middle nevertheless fails because, given the special non-truth-functional way truth-values are projected onto molecular sentences, both the sentences P and $\sim P$ may be 0.

This twofold classification of atomic sentences is then projected to molecular sentences, but in ways significantly different from matrix semantics. In particular, the truth-value of the whole relative to a 'world' is no longer completely determined by those of its parts. The details of this projection deserve some special comment because it is both new and not very intuitive.

Worlds in the relevant sense are understood to be states of information. It is assumed that these states are ordered in the sense that information can only increase. What information we have we do not lose, and the information we have may be augmented in various possible ways. Hence, these states form a tree structure such that if a sentence is provable relative to any state of information it remains provable at any states subsequent to or 'beneath' it on the tree. We also add an assumption that from a mathematical perspective any finitely distant improvement in information is in principle accessible. (We could, for example, just wait around until it comes.) The theory therefore assumes that any sentence P that is true at a finite distance in every possible refinement of an information state is also true at that state. We shall call such a sentence *finitely inevitable*. Moreover, if a sentence is finitely inevitable from

[17] See Michael Dummett, *Elements of Intuitionism* (1977), p. 172.

a state w, then we can collect a set of all worlds with a special property. Each of these worlds is a world in which P is true, and each is only a finite distance beneath w. This set in a sense forms a barrier across the tree structure beneath w. No matter which branch you descend from w, you will in a finite time run into at least one world in the barrier set. This set is said to 'bar' w. We begin by defining the general ideas necessary for talking about ordering states of information on trees. We assume a propositional logic syntax with the connectives $\sim$, $\wedge$, $\vee$, and $\rightarrow$.[18]

DEFINITION: A structure $\langle A, \leq \rangle$ is said to be a *partial ordering* iff $\leq$ is
(1) reflexive, i.e., for any x in A, $x \leq x$;
(2) transitive, i.e., for any x, y, z in A, if $x \leq y \wedge y \leq z$, then $x \leq z$; and
(3) antisymmetric, i.e., for any x and y in A, if $x \leq y \wedge y \leq x$, then $x = y$.

DEFINITION: For x and y in A, if $x \leq y$, x is said to *precede* y and y to *succeed* x, and if in addition there is no z between them (for any z in A if $x \leq z$ or $z \leq y$, then $x = z$ or $z = y$) then we say x is the $\leq$-*immediate predecessor* of y and y is the $\leq$-*immediate successor* of x.

As is customary in the subject, we refer to the set of possible worlds as K and to the possible worlds in K by lower case k, with and without prime marks and subscripts.

DEFINITION: A *world structure* is any $\langle K, \leq \rangle$ such that
(1) $\leq$ is a partial ordering on K;
(2) there is a unique maximal element E in K (i.e., E is in K, and all elements in K are $\leq$-predecessors of E);
(3) for each element k of K, there is a unique finite chain $k_n, \ldots, k_1$ such that $k = k_n$, $E = k_1$, and each element of the chain is an $\leq$-immediate successor of the previous element. (This chain is said to be the *branch ending with* k and to *contain* the branch $k_m, \ldots, k_1$ such that $m < n$.)

Let us adopt the convention that $\langle K, \leq \rangle$ ranges over world structures and k, k', k'' over elements of K. We are now in a position to define the assignment of truth-values to worlds. In the definition below the requirements on the assignment of values to atomic sentences state in a slightly more formal idiom the ideas that information does not degenerate and that finitely inevitable information is accessible.

The clauses assigning values to molecular sentences are less transparent. Their justification is in part that they work in the sense of yielding the later completeness result. But one of the requirements of semantic theory is supposed to be conceptual adequacy. The various clauses are supposed to state plausible analyses of the connectives, given the intended readings of the truth-values. In this theory, then, the clause for negation should say that we have a proof that the unnegated sentence is absurd. The clause for conjunction should embody the idea that given a proof for each part of a conjunction we may turn them into a proof for the whole conjunction. That for disjunction should say that a proof for either part of a disjunction may be transformed into a proof for the whole. That for the conditional should say that given a proof for the antecedent we may transform it into a proof of the consequent. Whether they do in fact express these ideas in a more formal idiom is somewhat doubtful.

