Lectures on Abstract Structures in the Semantics of Classical and Intuitionistic Logic

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Lecture 1

Abstract Structures

Structure. We all have a good intuitive idea of a "structure." Examples include buildings, governmental institutions, ecologies, and polyhedral. In the branch of mathematics known as ***abstract*** or ***universal algebra*** the general properties of structures are studied, and these ideas help explain the structures we find in logic like those of grammars, semantical interpretations, and inferential systems.

The raw intuition behind the mathematical definition of a structure is an architect's blue-print. The blue print succeeds in describing a building by first listing its various materials and then by a diagram describing the relations that must obtain among these "building blocks" in the finished structure. In algebra a structure is defined in a similar way. First a list of set $A_1,...,A_k$ is given. These may be viewed as list of building-blocks divided into various kinds or classes. Next are listed the relations $R_1,...,R_i$ and functions $f_1,...,f_m$ that hold among these materials. (Recall that functions are just a sub-variety of relations.) Lastly it is useful to list some specific individual building blocks $O_1,...,O_m$ that have special importance in the structure. It is customary to list all the elements of the structure in order, that is as an ordered tuple: $< A_1,...,A_k,R_1,...,R_i,f_1,...,f_m,O_1,...,O_m >$.

Definition 1. Abstract Structure

An ***abstract structure*** is any $<A_1,...,A_k,R_1,...,R_l,f_1,...,f_m,O_1,...,O_n>$ such that:

for each i=1…k, A_l is a set,
for each i=1…l, R_i is a relation on $C = \cup\{A_1,...,A_n\}$,
for each i=1…m, f_i is a function on $C = \cup\{A_1,...,A_n\}$, and
for each i=1…m, $O_i \in C = \cup\{A_1,...,A_n\}$.

It is also common to investigate a family of structures with similar properties, and to assign the family a name, e.g. group, ring, lattice, or Boolean algebra. The properties defining such a family are usually formulated as defining conditions on the type of sets, relations, functions and designated elements that fall into the family. Sometimes these restrictions are referred to as the "axioms" of the structure-type. Strictly speaking they are not part of a genuine axiom system. Rather they are clauses appearing in the abstract definition of a particular set (family) of structures. Let us review some familiar examples.

Sentential Syntax. The usual definition of syntax for sentential logic may be recast so that it is clear that the rules of syntax "define" a certain kind of syntactic "structure." Let us begin by stating a version of the definition of the sort that usually appears in elementary logic texts, and which does not use algebraic ideas explicitly:

Definition 2. Sentential Syntax. The set of $\mathbf{F}_{SL}$ of (***Well-Formed***-) ***Formulas of SL.***
Let $\mathbf{AF}_{SL}$ be the basis set {*A,B,C*} (of ***atomic formulas***) and let $\mathbf{R}_{SL}$ be the rule set {$R_{\sim}$,$R_{\wedge}$,$R_{\vee}$,$R_{\rightarrow}$,$R_{\leftrightarrow}$} (of ***grammar rules***) defined below:

a. (Rule $R_{\sim}$) The result of applying $R_{\sim}$ to *P* is ~*P*.
b. (Rule $R_{\wedge}$) The result of applying $R_{\wedge}$ to *P* and *Q* is $(P\wedge Q)$.
c. (Rule $R_{\vee}$) The result of applying $R_{\vee}$ to *P* and *Q* is $(P\vee Q.)$
d. (Rule $R_{\rightarrow}$) The result of applying $R_{\rightarrow}$ to *P* and *Q* is $(P\rightarrow Q)$.
e. (Rule $R_{\leftrightarrow}$) The result of applying $R_{\leftrightarrow}$ to *P* and *Q* is $(P\leftrightarrow Q)$.

Then, $\mathbf{F}_{SL}$ is the set inductively defined relative to $\mathbf{AF}_{SL}$ and $\mathbf{R}_{SL}$ as follows:

1. (Basis Clause) All formulas in $\mathbf{AF}_{SL}$ are in $\mathbf{F}_{SL}$.
2. (Inductive Clause) If *P* and *Q* are in $\mathbf{F}_{SL}$, then the results of apply the rules $R_{\sim}$,$R_{\wedge}$,$R_{\vee}$,$R_{\rightarrow}$,$R_{\leftrightarrow}$ from in $\mathbf{R}_{SL}$, namely ~*P*, $(P\wedge Q)$, $(P\vee Q)$, $(P\rightarrow Q)$, $(P\leftrightarrow Q)$, are in $\mathbf{F}_{SL}$.
3. (Closure) Nothing is in $\mathbf{F}_{SL}$ except by Clauses 1 and 2.

Using the idea of an abstract structure, it is possible to reformulate the definition in a way that makes the structural aspects of the grammar fully explicit:

Definition 3. Sentential Syntax, Algebraic Formulation.

By a ***sentential logic syntax*** is meant any structure $\mathbf{Syn}_{SL}$=< F_{SL},$R_{\sim}$,$R_{\wedge}$,$R_{\vee}$,$R_{\rightarrow}$,$R_{\leftrightarrow}$> such that:

1. $R_{\sim}$,$R_{\wedge}$,$R_{\vee}$,$R_{\rightarrow}$,$R_{\leftrightarrow}$ are functions on symbol strings defined as follows:
   $R_{\sim}$ constructs ~*x* from any sting *x*;
   $R_{\wedge}$ constructs $(x\wedge y)$ from strings *x* and *y*;
   $R_{\vee}$ constructs $(x\vee y)$ from strings *x* and *y*;
   $R_{\rightarrow}$ constructs $(x\rightarrow y)$ from strings *x* and *y*;
   $R_{\leftrightarrow}$ constructs $(x\leftrightarrow y)$ from strings *x* and *y*.
2. There is a denumerable set of strings AF_{SL} (called the set of ***atomic formulas***), such that F_{SL} (the set of ***Well-Formed-Formulas of SL***) is defined inductively as follows:
   a. (Basis Clause) All formulas in AF_{SL} are in F_{SL}.
   b. (Inductive Clause) If *P* and *Q* are in F_{SL}, then the results of apply the rules $R_{\sim}$,$R_{\wedge}$,$R_{\vee}$,$R_{\rightarrow}$,$R_{\leftrightarrow}$ from in R_{SL}, namely ~*P*, $(P\wedge Q)$, $(P\vee Q)$, $(P\rightarrow Q)$, $(P\leftrightarrow Q)$, are in F_{SL}.
   c. (Closure) Nothing is in F_{SL} except by clauses 1 and 2.

Once a syntax is defined as a structure, then algebraic ideas may be applied to "explain" it, as "explanation" is understood in mathematics: specific properties of grammar can be seen to hold not as a result of peculiarities of grammar but as consequences of the fact that grammars happens to be special cases of yet more abstract types of structures. Many features of grammars do in fact hold because grammars happen to be suspecies of more abstract structure-types. A particularly interesting and simple case is that of partial orderings.

Partial Orderings. The familiar "less than" relation on numbers, symbolized by ≤, and the subset relation on sets, symbolized by ⊆, are instances of what is know as partial ordering. In algebra such orderings are viewed as structures. To define such a structure, however, we mustfist define some standard properties of relations. We then define several common varieties of ordered-structures.

Definition 4. Properties of Relations and Ordered Structures

A binary relation ≤ is said to be:

reflexive iff for any *x*, $x \leq x$;
transitive iff for any *x*,*y*, and *z*, if $x \leq y$ and $y \leq z$, then $x \leq z$;
symmetric iff for any *x* and *y*, if $x \leq y$ then $y \leq x$;
asymmetric iff for any *x* and *y*, if $x \leq y$ then not ($y \leq x$);
antisymmetric iff for any *x* and *y*, if $x \leq y$ and $y \leq x$, then $x=y$;
complete iff for any *x* and *y*, either $x \leq y$ or $y \leq x$;
x is a ***≤-least element*** of *B* iff $x \in B$ and for any $y \in B$, $x \leq y$.

Any structure $<A,\leq>$ such that *A* is a non-empty set and ≤ is a binary relation on *A* is called:

1. a ***pre-*** or ***quasi-ordering*** iff ≤ is reflexive and transitive;
2. a ***partially ordering*** iff ≤ is a pre-ordering and antisymmetric;
3. a ***total*** or ***linear*** ordering iff ≤ is partial and complete;
4. a ***well-ordering*** iff, ≤ is a partial ordering and for any subset *B* of *A, B* has a ≤-least element.

Definition 5. The ***subformula relation*** ≤ (read "is a part of") is a defined on a sentential structure $\mathbf{Syn}_{SL} = < F_{SL}, R_{\sim}, R_{\wedge}, R_{\vee}, R_{\rightarrow}, R_{\leftrightarrow} >$ defined inductively as follows:

For any atomic formula, $A \leq A$;
For any molecular formula $R_i(A_1,...,A_n)$, each A_k, for k=1,…,n, is such that $A_k \leq R_i(A_1,...,A_n)$.

Theorem 1. The subformula relation ≤ of any $\mathbf{Syn}_{SL}$ is a partial ordering.

Another example of ideas from logic that lend themselves to algebraic formulation are truth-tables. Viewed algebraically truth-tables form a structure on the values {T,F} and each truth-table defines a specific function defining structure on this minimal set.

Definition 6. The Classical Bivalent Structure of Truth-Values

By the ***classical algebra of truth-values*** is meant the structure <{T,F},∧,∨,→,↔,–> such that

∧={<T,T,T>,<T,F,F>,<F,T,F>,<F,F,F>}
∨={<T,T,T>,<T,F,T>,<F,T,T>,<F,F,F>}
→={<T,T,T>,<T,F,F>,<F,T,T>,<F,F,T>}
↔={<T,T,T>,<T,F,F>,<F,T,F>,<F,F,T>}
–={<T,F>,<F,T>}

Often T is identified with 1 and 0 with F, and {0,1} with 2.

Standard Abstract Structures

The structure of truth-values is actually a special case of a more general (i.e. abstract) set of structures known as Boolean algebras, which includes the standard algebra of sets. There are a number of equivalent ways to define a Boolean algebra, some of which we shall encounter later, but for purposes of illustration here let us use a simple definition that employs the idea of partial ordering.

Definition 7. Properties of Binary Operations (aka Functions).

Let • be a binary operation on a set B, and let us write •(*x,y*) as *x*•*y*. Then,

B is ***closed under*** • iff for all *x,y* of B, *x*•*y*∈B,
• is ***associative*** iff for all *x,y* of B, *x*•*y*=*y*•*x*,
• is c***ommutative*** iff for all *x,y* of B, *x*•(*y*•*z*)=(*x*•*y*)•*z*,
• is ***idempotent*** iff for all *x,y* of B, *x*•*x*=*x*,

Definition 8. Varieties of Structures.

A structure <B,$\wedge$>/<B,$\vee$> is a ***meet/join semi-lattice*** iff $\wedge$/$\vee$ is a binary operation under which B is closed and $\wedge$/$\vee$ is associative, commutative, and idempotent.

If <B,$\wedge$> is a meet semi-lattice, then the ordering relation $\leq$ on B is defined as

$x \leq y$ iff $x \wedge y = x$.

If <B,$\wedge$> is a join semi-lattice, then the ordering relation $\leq$ on B is defined as

$x \leq y$ iff $x \vee y = y$.

The structure <B,$\wedge$,$\vee$> is a **lattice** iff <B,$\wedge$> and <B,$\vee$> are receptively meet and join semi-lattices, and the ordering relation $\leq$ on B is defined as: $x \leq y$ iff $x \wedge y = x$ iff $x \vee y = y$.

If <B,$\wedge$,$\vee$> is a lattice, then 0 is ***the least element*** of B iff

$0 \in B$
for any x in B, $0 \leq x$,
$0 \wedge x = 0$ and
$0 \vee x = x$.

If <B,$\wedge$,$\vee$> is a lattice, then 0 is ***the greatest element*** of B iff

$1 \in B$
for any x in B, $x \leq 1$,
$1 \wedge x = x$ and
$1 \vee x = 1$.

If <B,$\leq$> is a partially ordered structure and x and y are in B, then ***the greatest lower bound*** (briefly, ***glb***) of $\{x,y\}$ (if it exists) is the $z \in B$ such that

$z \leq x$ and $z \leq y$
for any w in B if $w \leq x$ and $w \leq y$, then $w \leq z$.

If <B,$\leq$> is a partially ordered structure and x and y are in B, then ***the least upper bound*** (briefly, ***lub***) of $\{x,y\}$ (if it exists) is the $z \in B$ such that

$x \leq z$ and $y \leq z$
for any w in B if $x \leq w$ and $y \leq w$, then $z \leq w$.

A lattice <B,$\wedge$,$\vee$> is **distributive** iff

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, and
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

If <B,$\wedge$,$\vee$,0,1> is a structure such that <B,$\wedge$,$\vee$> is a lattice and 0 and 1 are respectively its least and greatest elements, then – is a (***unique***) ***complementation operation*** on the structure iff

– is a one-place operation on B	$-1=0$
for any $x \in B$, $-x \in B$	$-0=1$
$x \wedge -x = 1$	$-(x \wedge y) = -x \vee -y$
$x \vee -x = 0$	$-(x \vee y) = -x \wedge -y$
$--x = x$	$x \leq y$ iff $-x \wedge y = 0$ iff $-y \leq -x$ iff $-x \vee y = 1$

A structure <B,$\wedge$,$\vee$,–,0,1> is a ***Boolean algebra*** iff

<B,$\wedge$,$\vee$> is a lattice
<B,$\wedge$,$\vee$> is distributive
0 and 1 are respectively the least and greatest elements of <B,$\wedge$,$\vee$>
– is a complementation operation on <B,$\wedge$,$\vee$,0,1>

Theorem 2. If <B,∧,∨> is a lattice, then <B, ≤> is a partial ordering

Theorem 3. If ≤ is a partial ordering on a set B and if for any x and y in B, the glb{x,y} and the lub{x,y} exist and are in B, and if ∧ and ∨ are binary operations on B defined as follows

$$x \wedge y = \text{glb}\{x,y\}, \text{ and } x \vee y = \text{lub}\{x,y\},$$

then the structure <B,∧,∨> is a lattice with ordering relation ≤.

Theorem 4. The classical structure of truth-values is a Boolean algebra.

Sameness of Structure. One of the most important ideas in algebra is sameness of structure. Two teacups from the same set and two pennies have the same structure. So too do two twins. In these cases the structures match very closely. But family members and even members of the same species have some features of structure in common. More abstractly, the reason maps work is that there is a similarity of structure between geographical features in the world and the symbols on the map that represent them. Blue-prints work for this reason too. Mathematically this sameness is explained by saying that there is a mapping from the entities of one structure into the entities of a second in such a way that the mapping "preserves structure." Informally, if we have two structures and entity x_1 in the first that "corresponds" to an entity x_2 in the second, we may call x_2 ***the representative*** of x_1. Often one structure may be more complex than the other, yet both exhibit some structural features in common. One way this happens occurs when elements of the more complex are "identified" or viewed as a unit in the second. This happens, for example, in our representative democracy in which all the citizens in an election district are represented by a single individual in Congress. Thus for a "similarity of structure" to obtain we require as a minimum that each entity of one structure corresponds to one and only one entity in the second. In mathematical terms, there is an into ***function*** that assigns a ***value*** in the second structure to each ***argument*** in the first. If θ is the mapping function then $\theta(x_1)=x_2$ and $\theta(x_1)$ is the representative of x_1. Such a mapping is called a ***homomorphism*** (from the Greek *homos = the same* and *morphos=structure*.)

Two structures S=<$A_1,...,A_k,R_1,...,R_i,f_1,...,f_m,O_1,...,O_n$> and S'=<$A'_1,...,A'_k, R'_1,...,R'_i,f'_1,...,f'_m,O'_1,...,O'_n$> are said to be of the ***same character*** or ***type*** iff

for each i=1,…,l, there is some n such that R_i and R'_i are both n-place relations, and

for each i=1,…,n, there is some n such that f_i and f'_i are both n-place functions.

Very often a discussion is clearly limited to structures of the same type. When this restriction is clear, it is tedious to keep mentioning it, and it is usually assumed without saying so explicitly.

Definition 9. Homomorphism.

If S=<A_1,...,A_k,R_1,...,R_i,f_1,...,f_m,O_1,...,O_n> and S'=<A'_1,...,A'_k,R'_1,...,R'_i, f'_1,...,f'_m,O'_1,...,O'_n> are structures of the same character, θ is called a ***homomorphism from*** S to S' iff θ is a function from U{A_1,...,A_n} into U{A'_1,...,A'_n} such that

1. for each i=1,…,k, if $x \in A_i$, then $\theta(x_i) \in A'_i$;
2. for each i=1,…,l, $<x_1,...,x_n> \in R_i$ iff $<\theta(x_1),...,\theta(x_n)> \in R'_i$;
3. for each i=1,…,m, $\theta(f_i(x_1,...,x_n) = f'_i(\theta(x_1),...,\theta(x_n))$;
4. for each i=1,…,m, $\theta(O_i)=O'_i$.

Sentential Semantics

One of the simplest and most elegant applications of algebraic ideas to logic is its use in formulating standard truth-functional semantics. We have already seen how to formulate syntactic structure and the structure of truth-values as algebras. It is now possible to formulate the idea of a "valuation," i.e. the traditional notion of an assignment of truth-values to formulas, as a homomorphism between the two structures. Many of familiar semantic the properties of classical valuations then follow directly as properties of morphisms.

Let us begin by restating the standard definition of a valuation in non-algebraic terms

Defintion 10. The Semantics for Sentential Logic

A (***classical***) ***valuation*** for the set $\mathbf{F}_{SL}$ of ***formulas*** of an ***SL* language** generated by $\mathbf{AF}_{SL}$ is any assignment **V** of a truth-values T or F to the formulas in $\mathbf{F}_{SL}$ that meets the following conditions:

V assigns to every atomic sentence in $\mathbf{AF}_{SL}$ either T or F;

V assigns to negations, conjunctions, disjunctions, conditionals and biconditionals the truth-value calculated by the truth-tables from the truth-values that $\Im$ assigns to its parts.

The formula *P* is a ***tautology*** (abbreviated $\models_{SL} P$) iff for all **V**, **V** assigns T to *P*.

The argument from P_1,...P_n,... to *Q* is ***valid*** (abbreviated, $P_1,...P_n,... \models_{SL} Q$) iff for any **V**, if **V** assigns T to all of P_1,...P_n,..., then **V** assigns T to *Q*.

The algebraic formulation is short and sweet.

Definition 11. Sentential Semantics, Algebraic Formulation.

If $\mathbf{Syn}_{SL}$=< F_{SL},$R_{\sim}$,$R_{\wedge}$,$R_{\vee}$,$R_{\rightarrow}$,$R_{\leftrightarrow}$> is a sentential syntax and **2**=<{T,F},∧,∨,→,↔,–> is the classical algebra of truth-values, then **V** is a ***classical valuation*** for $\mathbf{Syn}_{SL}$ iff **V** is a homomorphism from a $\mathbf{Syn}_{SL}$ to **2**.

We still have to define *tautology* and *validity*. The too a given equivalent definitions that are stated by reference to the strucure. Before stating these new definitions, however, let us perform an abstraction.

The algebraic formulation of classical semantics is so elegant that it invites immediate generalization or "abstraction" from the fact that the semantics has merely two truth-values. Indeed it is just such an abstraction that was made by Lukasiewicz and other Polish logicians in the 1920's and which has provided the standard framework for the development of valuational semantics ever since. In its abstract version valuational semantics in a special sort of algebraic structure called a logical matrix. This is very like the structure for the two-valued classical truth-values just employed, but in addition it singles out as a designated set a subset of truth-values, called the ***designated values***, that are those used to defining tautology and validity.

Definition 12. Sentential Semantics Formulated in Terms of Logical Matrices

A ***logical matrix*** is any structure =M=<U,D,$\wedge$,$\vee$,$\rightarrow$,$\leftrightarrow$,–> such that

U is non-empty (usually a subset of the real numbers)

D (the set of ***designated values***) is a non-empty subset of U

$\wedge$,$\vee$,$\rightarrow$,$\leftrightarrow$ are binary relations on U

– is a unary relation on U.

The set of valuations Val_M (relative to $\mathbf{Syn}_{SL}$) is the set of all homomorphisms **V** from $\mathbf{Syn}_{SL}$ to M.

A ***sentential matrix language*** *SL* is any < $\mathbf{Syn}_{SL}$,Val_M>.

The argument from P_1,...P_n,... to Q is ***valid*** in *SL* (abbreviated, $P_1,...P_n,... \models_{SL} Q$) iff for any **V**, if $\mathbf{V}(P_1)\in D$,..., $\mathbf{V}(P_n)\in D$, then $\mathbf{V}(Q)\in D$.

The formula P is a ***tautology*** in *SL* (abbreviated $\models_{SL}P$) iff for all **V**, $\mathbf{V}(P)\in D$.

Much of our discussion of the intensions will be formulated in terms of logical matrices.

Sameness of Kind. Sameness is one of the "great ideas." Aristotle was the first to clearly distinguish ***numerical identity*** (he coined the term) from other sorts of sameness. Algebra has a nice set of concepts that make all the relevant distinctions and it also a battery of extremely useful collateral ideas. Let us first distinguish numerical identity. This is the idea treated in "first-order logic with identity." It is given = as its own logical symbol in the syntax, and special *ad hoc* clauses in the definition of a semantic interpretation specifying that the symbol stands for the identity relation on the domain.. This identity relation is understood to be a theoretical primitive (part of the stock of primitives that metatheory incorporates from set theory). It is the idea that is then summarized in the two semantic metatheorems whose syntactic versions are used to axiomatize truths of numerical identity:

$\models_{FOL=} x=x$
$\{x=y, P\} \models_{FOL=} P[y//x]$

Sameness of kind has to do with classification into sets of individuals of the same "sort." One traditional way to discuss the idea is in terms of the sameness relation where this relation is understood to fold among more than one thing. Algebra specifies the properties that must hold of such a relation:

Definition 13. Equivalence Relation, Equivalence Class.

A binary relation $\equiv$ on a set A is said to be an ***equivalence relation*** on A iff $\equiv$ is reflexive, transitive and symmetric. The ***equivalence class*** of x under $\equiv$, briefly $[x]_{\equiv}$, is defined as $\{x| x\equiv x\}$.

Clearly numerical identity counts as an equivalence relation, but so do many other relations. Sameness of kind is also discussed in terms of sets. One way to do so is to put things into sets is, as it were manually, by means of set abstracts: we find and open sentences $P(x)$ that is true of all the "same" things. The " $P(x)$ " describes what they all have in common. We may go through everything there is and find such defining characteristics for "kinds" or "sorts" so that we can classify everything in to non-overlapping, mutually exclusive sets $\{x| P_1(x)\},...,\{x|P_n(x)\}$. Algebra provides a name for such a classification into "kinds:"

Definition 14. Partition.

A family F=$\{B_1,...,B_n\}$ of sets is said to be a ***partition*** of a set A iff A=U$\{B_1,...,B_n\}$ and no two B_i and B_j overlap (i.e. for each i and j, $B_i \cap B_j = \varnothing$).

There is moreover a way to generate a partition from a sameness relations and vice versa.

Theorem 5. If a family F=$\{B_1,...,B_n\}$ of sets is a partition of a set A, then the binary relation $\equiv$ on A defined as follows: $x\equiv y$ iff for some i, $x\in A_i$ and $y\in A_l$ is am equivalence relation.

Theorem 6. The family of all equivalence classes $[x]_{\equiv}$ for all x in a given set A is a partition of A.

The set of all entities from the first structure that have the same representative are in a sense "the same:" they form an equivalence class. For example, the set of citizens represented by the same congressman is a equivalence class. One of direct consequences of these ideas is the fact that equivalence classes do not overlap and that they exhaust all the entities of the first structure.

Theorem 7. If θ is a homomorphism from $S=\langle A_1,...,A_k,R_1,...,R_i,f_1,...,f_m,O_1,...,O_n\rangle$ to $S'=\langle A'_1,...,A'_k,R'_1,...,R'_i,f'_1,...,f'_m,O'_1,...,O'_n\rangle$, then the binary relation $\equiv_\theta$ on $C=\cup\{A_1,...,A_n\}$ defined $x\equiv_\theta y$ iff $\theta(x)=\theta(y)$ is an equivalence relation on C. Furthermore, if the equivalence class $[x]_\theta$ of x under θ is defined as $\{y|\ y\equiv_\theta x\}$, then the family F of all equivalence classes, i.e. $\{[x]_\theta \mid x\in C\}$, is a partition of C.

Identity of Structure. If a structural representation is so tight that it exhausts the elements of the second structure in the sense that all of its elements are representatives of some entity in the first the function, then the representation is said to be ***onto***. There are, for example, no voting members of Congress that do not represent some state. In Germany, however, where some members of Parliament are allotted to parties due to national voting percentages there are members that do not represent a specific district. We have seen, for example, that truth-value assignments (valuations) are onto homomorphisms from formulas onto the set {T,F} structures by the "truth-functions" specified in the truth-tables for the connectives.

In some instances the representation is so fine grained that not two entities of the first structure have the same representative. Such a mapping would be two cumbersome for Congress, but it is essential for social security numbers. Such mappings are said to be 1 to 1. Any mapping that is 1 to 1 and onto totally replicates the structure and entities of the fist structure and is called an isomorphism (from *isos=equal*).

Definition 15. Isomorphism.

If θ is a homomorphism from $\langle A_1,...,A_k,R_1,...,R_i,f_1,...,f_m,O_1,...,O_n\rangle$ to $S=\langle A'_1,...,A'_k,R'_1,...,R'_i,f'_1,...,f'_m,O'_1,...,O'_n\rangle$, then θ is said to be an ***isomorphism*** form S to S if θ is a 1-1 and onto mapping.

It follows from the definitions that given a homomorphism from a first structure to a second we can define a third structure made up of the equivalence classes of the first and this new structure can be made to have the exactly the same structure as (be isomorphic to) the second. This new structure is called the quotient algebra.