[18] See E. W. Beth, *The Foundations of Mathematics* (1968). The presentation here follows that of Dummett, *Elements of Intuitionism* (1977). Kripke has also proposed an alternative semantics which differs from Beth's in a number of important technical ways but which retains the intuitive interpretation of 'worlds' as states of information. See Saul Kripke, 'Semantical Analysis of Intuitionistic Logic, I' (1965).

DEFINITION: A subset K' of K is said to *bar* k relative to $\langle K, \leq \rangle$ iff there is some branch b of $\langle K, R \rangle$ ending with k such that for any branch b' of $\langle K, R \rangle$ containing b there is an element k' of K' that is a strict $\leq$-predecessor of k.

DEFINITION: The set $[R]$ of *possible worlds* relative to $\langle \mathrm{K}, \mathrm{R} \rangle$ is defined as $\{R \mid R(K \times \mathrm{Sen} \xrightarrow[\text{into}]{} \{0, 1\})$ such that for any P and k,

(1) if P is atomic, then
(a) $R(k, P) = 1$ only if $R(k', P) = 1$ for all $\leq$-precedessors k' of k;
(b) if some subset K' of K bars k and all k' of K' are such that $R(k', P) = 1$, then $R(k, P) = 1$;
(2) if P is molecular, then
(a) if P is some $\sim Q$, $R(k, P) = 1$ iff, for all $\leq$-predecessors k' of k, $R(k', Q) \neq 1$;
(b) if P is some $Q \wedge S$, $R(k, P) = 1$ iff $R(k, Q) = R(k, S) = 1$;
(c) if P is some $Q \vee S$, $R(k, P) = 1$ iff there is some subset K' of K that bars k and is such that for any k' of K', either $R(k', Q) = 1$ or $R(k', S) = 1$;
(d) if P is some $Q \rightarrow S$, $R(k, P) = 1$ iff, for all $\leq$-predecessors k' of k, $R(k', Q) = 1$ only if $R(k', S) = 1\}$.

Let R range over possible worlds, and instead of writing $R(k, P)$ we shall write $R_k(P)$. We may identify the (Beth) *intuitionistic language* (briefly, LI) with $\langle \mathrm{Sen}, [R] \rangle$.

We now state two useful and interesting properties of world structures.

DEFINITION: We define P to be *finitely inevitable* relative to $\langle K, \leq \rangle$, R, and k iff
(1) some subset K' of K bars k and
(2) for all k' of K', $R_{k'}(P) = 1$.

THEOREM. (1) If k is a least element of K, then R_k is classical (i.e., it is bivalent and assigns values to molecular sentences in accordance with the matrix C); and (2) $R_k(P) = 1$ iff P is finitely inevitable relative to $\langle K, \leq \rangle$, R, and k.

☐ **EXERCISES**

1. Prove the last metatheorem.
2. Explain which clause for molecular sentences in the defintion of [R] best captures the intuitive ideas about proof underlying intuitionistic semantics. Which clause seems to do least well?

We now state but do not prove Beth's completeness theorem.[19]

THEOREM. $X \vdash P$ (in intuitionistic natural deduction) iff $X \vDash_{\mathrm{LI}} P$.

7.4.4 Supervaluations

In all the examples of nonstandard semantics which we have so far considered, some classically valid inferences are rejected. The theory we shall now discuss has as its main virtue that it simultaneously allows for three truth-values and retains a perfectly classical account of entailment. The theory is due to Bas van Fraassen, and makes use of a special notion of a non-truth-functional reference relation. Before beginning our discussion, we shall adopt a change of vocabulary so as to use here the terms commonly found in the literature. In particular, it is common in logic generally, and always true in discussions of supervaluations, to call what we have been referring to as a reference relation, a function from sentences to truth-values, by the term *valuation*. A valuation is some assignment of truth-values to sentences of a propositional syntax, which we have hitherto been calling a 'possible world' or reference relation.