Definition 16. Quotient Algebra.

If θ is a homomorphism from S=<$A_1,\ldots,A_k,R_1,\ldots,R_i,f_1,\ldots,f_m,O_1,\ldots,O_n$> to S'=<$A'_1,\ldots,A'_k,R'_1,\ldots,R'_i,f'_1,\ldots,f'_m,O'_1,\ldots,O'_n$>, then ***the quotient algebra*** for <$A_1,\ldots,A_k,R_1,\ldots,R_i,f_1,\ldots,f_m,O_1,\ldots,O_n$> under θ is S''=<$A''_1,\ldots,A''_k,R''_1,\ldots,R''_i,f''_1,\ldots,f''_m,O''_1,\ldots,O''_n$> defined as follows:

given $x \equiv_\theta y$ iff $\theta(x)=\theta(y)$ and $[x]_\theta$ to be $\{y \mid y \equiv_\theta x\}$,

$A''_i = \{[x]_\theta \mid x \in A_i\}$

$<[x_1]_\theta,\ldots,[x_n]_\theta> \in R''_I$ iff $<x_1,\ldots,x_n> \in R_i$

$f_i([x_1]_\theta,\ldots,[x_n]_\theta) = [f_i(x_1,\ldots,x_n)]_\theta$

$O'' = [O_i]_\theta$

Theorem 8. If θ is a homomorphism from S=<$A_1,\ldots,A_k,R_1,\ldots,R_i,f_1,\ldots,f_m,O_1,\ldots,O_n$> to S=<$A'_1,\ldots,A'_k,R'_1,\ldots,R'_i,f'_1,\ldots,f'_m,O'_1,\ldots,O'_n$>, then S is homomorphic to its quotient algebra S'' under θ , and S' is isomorphic to S''.

Congruence and Substitution. We are familiar in logic with various sorts of substitutability. One of the most familiar kind is the substitutability of material equivalents *salve veritate.* This phenomenon is a special case of a much more general one that results from the homomorphic nature of valuations.

The formula P is a ***tautology*** (abbreviated $\models_{SL} P$) iff for all **V**, **V** assigns T to

Definition 17. If S=<$A_1,\ldots,A_k,R_1,\ldots,R_i,f_1,\ldots,f_m,O_1,\ldots,O_n$> is a structure with a binary relation $\equiv$ on $C=\cup\{A_1,\ldots,A_n\}$, $\equiv$ is said to have ***the substitution property*** and to be a ***congruence relation*** iff

if $x_1 \equiv y_1,\ldots, x_n \equiv y_n$, then $<x_1,\ldots,x_n> \in R_i$ iff $<y_1,\ldots,y_n> \in R_i$, and

if $x_1 \equiv y_1,\ldots, x_n \equiv y_n$, then $f_i(x_1,\ldots,x_n) \equiv f_i(y_1,\ldots,y_n)$.

Theorem 9. If θ is a homomorphism from S=<$A_1,\ldots,A_k,R_1,\ldots,R_i,f_1,\ldots,f_m,O_1,\ldots,O_n$> to S'=<$A'_1,\ldots,A'_k,R'_1,\ldots,R'_i,f'_1,\ldots,f'_m,O'_1,\ldots,O'_n$>, then the equivalence relation $\equiv_\theta$ is a congruence relation with the substitution property.

Corollary 1. (**Substitutability of Material Equivalents**.) If **V** is a classical valuation (i.e. a homomorphism from a sentential syntax $\mathbf{Syn}_{SL}$=<$F_{SL},R_\sim,R_\wedge,R_\vee,R_\rightarrow,R_\leftrightarrow$> to the classical truth-value structure **2**=<{T,F},$\wedge,\vee,\rightarrow,\leftrightarrow$,–>), then the equivalence relation $\equiv_{\mathbf{V}}$ is a congruence relation and has the substitution property.

Applications of These Ideas in Logic. Much of what we shall encounter in these lectures are generalization from these basic results. The techniques will be to treat syntaxes from sentential logic to modal and epistemic logic to first-order logic as algebras defined on "strings" of symbols. Semantics is then conducted by defining structures, and then defining what are more familiarly know as valuations and interpretations as various sorts of morphisms over these structures. Various subsitutablity results then follow.

Exercises.

1. Prove Theorem 7.
2. Prove Theorem 8.
3. Prove Theorem 9 (optional)

Definition. Subalgebra

If S=<A_1,...,A_k,R_1,...,R_i,f_1,...,f_m,O_1,...,O_n> and S'=<A'_1,...,A'_k,R'_1,...,R'_i,f'_1,...,f'_m,O'_1,...,O'_n> are structures of like character, then S is a ***subalgebra*** of S' iff C=$\bigcup\{A_1,...,A_k\}\subseteq \bigcup\{A'_1,...,A'_k\}$ and each R_1,...,R_i,f_1,...,f_m is the restriction repectively of R'_1,...,R'_i,f'_1,...,f'_m to C. When A'_l is empty it is customary to delete it from if it is clear from the context to which set it it corresponds.

Lecture 2

Matrix Semantics

1. Language and Logical Inference in the Abstract

Let us begin our abstract study of logic by defining the core notions of syntax, semantics, and proof theory in there broadest algebraic senses. We shall assume at a minimum that the language in question contains sentences and that these are the syntactic units that make up arguments to be appraised for their validity.

Syntax. It is sufficient to define a syntax as a structure on "expressions" organized by rules of grammatical construction. In logic, "expressions" are normally understood to be finite stings built up by "concatenation" from a finite set of signs by means of the grammar rules understood as 1 to 1 ("uniquely decomposable") operations on finite strings. As is customary let Σ stand for the set of signs used to construct the syntax.

Definition 1. Syntax.

By a ***syntax*** $\mathcal{S}yn$ is meant a structure $<A_1,...,A_k,f_1,...,f_m>$ such that for some finite set Σ of signs, each f_i is a 1-1 function defined in terms of concatenation (the operation $^\frown$ on signs and strings) that maps some subset of Σ^* 1 to 1 into Σ^*, where Σ^* is the set of all finite strings of signs in Σ.

We assume that there is some A_i intended to represent sentences, and we use $\mathcal{S}en$ as the preferred name of that A_i.

We let P and Q range over $\mathcal{S}en$, and X,Y and Z over subsets of $\mathcal{S}en$.

Example. 1 Sentential Logic.

By a $\mathcal{SL}$ ***syntax*** is meant a structure $<\mathcal{S}en,f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow}>$ such that there is some set $\mathcal{AS}en$ such that

1. $\mathcal{AS}en$ is an at most denumerable set (of "atomic sentences") constructed from some finite base of signs.
2. the operations are defined as follows:

   $f_{\sim}(x)= \sim^\frown x$

   $f_{\wedge}(x,y)= (^\frown x ^\frown \wedge ^\frown y ^\frown)$

   $f_{\vee}(x,y)= (^\frown x ^\frown \vee ^\frown y ^\frown)$

   $f_{\rightarrow}(x,y)= (^\frown x ^\frown \rightarrow ^\frown y ^\frown)$
3. $\mathcal{S}en$ is the least set (the set inductively defined) such that $\mathcal{AS}en \subseteq \mathcal{S}en$ and $\mathcal{S}en$ is closed under $f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow}$.

Substitution may also be defined for abstract syntaxes of this sort.

Definition 2. Substitution.

♦ is a ***(uniform) substitution operation*** for *Syn* iff ♦ is a homomorphism from *Syn* into itself.

The notion is extended to sets as follows: $♦(X)=\{♦(P)|P\in X\}$.

Let **Sub***Syn* be the set of all substitution operations for *Syn*.

Definition. Subalgebra.

If $S=<A_1,...,A_k,R_1,...,R_i,f_1,...,f_m,O_1,...,O_n>$ and $S'=<A'_1,...,A'_k,R'_1,...,R'_i,f'_1,...,f'_m,O'_1,...,O'_n>$ are structures of like character, then S is a ***subalgebra*** of S' iff $C=\bigcup\{A_1,...,A_k\}\subseteq \bigcup\{A'_1,...,A'_k\}$ and each $R_1,...,R_i,f_1,...,f_m$ is the restriction repectively of $R'_1,...,R'_i,f'_1,...,f'_m$ to C.

Definition. Sentential Subalgebra.

The sentential subalgebra *Syn*|*Sen* of a syntax *Syn* is its subalgebra in which all categoeries of expressions other than *Sen* are empty.

Definition. Sentential Substitution.

♦ is a ***(uniform) sentential substitution operation*** for *Syn* iff ♦ is a homomorphism from *Syn*|*Sen* into itself.

The notion is extended to sets as follows: $♦(X)=\{♦(P)|P\in X\}$.

Let **Sub***Sen* be the set of all sentential substitution operations for *Syn*.

Later we shall use ├ to represent the "deducibility" relation. We then use substitution to define "formal" inference patterns, and do so traditionally by means of tree diagrams that presuppose the working of substitution in a somewhat hidden way. Since we are stating relevant general concepts defined in terms of substitution, we shall explain this form of definition here.

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Definition 3. "Formal" Relations (Inference Patterns) by Tree Diagrams.

We call $<X,P>$ a "deduction" and usually write it as $X├P$. Let $♦(<X,P>)$ be $<♦(X),♦(P)>$.

By the tree R:

$$\frac{X_1├P_1>, \quad .... \qquad X_k├P_k}{Y├Q}$$

we refer to (define) the following relation R on deductions

$R =\{<♦(<X_1,P_1>), ..., ♦(<X_k,P_k>),♦(<Y,Q>)> \mid ♦$ is a sentential substitution for *Syn* and $♦(<X_1,P_1>), ..., ♦(<X_k,P_k>), ♦(<Y,Q>)$ are deductions in *Syn*}. [1]

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[1] Later in natural deduction theroy we shall state more complex rules in which specific form and term stubstitution is specified in the tree diagram. For example, the rules ∀+ defined by the tree

$\frac{X├P}{X├\forall xP^c_x}$ refers to $\{<♦(<X,P>), <♦(X),\forall x\ ♦(P)^c_x>)\mid ♦$ is a sentential substitution for *Syn* and $♦(<X,P>), <♦(X),\forall x\ ♦(P)^c_x>$ are deductions in *Syn*}.

Semantics. Characteristic of algebraic semantics is the interpretation of syntax by means of morphisms over structures of a character similar to that of syntax. It is also standard to interpret validity as some sort of "truth-preserving" relation holding between a set of premises and a conclusion. In general it is not necessary to specify the exact meaning of "truth", nor employ a single "truth-value" as the unique value preserved under valid inference.

Definition 4. Semantic Ideas.

By **a *semantic structure*** for $\mathcal{S}yn$=<A_1,...,A_k,f_1,...,f_m> for A_i=$\mathcal{S}en$ is meant any structure $\mathcal{S}em$ =<B_1,...,B_{k+1},f_1,...,f_m> such that

1. U{B_1,...,B_k}$\neq\varnothing$
2. $B_{k+1}\neq\varnothing$. (B_{k+1} is called the set of ***designated values****;* it is usually referred to as D. It is used below to define logical entailment.)
3. <B_1,...,B_k,f_1,...,f_m> is of the same character as <A_1,...,A_k,f_1,...,f_m>.

If $\mathcal{S}yn$=<A_1,...,A_k,f_1,...,f_m> is a syntax and $\mathcal{S}em$=<B_1,...,B_{k+1},f_1,...,f_m> is a semantic structure for $\mathcal{S}yn$, then the set **I**-$\mathcal{S}em$ of all ***semantic interpretations*** of $\mathcal{S}yn$ relative to $\mathcal{S}em$ is the set of all homomorphisms from <A_1,...,A_k,f_1,...,f_m> into <B_1,...,B_k,f_1,...,f_m>.

By a ***language*** $\mathcal{L}$ is meant any pair <$\mathcal{S}yn$,$\mathcal{F}$> such that $\mathcal{F}$ is family of semantic structures for $\mathcal{S}yn$.

If <$\mathcal{S}yn$,$\mathcal{F}$> is a language in which $\mathcal{F}$ is a singleton {$\Im$}, then we identity $\mathcal{F}$ with $\Im$.

Definition 5. Logical Ideas.

For any in **I**-$\mathcal{S}em$, $\Im$ is said to ***satisfy*** P iff $\Im(P)\in D$, and to ***satisfy*** X iff for all P in X, $\Im(P)\in D$.
X is said to ***semantically entail*** P in **I**-$\mathcal{S}em$ (briefly, $X\models_{\mathcal{S}em}P$) iff, for any in $\Im$ in **I**-$\mathcal{S}em$, $\Im$ satisfies X only if it satisfies P.
X is ***satisfiable*** in **I**-$\mathcal{S}em$ iff, for some $\Im$ in **I**-$\mathcal{S}em$, $\Im$ satisfies X,
X is ***unassailable*** in **I**-$\mathcal{S}em$ iff, for any $\Im$ in **I**-$\mathcal{S}em$, there is some $P\in X$, such that $\Im(P)\in D$.
X is said to ***semantically entail*** P in $\mathcal{L}$=<$\mathcal{S}yn$,$\mathcal{F}$> (briefly, $X\models_{\mathcal{L}}P$) iff for any semantic structure $\mathcal{S}em$ of $\mathcal{S}yn$ in $\mathcal{F}$, $X\models_{\mathcal{S}em}P$.
If $X\models_{\mathcal{L}}P$, the argument from X to P is said to be ***valid***, and $\models_{\mathcal{L}}$ is called ***entailment***.
$\models_{\mathcal{L}}$ is ***compact*** or ***finitary*** iff $X\models_{\mathcal{L}}P$ iff for some finite subset Y of X, $Y\models_{\mathcal{L}}P$
We let $\models$ stand for either $\models_{\mathcal{S}em}$ or $\models_{\mathcal{L}}$, and abbreviate $\varnothing\models P$ as $\models P$ and refer to P in this case as ***valid***. We abbreviate $\{P_1,...,P_n\}\models Q$ as $P_1,...,P_n\models Q$, and $X\cup Y\models P$ as $X,Y\models P$

Proof Theory. Intuitively deduction is a matter of determining by reference to precise syntactic rules the sentences that are deducible from other sentences. The rules are not invented arbitrarily but rather are designed to provide a syntactic characterization of the more fundamental semantic relation of logical entailment. If we are able to "deduce" a sentence P from a set of premises X, we say that the deducibility relation holds between X and P. We begin by characterizing some of the relational properties of this very special relation.

Definition 6. An Abstract Characterization of Deducibility. Let $\vdash$ be a relation that holds between sets of sentences and sentences in a syntax $\mathcal{Syn}$.

$\vdash$ is ***reflexive*** iff $P \vdash P$

$\vdash$ is ***transitive*** iff $X \vdash P$ and $Y \cup \{P\} \vdash Q$, then $X \cup Y \vdash Q$

$\vdash$ is ***monotonic*** iff $X \vdash P$ and $X \subseteq Y$, then $Y \vdash P$

$\vdash$ is a ***consequence relation*** iff $\vdash$ is reflexive, transitive and monotonic.

$\vdash$ is ***finitely axiomatizable*** iff $X \vdash P$ only if for some Y, $Y \subseteq X$ and $Y \vdash P$.

$\vdash$ is ***closed under substitution*** iff
($X \vdash P$ only if, for any substitution operation $\sigma \in$ **Sub**$\mathcal{Sen}$, $\sigma(X) \vdash \sigma(P)$).

$\vdash$ is a ***deducibility relation*** iff it is a finitely axiomatizable consequence relation closed under substitution.

Theorem 1. For any $\mathcal{Syn}$, $\models$ is a consequence relation.

Theorem 2. If $\vdash$ is a ***deducibility relation***, then ($X \vdash P$ iff, for any substitution operation ♦∈ **Sub**$\mathcal{Sen}$, $\sigma(X) \vdash \sigma(P)$).

Definition 7. Mutual Deducibility

Definition. $P \dashv\vdash Q$ iff, $P \vdash Q$ and $Q \vdash P$.

Theorem 1. $\dashv\vdash$ is an equivalence relation on $\mathcal{S}en$.

Provability. A notion narrower than deducibility is provability. A sentence is provable intuitively if its deduction does not depend on the truth of anything unproven. That is, *P* is provable from *X* iff if *X* is provable, so is *P.* But this account is not quite general enough. For *P* to be provable from *X* it is required that the proof be a matter of form. That is, if anything of the same form as *X* is provable, then anything of the same form as *P* should also be provable. We capture the idea of "sameness of form" by appeal to substitution.

Definition 8. The Provability Relation.

P is ***provable from*** *X* relative to a deducibility relation $\vdash$ (briefly, $X \Vdash P$), iff for all substitution operation $\sigma \in$ **Sub**$_{\mathcal{S}en}$, (for all $Q \in X$, $\vdash \sigma(Q)$) only if $\vdash \sigma(P)$.

Theorem 4. $X \Vdash P$ only if $X \vdash P$, but not conversely.

Rules of Proof and Provability

The tree $\dfrac{\varnothing \vdash P}{\varnothing \vdash Q}$ aka $\dfrac{\vdash P}{\vdash Q}$ is the normal form used to stipulate a rule of proof.

Examples

1. Necessitation in Modal Logic. $\dfrac{\vdash P}{\vdash \Box P}$

2. Theoremhood in Classical Sentential Logic: $\dfrac{\vdash P}{\vdash \mathbf{Th}P}$

3. Theoremhood in $\mathcal{PM}$: $\dfrac{\vdash P}{\vdash \mathcal{Th}(\underline{\mathbf{n}}_P)}$

Remark. Rule 2 (and Rule 1 if $\Box$=**Th**) is classical sound:

$$\varnothing \models_C P \quad \text{iff} \quad \varnothing \models_C (P \leftrightarrow (Q \vee \sim Q))$$

Rule 3 is not classically sound because neither $\mathcal{Th}$ nor $\underline{\mathbf{n}}_P$ have logically fixed referents. But if $\mathtt{Val}_{\mathcal{PM}}$ is that subset of $\mathtt{Val}_C$ that satisfies the axioms of $\mathcal{PM}$ and $\models_{\mathcal{PM}}$ is the restriction of $\models_C$ to $\mathtt{Val}_{\mathcal{PM}}$, then

$$\varnothing \models_{\mathcal{PM}} P \quad \text{iff} \quad \varnothing \models_{\mathcal{PM}} \mathcal{Th}(\underline{\mathbf{n}}_P)$$

Theorem 5. $\dfrac{\varnothing \vdash P}{\varnothing \vdash Q}$ iff $P \Vdash Q$

Inductive Systems. The deducibility relations we shall be studying in these lectures are primarily those of classical and intuitionistic logic. They both exhibit a good deal more structure than is captured in the abstract notion of a deducibility relation. They fall into the class of deducibility relations, familiar to students of elementary logic, that are characterizable in terms of axiom and natural deduction systems. In order to characterize this kind of deducibility relation, we begin by defining the concepts that abstract their special structural features. The ideas come from the theory of inductive sets.

Definition 9. Inductive System, Derivation and Proof.

An ***inductive system*** is any structure $<B,C,\{R_1,...,R_n\}>$ such that:

1. B (the set of *basic elements* of the system) and C (the set ***constructed** by* the system) are at most denumerable sets;
2. each R_i (a ***construction rule*** of the system) is a finite relation on $B\cup C$;
3. C is the least set X such that $B\subseteq X$ and, for any R_i , if R_i is an m+1-place relation, $<e_1,...,e_{m+1}>\in R$ and $<e_1,...,e_m>\in C$, then $e_{m+1}\in C$.

Relative to an at most denumerable set B, and a set of finitary relations $\{R_1,...,R_n\}$ defined for tuples in B, a ***derivation (tree)*** relative to B and $\{R_1,...,R_n\}$ is defined as any finite labeled tree Π such that:

1. every leaf node of Π is labeled by an element in B,
2. for any node n of Π with immediate predecessor nodes $m_1,...,m_k$,
   a. each m_i (for $i\leq k$) is labeled by some element e_i,
   b. n is labeled by some rule R_i such that $<e_1,...,e_k,e>\in R_i$.

If the leaf nodes of a deduction tree Π are labeled respectively $e_1,...,e_k$, its root node is labeled by e, and if $\{R_1,...,R_n\}=\{R| R$ is a finitary relation on Sen that labels some node of Π }, we say Π is a **derivation (tree) of** e from $< e_1,...,e_k>$ relative to $\{R_1,...,R_n\}$. If in addition all the leaf nodes of Π are in B, then Π is called a ***proof (tree) of*** e from $<e_1,...,e_k>$ relative to B and $\{R_1,...,R_n\}$.

Theorem 6. $e\in C$ for an inductive system $<B,C,\{R_1,...,R_n\}>$ iff there is some proof tree of e relative to B and some subset of $\{R_1,...,R_n\}$.

Axiom Systems. From an abstract perspective an axiom system is identified with an inductive system

Definition 10. Axiom System.

An ***axiom system*** for $\mathcal{Syn}=<A_1,...,A_k,f_1,...,f_m>$ is any inductive system such that $<Ax,\vdash,\{R_1,...,R_n\}>$ such that Ax and $\vdash$ are subsets of $\mathcal{Syn}$.

An axiom system $<Ax,\vdash,\{R_1,...,R_n>$ is ***finite*** and $\vdash$ is said to be ***finitely axiomatizable***, iff Ax is finite.

One weakness of analyzing $\vdash$ as a set is that in order to capture the more general idea of a deducibility, it must then be extended in some manner to a relation. In languages which have a semantic entailment relation that is compact and a conditional $\rightarrow$ that yields a "deduction theorem" (i.e. $\{P_1,...,P_n\}\vdash Q$ iff $\vdash(P_1\wedge...\wedge P_n)\rightarrow Q$), then the extension is possible. Conceptually the analysis is not very convincing because of its lack of generality: it depends on specific features of the syntax (on the right sort of connectives $\wedge$ and $\rightarrow$) and on compactness, a property not exhibited by some interesting logical systems.

Definition 11. Deducibilty in an Axiom System

The set $\vdash$ is extended to a relation as follows: $\{P_1,...,P_n\}\vdash Q$ iff $\vdash(P_1\wedge...\wedge P_n)\rightarrow Q$,

$X \vdash Q$ iff, there is some finite subset $\{P_1,...,P_n\}$ of X such that $P_1,...,P_n\vdash Q$.

Whether this $\vdash$ relation is a deducibility relation will depend on the properties of $\wedge$ and $\rightarrow$.

Example. The System C for Classical Sentence Logic.

C is defined as the indiuctive system $<Ax_C,\vdash_C,\{MP\}>$ such that :

1. Ax_C is any instance of the following three schemata:
   i. $P\rightarrow(Q\rightarrow P)$
   ii. $(P\rightarrow(Q\rightarrow R))\rightarrow((P\rightarrow Q)\rightarrow(P\rightarrow R))$
   iii. $(\sim P\rightarrow\sim Q)\rightarrow(Q\rightarrow P)$
2. *MP* (*modus ponens*) is $\{<P,Q,R>|\ Q=P\rightarrow R\}$

Theorem 7. The relation $\vdash_C$ is a deducibility relation.

Natural Deduction Systems. Natural deduction systems too are inductive systems, but in this case the elements included in the inductive sets are "deductions," i.e. pairs $<X,P>$ consisting of a set of premises and a conclusion that follows from them. The basic elements of the construction, therefore, must be a special selected set of deductions and the rules of construction must be rules that take deductions as arguments and yield deductions as values.

Definition 11. Natural Deduction Systems.

*B*y a ***deduction*** in $\mathcal{Syn}$ is meant any pair $<X,P>$ such that $P\in\mathcal{Sen}$ and X is a finite subset of $\mathcal{Syn}$. Here X is called the ***premise set*** of the deduction and P the ***conclusion****.*

An ***inference rule*** for $\mathcal{Syn}$ as any finitary relation on deductions in $\mathcal{Syn}$. In addition a special set *BD* of deductions is distinguished, called the set of ***basic deductions***.

By a ***natural deduction system*** for $\mathcal{Syn}$ is meant any inductive system $<BD,\vdash,RL>$ such that
1. *BD* is a distinguished set of deductions for $\mathcal{Syn}$, and
2. *RL* is a set of derivation rules for $\mathcal{Syn}$.

The inductively defined relation $\vdash$ is called the set of ***provable deductions*** for $\mathcal{Syn}$ relative to *BD* and *RL*.

We write $X\vdash P$ for $<X,P>\in\vdash$, and adopt the customary abbreviations:

$X,P\vdash Q$	means	$X\cup\{P\}\vdash Q$
$P_1,...,P_n\vdash Q$	means	$\{P_1,...,P_n\}\vdash Q$
$\vdash P$	means	$\varnothing\vdash P$

Theorem 8. $<X,P>$ is a provable deduction in $<BD,\vdash,RL>$ iff there is some proof tree of $\mathcal{Syn}$ relative to BD and some subset of RL such that its root node is labeled by $<X,P>$.

Example. A Natural Deduction Systems C for the Classical Sentential Logic

C=$<BD_C,\vdash_C,R_{\perp+},R_{\perp-},R_{\sim+},R_{\sim-},R_{\wedge+},R_{\wedge-},R_{\vee+},R_{\vee-},R_{\rightarrow+},R_{\rightarrow-},R_{Th}>$ is the inductive system such that

1. Let $<X,P>$ be a ***deduction*** iff $X\subseteq\mathcal{Sen}$ and $P\in\mathcal{Sen}$. We adopt these abbreviations:

$X\vdash_C P$	for	$<X,P>$ is in $\vdash_C$;
$X,Y\vdash_C P$	for	$X\cup Y\vdash_C P$;
$X,P\vdash_C Q$	for	$X\cup\{P\}\vdash_C Q$;
$P_1,\ldots,P_n\vdash_C Q$	for	$\{P_1,\ldots,P_n\}\vdash_C Q$;
$\vdash_C P$	for	$\varnothing\vdash_C P$.
$\perp$	for	$P_1\wedge{\sim}P_1$ (Here P_1 is the 1st atomic sentence.)