The theory of supervaluations was originally developed to represent failures of presupposition, but has been applied in interesting ways to all the problems that have motivated three-valued semantics. Suppose, for whatever reason, that some of the atomic sentences lack a classical truth-value. The remaining sentences that are bivalent (are either T or F) determine in a straightforward sense a partial world. Some but not all the facts of that world are decided. Those which are decided are the ones described by the bivalent sentences; those as yet undecided are those corresponding to the sentences that lack either T or F. We might attempt to complete a semantic theory with this information alone by proposing some way to calculate the truth-values for molecular sentences from this partial atomic valuation. The various three-valued matrix theories offer alternative ways to do this. Van Fraassen's idea is to retain as much of classical semantics as possible. He observed that any partial assignment, call it $R*$, of T or F to atomic sentences determines a

[19] For an explanation of how Beth's semantics may be reformulated in an equivalent, non-truth-functional three-valued epistemic logic, that is a special case of supervaluations, see John N. Martin, 'Epistemic Semantics for Classical and Intuitionistic Logic' (1984).

unique set of classical reference relations (valuations), namely the set of all classical valuations R that agree with $R*$ as far as $R*$ is defined. A novel, for example, is determinate only about some elementary facts, those which it explicitly says are true or false. There will also be other facts about which we may speculate but which the novel leaves open. The set of classical worlds consistent with the novel may be quite large. What is interesting is that, in a sense, this set of worlds contains all the information of the novel itself. Given the novel we can define the set of worlds and given a set of classical worlds we can determine a novel.[20]

DEFINITION: By a *partial atomic valuation* relative to a propositional syntax Syn = $\langle Sen, B, F\rangle$ is meant any function $R*$ such that for some B', B' is a subset of B and $R*(B' \xrightarrow[\text{into}]{} \{\mathrm{T}, \mathrm{F}\})$.

DEFINITION: If $R*$ is a partial atomic valuation, then the set $[[R*]]$ of completions of $R*$ is defined as

$$\{R \mid R \in [\mathrm{RC}] \text{ and for any } P \text{ in } B, \text{ if } R*(P) \in \{\mathrm{T}, \mathrm{F}\}, \text{ then } R(P) = R*(P)\}.$$

Here [RC] is the familiar set of two-valued classical valuations determined by the classical matrix C for Syn. Clearly a partial atomic valuation $R*$ determines a unique set of classical completions. A set of classical completions also determines a unique partial atomic valuation.

THEOREM. For every subset X of [RC], there is a partial atomic valuation $R*$ such that X is $[[R*]]$, the set of classical completions of $R*$.

☐ **EXERCISE**

Prove the metatheorem.

Mathematically, then, a set of classical worlds can be understood as representing a single partially undefined world, namely that world that all the classical worlds agree about. Van Fraassen's idea is then to let this agreement determine an assignment to molecular sentences as well. A molecular sentence is assigned T or F if the classical worlds consistent with a partial atomic valuation are unanimous in assigning it T or F, and it is not assigned anything (or, equivalently, is assigned N) if there is no unanimity among its classical completions. A three-valued assignment (called a *supervaluation*) is thus defined in two steps. First a partial atomic valuation $R*$ is given, then its set of classical completions $[[R*]]$ is determined, and finally a supervaluation S is defined as the function recording the unanimous assignments

[20] For the theory in its original form see Bas C. van Fraassen, 'Singular Terms, Truth-Value Gaps, and Free Logic' (1966), and *Formal Semantics and Logic* (1968). The presentation here follows that of Hans G. Herzberger, 'Canonical Superlanguages' (1975).

of [[$R*$]]. To sentences for which there is no unanimity, the supervaluation does not assign either T or F.

In the usual development of the theory the fact that the function assigns neither T nor F to a sentence P is formalized by making S a partial function on Sen and making S undefined for P. That is, P is placed outside the domain of S. In this case the domain of S is some proper subset of Sen, P is not a member of the domain, and S is literally undefined for P. If S is undefined in this way for some sentence P, S is said to have a *truth-value gap*. The same idea might equally well be formalized by making S a three-valued function and having it assign a third value N in those cases in which it would be undefined by the usual theory. Thus, in all important respects the semantics is three-valued.

DEFINITION: A *base for a superlanguage* is any family B such that each element of B is some set [[$R*$]] of classical completions of some partial valuation $R*$.