2. BD_C is the set of all deductions $<X,P>$ such that $P\in X$.
3. The rules in $\{R_{\perp+},R_{\perp-},R_{\sim+},R_{\sim-},R_{\wedge+},R_{\wedge-},R_{\vee+},R_{\vee-},R_{\rightarrow+},R_{\rightarrow-},R_{Th}\}$ are defined as follows:

	Introduction (+) Rules	***Elimination (-) Rules***
$\perp$	$\dfrac{X\vdash_C P \quad Y\vdash_C {\sim}P}{X,Y\vdash_C\perp}$	$\dfrac{X\vdash_C\perp}{X-\{{\sim}P\}\vdash_C P}$ (for $P\neq{\sim}Q$)
$\sim$	$\dfrac{X\vdash_C\perp}{X-\{P\}\vdash_C{\sim}P}$	$\dfrac{X\vdash{\sim}{\sim}P}{X\vdash_C P}$
$\wedge$	$\dfrac{X\vdash_C P \quad Y\vdash Q}{X\vdash_C P\wedge Q}$	$\dfrac{X\vdash_C P\wedge Q}{X\vdash_C P}$ $\quad$ $\dfrac{X\vdash_C P\wedge Q}{X\vdash_C Q}$
$\vee$	$\dfrac{X\vdash_C P}{X\vdash_C P\vee Q}$ $\quad$ $\dfrac{X\vdash_C P}{X\vdash_C P\vee Q}$	$\dfrac{X\vdash_C P\vee Q \quad Y\vdash_C R \quad Z\vdash_C R}{X,Y-\{P\},Z-\{Q\}\vdash_C R}$
$\rightarrow$	$\dfrac{X\vdash_C P}{X-\{Q\}\vdash_C Q\rightarrow P}$	$\dfrac{X\vdash_C P \quad X\vdash_C P\rightarrow Q}{X\vdash_C Q}$

Thinning $\quad \dfrac{X\vdash_C P}{X,Y\vdash_C P}$

We extend the notion of deduction to possibly infinite sets of premises X by saying $X\vdash_C Q$ relative to $\vdash_C$ iff, there is some finite subset $\{P_1,\ldots,P_n\}$ of X such that $P_1,\ldots,P_n\vdash_C Q$.

Theorem 9. The relation $\vdash_C$ is a deducibility relation.

In cases in which the notion of a uniform substitution $\blacklozenge$ is defined for $\mathcal{Syn}$, it is customary to define a derivation rule R for $\mathcal{Syn}$ by a tree diagram.

Recall that by the tree $\quad R: \quad \dfrac{X_1\vdash P_1>, \quad \ldots\ldots \quad X_k\vdash P_k}{Y\vdash Q}$

we refer to the relation

$$R \quad = \quad \{\langle \blacklozenge(\langle X_1,P_1\rangle), \ldots, \blacklozenge(\langle X_k,P_k\rangle),\blacklozenge(\langle Y,Q\rangle)\rangle \mid \blacklozenge \text{ is a sentential substitution for } \mathit{Syn} \text{ and } \blacklozenge(\langle X_1,P_1\rangle), \ldots,\blacklozenge(\langle X_k,P_k\rangle), \blacklozenge(\langle Y,Q\rangle) \text{ are all deductions in } \mathit{Syn}\}.$$

Since elegance and brevity are theoretical ideals of proof theory, finding the minimal set of rules necessary is often a goal. Some basic notions in terms of which systems are simplified and compared can now be defined.

Definition 12.

A relation R is said to be ***definable*** relative to rules $R_1,\ldots,R_n$ and is called a ***derived rule*** in $\langle BD,\vdash,RL\rangle$, where $\{R_1,\ldots,R_n\}\subseteq RL$, iff there is a derivation tree Π of d_{n+1} from $d_1,\ldots,d_n$ relative to BD and $\{R_1,\ldots,R_n\}$, and $R=\{\langle\sigma(d_1),\ldots, \sigma(d_{n+1})\rangle \mid \sigma$ is a substitution for Sen }.

A natural deduction system $\langle BD,\vdash,RL\rangle$ is said to be ***reducible to*** a natural deduction system $\langle BD',\vdash',RL'\rangle$ iff, $BD\subseteq BD'$ and every $R\in RL$ is a derivable rule in $\langle BD',\vdash',RL'\rangle$.

Two systems are ***strictly equivalent*** iff they are mutually reducible. Let two systems $\langle BD,\vdash,RL\rangle$ and $\langle BD',\vdash',RL'\rangle$ be called ***constructively equivalent*** iff $\vdash=\vdash'$.

2. Logical Matrices for Sentential Logic.

One of the oldest and most productive branches of logic is the investigation of the semantic properties of sentential logic by means of structures known as logical matrices. Logical matrices are algebras of "truth-values" and the interpretations they spawn are homomorphisms between syntax and these structures.

Definition 13. Logical Matrices

A ***logical matrix*** for any $\mathcal{SL}$ syntax $\mathit{Syn}=\langle \mathit{Sen},f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow}\rangle$ is any semantic structure structure $M=\langle U,D, g_{\sim},g_{\wedge},g_{\vee},g_{\rightarrow}\rangle$ for Syn such that U and D are non-empty, and $D\subseteq U$.

Frequently U is some set of ordered numbers starting with 0, e.g. $\{0,1\}$, $\{0,{}^1/_2,1\}$, $\{0,1,\ldots,n\}$ starting with 0, in which case M is said to be ***m-valued*** where m is the cardinality of U.

A semantic interpretation relative to a logical matrix M is called a ***valuation*** of M, and the set of all semantic interpretations **I-**M of Syn relative to M is traditionally called the ***set of valuations*** of M, which we abbreviate $\mathtt{Val}_M$. We let $\mathtt{V}$ range over $\mathtt{Val}_M$. Clearly, $X\models_M P$ is well defined.

A ***matrix language*** is any language $\langle \mathit{Syn},\mathcal{F}\rangle$ such that $\mathcal{F}$ is a family of logical matrices.

It is customary to refer to both the series of syntactic operations $f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow}$ and the series of semantic operations $g_{\sim},g_{\wedge},g_{\vee},g_{\rightarrow}$ by $\sim,\wedge,\vee,\rightarrow$. In some contexts where it would be unclear which is meant, we shall distinguish one series from the other by the use of prime marks.

One of the most useful investigations in matrix semantics is the "representation" of one matrix in another. Such representations are used to simplify the semantics by replacing a broad set of valuations (and its characterization of entailment) with a narrower one, generated by a simpler matrix which is also characteristic of the entailment relation in question. We shall see several important examples of such representations in the course of these lectures.

The relevant concept of representation is captured by the idea of homomorphism. Designated values play no role in the definition of valuations. As a result there is one sense of representation in which they are ignored, and a stricter sense in which they are not.

Definition 14. Matrix Morphisms

φ is a (***matrix***) ***homomorphism*** (in the weak sense) from a logical matrix M=<U,D, ∼,∧,∨,→> to another matrix M′=<U′,D′, ∼′,∧′,∨′,→′> (of the same character) iff φ is a homomorphism from to M=<U,∼,∧,∨,→> into <U′, ∼′,∧′,∨′,→′> .

φ is a ***strict*** (***matrix***) ***homomorphism*** from M=<U,D, ∼,∧,∨,→> to M′=<U′,D′, ∼′,∧′,∨′,→′> (of the same character) iff φ is a homomorphism and φ ***preserves designation and non-designation*** in the sense that for any *x* in U,

$x \in D$, then $\varphi(x) \in D'$, and
if $x \notin D$, then $\varphi(x) \notin D'$,

We shall call these morphisms ***onto*** and ***1 to 1*** if φ is an onto or 1 to 1 function respectively.

Note two additonal formulations that are equivalent to the condition that φ preserves designation and non-designation:

1. for any *x* in U, $x \in D$ iff $\varphi(x) \in D'$.
2. φ maps D into D′ , and U– D into U′– D′.

Notice also that if we interpret a syntax by a matrix M and there is a second matrix M′ to which M is homomorphic under φ, then we can interpret the syntax by M'. For any sentence *P*, we assign it a value v(*P*) in M, and then using φ we assign it to φ (v(P)). We call composition the process of defining a third function by taking the an argument's value under one function, turning it into the argument of a second function, and then calculating its value.

Definition 15. If f and g are (one-place) functions, their ***composition*** $f{\circ}g$ is defined: $f{\circ}g(x) = g(f(x))$.

Theorem 10. If M=<U,D,~,∧,∨,→> is a logical matrix for *Syn*=<*Sen*,~,∧,∨,→> and φ is a matrix homomorphism from M to M′, then $\{v{\circ}\varphi \mid v{\in}\mathtt{Val}_M\} \subseteq \mathtt{Val}_{M'}$.

Proof. Consider an arbitrary $v{\circ}\varphi$ such that $v\in \mathtt{Val}_M$. We show it meets the conditions for membership in $\mathtt{Val}_{M'}$. If P is atomic, then φ is defined for $v(P)$ and the range of φ is a subset of U′. Thus, $\varphi(v(P))\in$ U′. For the molecular case consider an arbitrary complex sentence $O_i(P_1,...,P_n)$ such that O_i is the grammatical operation generating the sentence, and the operations in M and M′ corresponding to O_i are respectively g_i and g'_i. Then by the relevant definitions, $v{\circ}\varphi\ (O_i(P_1,...,P_n)) = \varphi(v(O_i(P_1,...,P_n))) = \varphi(g_i(v(P_1),..., v(P_n))) = g'_i($❑$(v(P_1),\ldots, \varphi(v(P_n)) = g'_i(v{\circ}\varphi(P_1),\ldots, v{\circ}\varphi(P_n))$. Hence, $v{\circ}\varphi\in\mathtt{Val}_{M'}$.

QED

Theorem 11. If ❑ is a strict matrix homomorphism from M=<U,D, ~,∧,∨,→> to M′=<U′,D′, ~′,∧′,∨′,→′>, then $X \models_{M'} P$ only if $X \models_M P$.

(**Analysis.** Assume:

1. φ is a strict matrix homomorphism from M=<U,D, ~,∧,∨,→> to M′=<U′,D′, ~′,∧′,∨′,→′>
2. $X \models_{M'} P$ i.e. for any $v'\in\mathtt{Val}_{M'}$, if for all Q in X, $v'(Q)\in$D′ then $v'(P)\in$D′
3. that v is arbitrary, that $v\in\mathtt{Val}_M$ and that for any Q in X, $v(Q)\in$D

Show: $v(P)\in$D.
The trick is to apply 1 to 3 and derive that $v{\circ}\varphi(Q)\in$D′, for all $Q\in X$. Then apply Theorem 10, and deduce that $v{\circ}\varphi\in\mathtt{Val}_{M'}$, and hence by 2, $v\varphi$ satisfies P in the relevant sense. Show then that v satisfies (in the relevant sense) P.

Theorem 12. If φ is a strict matrix homomorphism from M=<U,D, ~,∧,∨,→> onto M′=<U′,D′, ~′,∧′,∨′,→′>, then $\{v{\circ}\varphi \mid v{\in}\mathtt{Val}_M\} = \mathtt{Val}_{M'}$.

Proof. By theorem 10 all we need show is $\mathtt{Val}_{M'} \subseteq \{v{\circ}\varphi \mid v{\in}\mathtt{Val}_M\}$. Assume $v'\in\mathtt{Val}_{M'}$. We show that that $v'\in \{v{\circ}\varphi \mid v{\in}\mathtt{Val}_M\}$. We construct a some v, such that $v'=v{\circ}\varphi$ and $v\in\mathtt{Val}_M$. Let P be an atomic sentence. Since φ is onto we know that whatever $v'(P)$ is, let's call it x, there is some $y\in$U such that $\varphi(y)=x$. We define $v(P)$ to be that y. We do so for each atomic sentence, and then project these values to molecular sentences by the operations in M. That is, we define v to be that $v\in\mathtt{Val}_M$ such that for any atomic sentence P, $\varphi(v(P)) = v'(P)$. We now show that $v'=v{\circ}\varphi$, i.e. that for any sentence Q, $v'(Q) = \varphi(v(Q))$. Proof is by induction. The atomic case is ture by the definition of v. For the molecular cases we assume the identity holds for the immediate parts of the sentence and show it is true for the whole. Consider the case of conjunction $R\wedge S$. Assume (as the induction hypo.) that $v'(R)= \varphi(v(R))$ and $v'(S)=$ ❑$(v(S))$. Now, $v'(R\wedge S)$ = [by membership of v' in $\mathtt{Val}_{M'}$]) $\varphi(v(R))\wedge' \varphi(v(S))$ = [since φ is a homomorphism from M to M′]) $\varphi(v(R)\wedge v(S))$ = [since v is a homomorphism from *Syn* to M] $\varphi(v(R\wedge S))$. The cases of the other connectives are similar.

QED

Theorem 13. If φ is a strict matrix homomorphism from M=<U,D, ~,∧,∨,→> onto M′=<U′,D′, ~′,∧′,∨′,→′>, then $X \models_{M'} P$ iff $X \models_M P$.

Examples of Traditional Matrix Logics. Lukasiewicz and his colleagues were largely motivated by philosophical issues in developing matrix semantics, particularly their doubts about classical bivalence. Logical issues too are central. Both the matrix and the resulting entailment relation must be acceptable. Acceptability here is rather complex matter.

Acceptability is partly conceptual. The definitions offered by the theory must be "conceptually adequate." Roughly this is a requirement that the definitions conform with prior usage, both in ordinary language and in the earlier literature of logical and philosophy. For example, if matrix elements are intended to be "truth-values," then metatheorems concerning them should translate into plausible claims about truth. While the law of bivalence (every sentence is either true or false) may be doubted, the law of non-contradiction (no sentence and its negation can both be true) is less so. It is issues of this sort that are of concern to philosophers of language when they evaluate many-valued semantics.

Logical issues, however, are equally important. By their nature they tend to be the focus of logicians rather than philosophers. Logicians hone their intuitions about which inferences are valid. Doing so is a matter partly of common sense, partly of thinking about the meanings of the "logical terms" at play, and partly of tradition, logical tradition itself being one of the major determinants of the meaning of logical terms. Because classical two-valued logic has been the standard theory throughout this tradition, logical issue largely centers on how much, if at all, a matrix entailment relation departs from classical logic, and whether these departures are desirable. It has proven to be very difficult to give a simple matrix semantics that is both conceptually plausible and yields an intuitively acceptable entailment relation.

A third criterion that is of less concern to the non-mathematical is elegance. Matrix semantics are very elegant indeed, and the goal of revising classical semantics using matrices has been a serious research enterprise, involving some of the best logicians, for almost eighty years. One of the large chapters of this story concerns the matrix characterization of intuitionistic logic, one of the century's major revisions of classical logic. We will take up intuitionistic semantics in detail later. At this point is will be instructive to illustrate the methods by citing some of the simpler and more famous many-valued theories.

Defintion 16. Truth-Tables for Standard Matrices

The Classical Bivalent Matrix **C**

	~	∧ T	∧ F	∨ T	∨ F	→ T	→ F
T	F	T	F	T	T	T	F
F	T	F	F	T	F	T	T

Klenne's Weak (Bochvar's Internal) Matrix **KW**

	~	∧ T	∧ F	∧ N	∨ T	∨ F	∨ N	→ T	→ F	→ N
T	F	T	F	N	T	T	N	T	F	N
F	T	F	F	N	T	F	N	T	T	N
N	N	N	N	N	N	N	N	N	N	N

Klenne's Strong Matrix **KS**

	~	∧ T	∧ F	∧ N	∨ T	∨ F	∨ N	→ T	→ F	→ N
T	F	T	F	N	T	T	T	T	F	N
F	T	F	F	F	T	F	N	T	T	T
N	N	T	F	N	T	T	N	T	N	N

Lukasiewicz' 3-valued Matrix **L3**

	~	∧ T	∧ F	∧ N	∨ T	∨ F	∨ N	→ T	→ F	→ N
T	F	T	F	N	T	T	T	T	F	N
F	T	F	F	F	T	F	N	T	T	T
N	N	T	F	N	T	T	N	T	N	T

Jaskowski's **C^2**-valued Matrix

	~	∧ 11	∧ 10	∧ 01	∧ 00	∨ 11	∨ 10	∨ 01	∨ 00	→ 11	→ 10	→ 01	→ 00
11	00	11	10	01	00	11	11	11	11	11	10	01	00
10	00	10	10	00	00	11	10	11	10	11	11	01	01
01	00	01	00	01	00	11	11	01	01	11	10	11	10
00	00	10	10	00	00	11	10	11	10	11	11	11	11

None of these matrices with the exception of C^2 is classical. Whether the classical inferences they reject are in fact invalid is a further issue which we do not have time to go into here. Suffice it to say that none of these has proven very convincing.

Definition 17. We shall use M^D to refer to a matrix M with desiganted values D. It is also traditional to identify T with 1, 0 with F, N with $^1/_2$, 2 with the set {0,1}, and in general *n* with $\{m|\ 0\leq m<n\}$. As the universe for the matrix in question we take the set of all values appearing in the truth-table.

By **Ln**=<U,D, ~,∧,∨,→> we mean the generalization of L3 in which D={1} and the operations conform to these rules:

$\sim x = 1\text{-}x$
$x\wedge y = \min\{x,y\}$
$x\vee y = \max\{x,y\}$
$x\rightarrow y = \min\{1, (1\text{-}x)+y\}$

For finite matrices $\mathbf{L_n}$ its domain U={ $\frac{x}{n}$ | x is a natural number and $0\leq x\leq n$ }. $\mathbf{L_\omega}$ is the limiting case in which U = ω = { x | x is a rational number and $0\leq x\leq 1$ }. One can also set U = [0,1].

By $\mathbf{M^n}$=<U^n,D^n, $\sim^n$,$\wedge^n$ $\vee^n$,$\rightarrow^n$>, we mean the generalization of M=<U,D, ~,∧,∨,→> in which the operations $f^n{}_i$ corresponding to conform to the following rules:

U^n and D^n are respectively the n-th Cartesian products of U and D, and
$f^n{}_i(<x_{1,1},...,x_{1,n}>,\ldots,<x_{m,1},...,x_{m,n}>)=<f_i(x_{1,1},...,x_{1,n}),\ldots, f_i(x_{m,1},...,x_{m,n})>$

Definition 18. Let C be the classical matrix <{T,F}{T},~,∧,∨,→>. Then, a matrix M=<U,D,~′,∧′,∨′,→′> is ***normal*** iff, {T,F}⊆U, {T}⊆D, and for any *x* and *y* in {T,F}, $\sim x=\sim' x$, $\wedge(x,y)=\wedge'(x,y)$, $\vee(x,y)=\vee'(x,y)$, $\rightarrow(x,y)=\rightarrow'(x,y)$.

Theorem 14.

$KW^{\{T\}}$, $KW^{\{T,N\}}$, $KS^{\{T\}}$, $KS^{\{T,N\}}$ are normal.

$\models_{KW\{T\}}$, $\models_{KW\{T,N\}}$, $\models_{KS\{T\}}$, $\models_{KS\{T,N\}}$, are proper subsets of $\Vdash_{C\{T\}}$.

$\{P|\models_{KW\{T\}}P\}$ and $\{P|\models_{KW\{T\}}P\}$ are empty.

$\models_{M^nD^n} = \models_{M}D$, and hence $\models_{C^n\{T\}^n} = \models_{C\{T\}}$,

3. Lindenbaum Algebras

The first abstract results which we shall actually prove in which we use matrix semantics to characterize a proof theoretic idea will consist of ways to characterize the relatively weak provability relation ╟. They consist of constructing the relevant matrix from the syntax itself. Since these matrices are relate purely sytaxtic entities (sentences) they fall short of what the philosphers have traditionally thought of as a "world" or a "semantic interpretation." They are nevertheless excellent illustrations of algebraic ideas we have been introducing, so sucessful in fact that they may give philosphers pause.

We shall begin with an utterly trivial matrix, interpreting the syntax literally by the syntax itself. That is, we shall assign sentences to other sentences in a way that preserves syntactic structure. The sentence assigned to a whole will be that of like construction generated from those assigned to its parts. A representative, therefore will be a negation, conjunction, disjunction, etc. if the sentence it represents is, but the representative may have more structure because the atomic sentences of the original may be assigned to molecular sentences with internal structure. As designated elements let us use the set of provable sentences, i.e. the theorems of $\vdash$.

Definition 19. $\mathbf{Th}_{\vdash}=\{P \mid \Vdash P\}$

$[P]_{\vdash}=\{Q \mid Q \dashv\vdash P\}$ (Recall that $\dashv\vdash$ is an equivalence relation.)

M-$\mathcal{Syn}$ be $=<\mathcal{Sen},\mathbf{Th}_{\vdash},\sim,\wedge,\vee,\rightarrow>$

Theorem 15. For any $\Vdash$ for $\mathcal{Syn}=<\mathcal{Sen},\sim,\wedge,\vee,\rightarrow>$, there exists a denumerable matrix M such that M

$X \Vdash P$ iff $X \models_{M} P$

Proof. For the matrix in question let us take $\mathcal{Syn}$ itself with all the theorems of $\vdash$ as designated elements. Let M-$\mathcal{Syn}$ be $=<\mathcal{Sen},\mathbf{Th}_{\vdash},\sim,\wedge,\vee,\rightarrow>$ where $\mathbf{Th}_{\vdash}=\{P \mid \Vdash P\}$. Observe that valuations over this matrix are just substitution relations: $\mathtt{Val}_{\text{M-}\mathcal{Syn}} = \mathbf{Sub}_{\mathcal{Sen}}$

Now, $X \models_{\text{M-}\mathcal{Syn}} P$ iff $\forall\sigma\in\mathtt{Val}_{\text{M-}\mathcal{Syn}}, [\forall Q\in X, \sigma(Q)\in\mathbf{Th}_{\vdash}] \Rightarrow \sigma(P)\in\mathbf{Th}_{\vdash}$

iff $\forall\sigma\in\mathtt{Val}_{\text{M-}\mathcal{Syn}}, [\forall Q\in X, \vdash\sigma(Q))] \Rightarrow \vdash\sigma(P)$

iff $\forall\sigma\in \mathbf{Sub}_{\mathcal{Sen}}, [\forall Q\in X, \vdash\sigma(Q))] \Rightarrow \vdash\sigma (P)$

iff $X \Vdash P$. **QED**

A more elegant syntactic matrix, called a Lindenbaum algebra, is that formed by the equivalence classes generated by $\vdash$. In such a structure the set of its logical equivalents "represent" a sentence. Such a class does indeed "stand proxy" for something like a "meaning" or "propositions," at least if we grant that "sameness of meaning" is at some level of abstraction the same as logical equivalence. If such a matrix is well-defined, it is in fact characteristic of the provability relation. In general, however, not all $\vdash$ relations generate such a structure. Though $\dashv\vdash$ is trivial an equivalence relation, to generate the structure in question it must also be a congruence relation (have the substitution property.

Definition 20. If $\dashv\vdash$ is a congruence relation on M-$\mathcal{Syn}=<\mathcal{Sen},\mathbf{Th}_{\vdash},\sim,\wedge,\vee,\rightarrow>$, then the quotient algebra determined by $\dashv\vdash$, namely

$M_{\vdash}=<\{ [P]_{\vdash} \mid P\in\mathcal{Sen}\},\{\mathbf{Th}_{\vdash}\},\sim,\wedge,\vee,\rightarrow>$,

is called the ***Lindenbaum algebra*** for M-$\mathcal{Syn}$.

Notice that corresponding to the set $\mathbf{Th}_{\vdash}$ of designated values in M-$\mathcal{Syn}$ is the set of 's designated values

$$\{\mathbf{Th}_{\vdash}\} = \{ [P]_{\vdash} \mid P\in\mathbf{Th}_{\vdash} \},$$

that this set contains one entity only that can serve as a designated value in $M_{\vdash}$ value, and that this single entity, namely $\mathbf{Th}_{\vdash}$, is itself a set, the set of $\vdash$ theorems.

of in Furthermore, the operation $[\ ]_{\vdash}$ preserves designation and non-designation: $P\in\mathbf{Th}_{\vdash}$ iff $[P]_{\vdash}\in \{\mathbf{Th}_{\vdash}\}$. The following theorems follow directly from (and illustrate how to apply) the general results we have already proven about congruence relations and strict homomorphisms between matrices.

Theorem 16. If $\dashv\vdash$ is a congruence relation on M-*Syn*, then the mapping $[\ \ ]_{\vdash}$ is a strict homomorphism from M-*Syn* to $M_{\vdash}$..

Theorem 17. If $M_{\vdash}$ exists, then $\mathtt{Val}_{M_{\vdash}} = \{ \sigma\circ[\ ]_{\vdash} \mid \sigma\in \mathbf{Sub}_{Sen}\}$

Theorem 18 (Lindenbaum)[2] If $M_{\vdash}$ exists, then $X\Vdash P$ iff $X\models_{M_{\vdash}}P$

Theorem 19. If $\leq$ is the syntactic part-whole relation, then in general $[\ ]_{\vdash}$ is not an $\leq$-order preserving homomorphism: for some $\vdash$, P, and Q, it is not the case that $([P]_{\vdash} \leq [Q]_{\vdash}$ iff $P\leq Q)$.

[2] This theorem is not the more famous Lindenbaum Lemma which says that every consistent set may be extended to a maximally consistent set.

Exercises.

1. Prove Theorem 11.

2. Prove $\models_{KW\{T\}}$ is a proper subsets of $\models_{C\{T\}}$. Show first that it is a subset by first defining the right sort of homomorphism between the structures and then appealing to a previous theorem. Then show that it is a proper subset by finding some inference valid in the second that is not valid in the first. (This is part of Theorem 14.)