DEFINITION: If X is a subset of [RC], then the supervaluation $X+$ established by X is defined as that function from a subset A of Sen into {T, F} such that

(1) for any P of Sen, if for all R of X, $R(P) = \mathrm{T}$, then $X+(P) = \mathrm{T}$;
(2) for any P of Sen, if for all R of X, $R(P) = \mathrm{F}$, then $X+(P) = \mathrm{F}$;
(3) $X+$ is undefined for all other sentences.

A simplifying advantage obtained from representing non-unanimity by undefinedness rather than the assignment of a third value is the result that a supervaluation turns out to be merely the intersection of the family of classical completions establishing it.

THEOREM. For any subset X of [RC], $X+ = \bigcap X$. (Here, $\bigcap X$ is defined in the usual set theoretic way as $\{x \mid \text{for any } R \text{ in } X, x \in R\}$.)

DEFINITION: The set [S] of supervaluations established by elements of a base B for a superlanguage is defined as

$$\{[[R*]]+ \mid [[R*]] \in B\}.$$

DEFINITION: The superlanguage LS relative to the superlanguage base B is defined as $\langle \mathrm{Syn}, [S] \rangle$ such that [S] is the set of supervaluations established by elements of B.

Two of the most important features of superlanguages are that they preserve classical logic and are non-truth-functional. We define entailment for a superlanguage as a truth-preserving relation.

DEFINITION: $X \vDash_{\mathrm{LS}} P$ iff for any S in [S] of LS, if $S(X) = \mathrm{T}$, then $S(P) = \mathrm{T}$.

THEOREM. If $X \vDash_{\mathrm{LC}} P$ then $X \vDash_{\mathrm{LS}} P$.

Proof. Assume the antecedent and that for an arbitrary supervaluation S, $S(X) = \mathrm{T}$. Then relative to some [[$R*$]] of B, all R in [[$R*$]] are such that $R(X) = \mathrm{T}$. But since [[$R*$]] is a subset of [RC]. and $X \vDash_{\mathrm{LC}} P$, each such R must be such that $R(P) = \mathrm{T}$. Then elements of [[$R*$]] are unanimous and $S = [[R*]]$ is such that $S(P) = \mathrm{T}$. End of Proof.

THEOREM. Some superlanguage is not truth-functional.

Proof. Let B be any superlanguage base in which there are atomic sentences P and Q, and an atomic partial valuation $R*$ such that $R*$ is not defined for either. Then consider the supervaluation $S = [[R*]]+$ and the evaluation of conjunctions. $S(P \wedge Q)$ is undefined because there are classical completions R and R' of $R*$ in [[$R*$]] such that $R(P) = R(Q) = \mathrm{T}$, and $R(P \wedge Q) = \mathrm{T}$, and $R'(P) = R'(Q) = \mathrm{F}$ and $R'(P \wedge Q) = \mathrm{F}$. Thus there is a case in which $S(x)$ is undefined, $S(y)$ is undefined, and $S(x \wedge y)$ is undefined. Now consider the conjunction $P \wedge {\sim}P$. Clearly $S({\sim}P)$ is undefined, because those completions in [[$R*$]] that disagree about P will also disagree about ${\sim}P$. Moreover, $S(P \wedge {\sim}P) = \mathrm{F}$, because all completions R in [[$R*$]] are such that $R(P \wedge {\sim}P) = \mathrm{F}$ since they are all in [RC]. Hence there is a case of x and y such that $S(x)$ is undefined, $S(y)$ is undefined, yet $S(x \wedge y) = \mathrm{F}$. Thus the S-assignments to the immediate parts of a conjunction do not determine in a unique way the S-assignment to the whole. End of Proof.

□ EXERCISES

1. Prove the metatheorem that for $X \subseteq$ [RC], $X+ = \bigcap X$.
2. Prove that some superlanguages are truth-functional by proving the following more general theorem.

   There is a superlanguage LS = $\langle \mathrm{Syn}, [S] \rangle$ relative to a base B such that [S] = [RC]. (That is, some superlanguage has as its valuations exactly the classical valuations.)