3. Show $\{P | \models_{KW\{T\}} P\} = \varnothing$. (This is part of Theorem 14.)

4. Explain why theorem 16, 17, and 18 all follow directly from earlier results about matricies and valuations.

5. (Thought Question.) Let us call a puported inference relation $\vdash$ ***conceptually plausible*** if its definition consists of some principle ($X \vdash P$ iff ☞(X,P)) that is true of the inference relation $\vdash_C$ of classical logic. Such a definition would be an "abstraction" from that of classical logic. Think up a definition for a conceptually plausible inference relation that is not transitive. Think up one that is not montonic. Think up one that fails for substitutions. What implication does failure of subsitutivity have for finite axiomatizability? What would these failures do to the ordinary notion of proof?

Lecture 3

Boolean Algebras and Classical Logic

1. Boolean Algebras

In this lecture we shall be concerned with what is probably the most important structures used in used in semantics. These are the Boolean algebras used in the interpretation of classical logic. As operations on sets they were studied by Boole, and as truth-functions by Pierce and Wittgenstein. They are basic to standard set theory and elementary logic, and as a class of algebras have many interesting properties that have inspired fruitful generalizations.

Definition 1. Boolean Algebra.

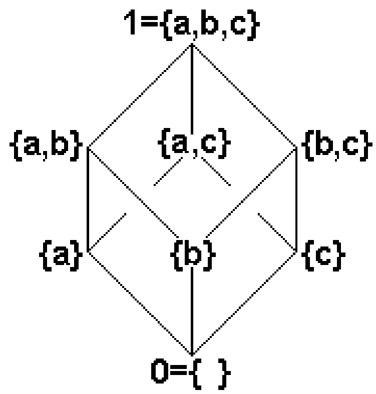
A structure $\langle B,\wedge,\vee,-,0,1\rangle$ is a ***Boolean algebra*** iff it is a structure satisfying the following conditions. Let *x, y* and *z* be arbitrary members of B.

1. $\langle B,\wedge,\vee\rangle$ is a lattice, i.e.
   - L1. $x\wedge y=y\wedge x$; $x\vee y=y\vee x$
   - L2. $(x\wedge y)\wedge z=x\wedge(y\wedge z)$; $(x\vee y)\vee z=x\vee(y\vee z)$;
   - L3. $x\wedge x=x=x\vee x$;
   - L4. $x\vee(x\wedge y)=x=x\wedge(x\vee y)$.
2. $\langle B,\leq\rangle$ is a partially ordered structure, i.e. by definition $x\leq y \Leftrightarrow x\wedge y=x \Leftrightarrow x\vee y=y$ and
   - P1. $x\leq x$;
   - P2 $x\leq y$ & $y\leq z$.$\Rightarrow$ $x\leq z$;
   - P3. $x\leq y$ & $y\leq x$.$\Rightarrow$ $x=y$.
3. $\langle B,\wedge,\vee\rangle$ is distributive, i.e.
   - D1. $x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$;
   - D2. $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$.
4. 0 and 1 are respectively the least and greatest element of B in $\langle B,\wedge,\vee,0,1\rangle$, i.e.
   - G1. $0\leq x\leq 1$;
   - G2. $1\wedge x=x$;
   - G3. $1\vee x=1$;
   - G4. $0\wedge x=0$;
   - G5. $0\vee x=x$.
5. $-$ is a unique complementation operation on one-place operation on $\langle B,\wedge,\vee,-,0,1\rangle$, i.e.
   B is closed under $-$ and
   - C1. $x\wedge -x=0$
   - C2. $x\vee -x=1$
   - C3. $--x=x$, $-1=0$, $-0=1$;
   - C4. $x\leq y \Leftrightarrow x\wedge -y=0 \Leftrightarrow -y\leq -x \Leftrightarrow -x\vee y=1$
   - C5. $-(x\wedge y)=-x\vee -y$, $-(x\vee y)=-x\wedge -y$.

Theorem 1. <B,∧,∨,–,0,1> is a Boolean algebra iff ∧ and ∨ are binary and – a unary operation on B under which B is closed, 1,0∈B and

L1. $x\wedge y=y\wedge x$; $x\vee y=y\vee x$;	C2. $x\vee -x=0$
D1. $x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$;	G2. $1\wedge x=x$;
D2. $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$;	G5. $0\vee x=x$;
C1. $x\wedge -x=1$	

Example. A three element Boolean Algebra

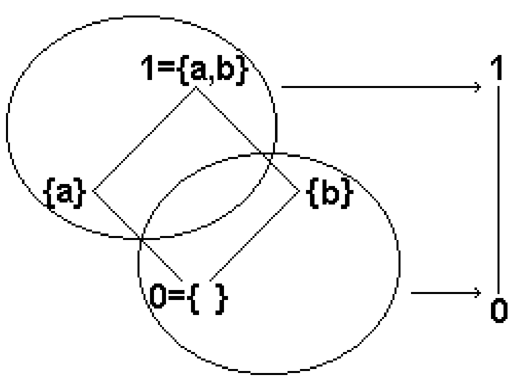


A Boolean Algebra of the Power set of {a,b,c}

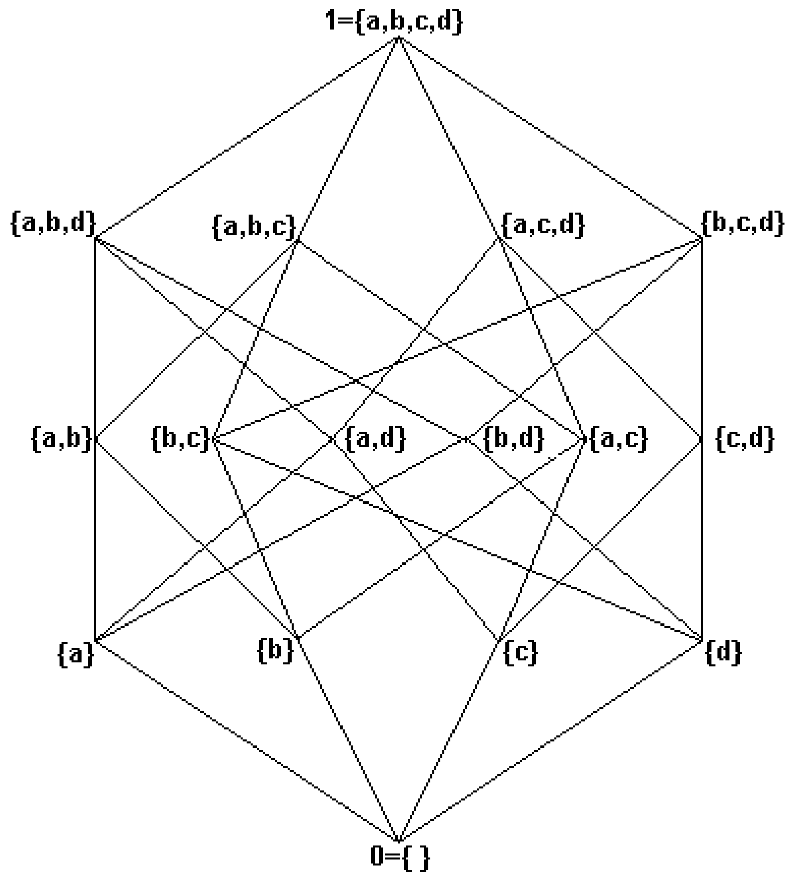
We shall let **B**=<B,∧,∨,–,0,1> range over Boolean algebras, distinguish one algebra from another by prime marks on its various components.

Theorem 2. Although any congruence relation for a Boolean Algebra <B,∧,∨,–,0,1> has (by definition) the substitution property for ∧,∨,–, it does not in general have the substitution property for ≤. That is, there are some Boolean Algebras with congruence relation ≡ such that for some a,b,c in B, a≡b, c≡d, and a≤c, yet not(b≤c).

Consider the function ϕ diagrammed below mapping one Boolean algebra to another and hence determining a congruence relation $\equiv_\phi$. That is ϕ is defined: Here ϕ(1)=1
ϕ(a)=1, ϕ(b)=0, ↗(0)=0.) Here $\phi(x\wedge y)=\phi(x)\wedge\phi(y)$ and $x\equiv_\phi y$ & $z\equiv_\phi w$.⇒ $x\wedge z\equiv_\phi y\wedge w$, and likewise for ∨. But, 0$\equiv_{\nearrow}$b & 1$\equiv_\phi$a & 0≤1, yet not(b≤0). **QED**



Example. A four element Boolean Algebra



A Boolean Algebra of the Power set of {a,b,c,d}

2. Filters, Ideals and the Binary Representation Theorem

A important subset of the universe of a Boolean algebra is the set of elements above *x*, or dually the elements below *x.* The former is called a *filter*, the latter an *ideal.* A maximal filter of *x* and dual maximal ideal of –*x* have the very nice property that they partition the algebra into just two equivalence classes that also determine a congruence relation. In other words, they proved a two element Boolean algebra with the "same structure" as the original. This binary structure "represents" the original and allows all Boolean algebras to be simplified into the structure on {0,1}. In the next section we shall apply this representation to the matrix interpretations of classical logic, where we shall find that the family of Boolean algebras is characteristic of classical deducibility, but by means of the representation theorem these may all be simplified to the familiar classical matrix on {T,F}.

Definition 2. Filters and Ideals.[1]

Let **B**=$\langle B,\wedge,\vee,-,0,1\rangle$ be a Boolean algebra and $A \subseteq B$.

A is a ***filter*** on **B** iff

1. $\forall x,y \in B,\ x \in A \Rightarrow x \vee y \in A$, and
2. $\forall x,y \in B,\ x,y \in A \Rightarrow x \wedge y \in A$

(equivalently, iff $\forall x,y \in B,\ x,y \in A \Leftrightarrow x \wedge y \in A$).

A is an ***ideal*** on **B** iff

1. $\forall x,y \in B,\ x \in A \Rightarrow x \wedge y \in A$, and
2. $\forall x,y \in B,\ x,y \in A \Rightarrow x \vee y \in A$

(equivalently, iff $\forall x,y \in B,\ x,y \in A \Leftrightarrow x \vee y \in A$).

For any $x \in B$, by **[*x*]↑** we mean $\{y | x \leq y\}$ and by **[*x*]↓** we mean $\{y | y \leq x\}$

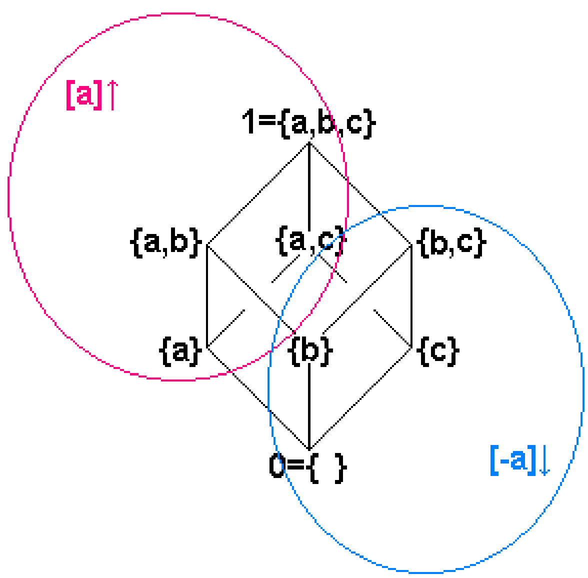
Theorem 3. For any Boolean algebra **B**=$\langle B,\wedge,\vee,-,0,1\rangle$a and any $x \in B$,

[*x*]↑ is a filter on **B**, and
[*x*]↓ is an ideal on **B**.

Definition 3. For any Boolean algebra **B**=$\langle B,\wedge,\vee,-,0,1\rangle$a and any $x \in B$,

[*x*]↑ is ***the prime*** (or ***principle***) ***filter on* B *relative to*** *x* and
[*x*]↓ is ***the prime*** (or ***principle***) ***ideal on* B *relative to*** *x*.

Example. The prime filter of a and the prime ideal of its complement –a={b,c}.



Theorem 4. For any Boolean algebra **B**=$\langle B,\wedge,\vee,-,0,1\rangle$, every filter/ideal of **B** is prime iff B is finite.

[1] A note on symbolism. We abbreviate the conjunction $x \in A$ & $y \in A$ as $x,y \in A$.

Definition 4. A filter/ideal of a Boolean algebra **B** is ***maximal*** iff

1. for some filter/ideal H, B⊂H, and
2. for any filter/ideal G, if there is a filter/ideal H such that G⊂H, then, if B⊆G, B=G (i.e. if G is a proper filter/ideal then B is not properly contained in it.)

Theorem 5. For any Boolean algebra **B**=<B,∧,∨,–,0,1>,

1. F is a maximal filter/ideal of **B** iff, d ∀*x*∈F, not(*x*∈F ⇔ –*x*∈F).
2. F is a maximal filter/ideal of **B** iff, B–F is a maximal ideal/filter of **B**
3. F is a maximal ideal of **B** iff, the function ϕ from B into its power set **P**(B) defined as follows: ∀*x*∈B,

   ϕ(*x*)=F if *x*∈F, and
   ϕ(*x*)=B–F if *x*∉F

   is a homomorphism from **B** onto the Boolean

   <{F,B–F},∩,∪,–,F,B–F>

Definition 5. Let <*X*,≤> be a partially ordered structure.

A ***chain*** in <*X*,≤> is any non-empty subset *Y* if *X* such that if *x,y*∈*Y* then *x*≤*y* or *y*≤*x*.

An ***upper bound*** of a chain *Y* is <*X*≤> is a member *x* of *X* such that for all *y*∈*Y*, *y*≤*x*.

An element *x* of is a ***maximal element*** of <*X*,≤> iff, for *x,y*∈*X*, *x*≤*y* ⇒ *x*=*y*

Axiom. (Zorn's Lemma, equivalent to the **Axiom of Choice)**
If every chain of a partially ordered structure <*X*,≤> has an upper bound, then <*X*,≤> has a maximal element.

Theorem 6. For any Boolean algebra **B**=<B,∧,∨,–,0,1>, any *x*∈B and any ideal H of **B** that does not contain *x,* there exists a maximal ideal M of **B** such that H⊆M and *x*∉M.

Theorem 7. For any Boolean algebra **B**=<B,∧,∨,–,0,1>, and any *x* and *y* of B, if not(*y*≤*x*), then there exists a maximal ideal M of **B** such that *x*∈M and *y*∉M.

Theorem 8. Every Boolean algebra is homomorphic to some two element Boolean algebra.

2. Boolean Interpretations of Classical Logic

Like any structure, a Boolean algebra if it has the same caharacter as a syntax may be used to fashion a logical matrix for the interpretation of the syntax. To do so we must specify in additiona a set of desiganted elements. Boolean algebras have the very nice property that the ordering relation within maximal filters replicate classical entailment. That is, if we sepcify a mximal filter as the set set of designated values, it will happen that when ever the premises of a classical valid argument are assigned values in the filter, the value assigned to the conclusion will also be in ther filter.

This replication, which is stated precisely in Theorem 9, is the semantic foundation that underlies the fact that Boolean algebras (with maximal filters as designated) are characteristic of classical deducibility. This "characterization" is spelled out in a soundness and completeness theorem. One appraoch is to adapt the Henkin compleness proof for sentential logic, which is familiar from elementary logic. The proof divides into one for soundness and one for completeness. The completeness proof remains unchanged, because M_C is itself a Boolean algebra, and hence the proof that any maximally consistent set is satifiable in an M_C-valuation automatically establishes that it is satisfiable in a Boolean valuation (with {1} as the maximal filter of designated elements.)

The proof of the soundness theorem needs to be adapted to Boolean algebras but the structure of the proof remains the same and the steps are just as straightforward as they are it is in the case of M_C. Soundness, recall, is estabilished by an induction that shows every provable deduction is valid. First every basic deduction is shown to be valid, Then, assuming (the induction hypothesis) that the arguments for a derivation rule are valid, it is shown that the value for the rule is valid. By the Boolean replication of $\models$ by $\leq$, these facts about validity translate into facts about $\leq$ in the Boolean structure, and going from $\leq$-facts about the inputs of a derivation rule to the relevant $\leq$-facts about the output becomes an exercise in applying the properties of the Boolean operations.

We are now ready for the definitions and theorems.

Definition 6. Boolean Matrices and Sentential Languages.

In this section we shall let $\mathcal{S}yn=<\mathcal{S}en,f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow}>$ range over $\mathcal{SL}$ syntaxes .

By a ***Boolean matrix*** we mean any $M=<B,F,\wedge,\vee,-,0,1>$ such that

1. $<B,\wedge,\vee,-,0,1>$ is a Boolean albegra, and
2. F is a maximal filter on $<B,\wedge,\vee,-,0,1>$.

The set of all Boolean matrices is $\mathcal{BM}$.

By a ***Boolean*** (***sentential***) ***language*** is meant $<\mathcal{S}yn,\mathcal{BM}>$ for any sentential syntax $\mathcal{S}yn$.

By the ***classical matrix*** M_C we mean the Boolean matrix $<\{0,1\},\{1\},\wedge,\vee,-,0,1>$ in which the operations are defined as follows:

	~	∧ T	∧ F	∨ T	∨ F	→ T	→ F
T	F	T	F	T	T	T	F
F	T	F	F	T	F	T	T

We shall continue to use $\vdash_C$ to refer to the natrual deduction deducibility relation relation for classical logic defined in Lecture 2.

If Y is some finite subset $\{y_1,...,y_n\}$ of B, then we shall use $\bigwedge\{f(x)\}_{x\in Y}$ to mean $f(y_1)\wedge...\wedge f(y_n)$

Theorem 9. In any Boolean matrix M, $X\models_M P$ iff $\bigwedge\{\mathrm{v}(Q)\}_{Q\in X}\leq \mathrm{v}(P)$

Theorem 10. If $\mathcal{L}$ be a Boolean sentential langauge, then $X\vdash_C P$ iff $X\models_{\mathcal{L}} P$.

(Proof is as sketched above)

The ordinary completeness proof which states that classical deducibility is characterized by entailment over the two-valued matrix M_C is then a corollary of the this Boolean characterization theorem plus the bivalent representation theorem.

Theorem 11. If $\mathcal{L}$ be a Boolean sentential langauge, then $X \vdash_C P$ iff $X \models_{M_C} P$.

Of course, there is a clear sense in which Theorem 11 is stronger than Theorem 10, and the stremlined Henkin completeness proof that estabilishes completeness for interpretations over just M_C is a more direct route to it. The weaker interpretaion, however, is interesting for two reasons. The first is conceptual, to which we now turn.

3. Frege's Intensional Semantics

There has long been a tradition in logic and philosophy that logic and the "propositions" it expresses are not entities that exist in the common and garden material world but rather have a special status as intensional entities. Aristotle spoke about genera and species which are not the same as sets, and are indeed antitonic to them. Conceptually the genus G and differential D are "contained" in the species because they are used to define it: if we use the symbol + to indicate the process of "conceptual addition" operation, then we might symbolize the relationship as S=D+G, and hence G≤S. But the extensions of genera and species fall in a reverse ordering: Ext(S)=Ext(D)∩Ext(G), and hence S⊆G.[2]

This Aristotelian tradition lasted through the Middle Ages. The rationalists too spoke of logical truths as describing conceptual inclusion. But it is Frege's use in the nineteenth century of intensional entities as part of an informal semantics of indirect statements (*S believes that P*) that is the inspiration for the study of intensions in modern logic.

Belief statements have the distinctive logical property that substitution of material equivalents and identities within the belief clause is invalid.
It is invalid to infer (3) from (1) and (2). (The example is Russell's)

(1) *George III believes Scott is Scott.*
(2) *Scott is the author of Waverley.*
(3) *George III believes Scott is the author of Waverley.*

Likewise {*S believes P, P↔Q*} does not entail *S believes P*. What is the explanation of this failure? The answer can be put in algebraic terms.

We know that substitutivity of material equivalents is a manifestation of homomorphic structure. In more traditional semantic terms this property is called the **compositionality of extension** or **reference**.

Principle 1. The Compositionality of Reference. The referent of a whole expression is determined in a rule-like way from the referents of its parts.

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[2] A patially ordered structure <*X*,≤> is ***antitonic*** relative to ϕ to a partially ordered structure <*Y*,≤′> (and ϕ is an ***antitone mapping*** from the first to the second) iff for any $x,y \in X$, $x \leq y$ iff $\phi(y) \leq \phi(x)$.

Algebraically, the principle asserts that there is a function *g* on "referents" that corresponds to a grammatical operation *f*, and that the reference relation Ext is a homomorphism mapping expressions to referents: $\text{Ext}(f(e_1,...,e_n)) = g(\text{Ext}(x_1),\ldots,\text{Ext}(x_n))$.[3] Fore example, if *f* is a grammatical operation that generates sentences then the referent of the sentences would be a truth value, i.e. $\text{Ext}(f(e_1,...,e_n)) \in \{T,F\}$. The compositionality of reference then would require that *g* would be a function mapping the semantic values of $\text{Ext}(e_1),...,\text{Ext}(e_n)$ to that very truth-value.

Belief statements, however, appear to be a counter-example to the principle. Let $f(a,P)=Bel_aP$. (We read Bel_aP as "a believes that *P*".) Let Ext(*a*) be an object in the domain and Ext(*P*) be a truth-value. Let *g* be the semantic "rule" corresponding to *f*. Then it is easy to show, it seems, that *g* is not a function.

On the one hand:

T	=	Ext($Bel_{Georege\ III}$*Scott is Scott*)
	=	*g*(Ext(*George III*),Ext(*Scott is Scot*))
	=	*g*(Ext(*George III*),T)

But on the other hand:

F	=	Ext($Bel_{Georege\ III}$*Scott is the author of Waverley*)
	=	*g*(Ext(*George III*),Ext(*Scott is the author of Waverley*))
	=	*g*(Ext(*George III*),T)

The semantic "rule" *g* for belief sentences is a relation not a function.

Frege's actually tries to avoid this conclusion and proposes an ingenious analysis that preserves functional compositionality of reference. He suggests that words, simple and complex, have intensions (he calls them *senses* but we might also call them "meanings") as well as extensions (which he calls referents). Senses too obey a rule of compositionality.

Principle 2. The Compositionality of Intensions. The intension of the whole expression is determined in a rule like manner from that of its parts.

Algebraically, Frege is postulating a structure of intensions homomorphic to grammatical structure. The ordinarily language terms we employ to name this homomorphism is the verb "express." We say a term or sentence "expresses" an idea or thought. Frege suggests that such locutions are informal ways of indicating the important semantic that holds between expressions and their intensions.

The algebras of intensions and extension, moreover, are related. Senses, Frege says, determine referents.

[3] The notation Ext for the reference function, and Int (below) for assigments of intensions is due to Richard Montague. Rudolf Carnap is responsiblle for fixing the tersm of art *extension* and *intension* to these entities.

Principle 3. Sense Determines Reference. The extension of an expression is determined in a rule-like way from its intension.

Algebraically, the principle says that there is a homomorphism from intensional structure to extensional structure. There is no traditional name for this mapping, though some philosophers of language have called it the "reality function." Here we shall merely call it ρ.

Frege now applies these principles to explain the logical workings of belief. His idea briefly is that words which occur within the scope of the *belief*-predicate (or verb or operator or whatever it should be called) do not in those occurrences have there usual referents. Rather they refer to the intension that they usually "express." Terms in Frege's view, then, are systematically ambiguous. Most of the time, outside the scope of verbs like *believes*, they stand for their normal referent. One they are understood this way, they do not violate the three basic principles of compositional semantics.

The details of Frege's theory, however, we shall postpone to the next lecture -- they are actually rather controversial. In the remaining part of this lecture, we shall focus on developing these ideas for the more ordinary example of sentential logic.

One of the lovely properties of Boolean interpretations of sentential logic is that they may be used to provide a detailed mathematical theory of the operations of extensions and intensions. Extensions are organized in the usual structure of truth-values, i.e. the classical matrix M_C. Intensional structure, however, is one that organizes the intensions of sentences. In modern logic these are usually called propositions. (Frege called them thoughts as well. In traditional logic a proposition is a sentence.) But what are propositions?

One interpretation is in terms of "possible worlds." A sentence's truth limits what is possible to those situations in which it is true. A more detailed, informative, meaningful sentence -- these terms are roughly synonymous -- restricts possibilities more than an less detailed, less informative, or less meaningful sentence. The "set of worlds in which a sentence *P* is true" is roughly the "information content" of *P*, and one such set is characteristic of each "proposition." Indeed, modern semantics employs just such world-sets as proxies for informal "senses." The world-sets may be put into structures and these structures made to exhibit all the structural features intuitively attributed to propositions. Moreover, to an algebrist if two sorts of entities exhibit exactly the same structure they are essentially identical. Naïve "sense" are reduced to or replaced by mathematically constructed proxies.

The intensional structures of world-sets appropriate for sentential logic is the Boolean algebra that has a universe consisting of sets of worlds called *propositions*) and which relates these sets by the Boolean operations on sets.

Let us now state the formal version of Frege's sentential semantics. We will postulate a set of possible worlds, traditionally called K. K will be inhabited by propositions. These propositions will be world-sets, and are intuitively the "information content" of some sentence. Since propositions are world-sets, they are subsets of K. The universe of the intensional structure, therefore, is inhabited by subsets of K, and "the universe of intensional structure" is identical to the set of all of K's subsets. This set is called *the power set* of K. Since propositions are sets, the operations in the structure organaizing them may be seen as set theoretic intersections, union, and complimentation. In addition there is a special operation ⇒ used to interpret the conditional, and defined in terms of complementation and union. An intensional interpretations that assign a proposition to each sentence is then simply a valuation (homorphism) from syntax to intensional structure. With the proposition that *P*, symbolized Int(*P*), in hand, it is possible to find *P*'s extension (truth-value) in a world: *P* is true in *k* iff $k \in$Int(*P*). The reality function, called ρ_k, that assigns to each *P* in *k* a truth-value is then easily defined: (Int(*P*))=T if $k \in$Int(*P*) and ρ_k(Int(*P*))=F if $k \notin$Int(*P*).

Frege's Semantics of Intensional and Extensional Structures for Classical Sentential Logic.

Definition 7. An ***intensional structure for sentential logic*** relative to a set K (called the set of possible worlds) is <**P**(K),$\cap$,$\cup$,$\Rightarrow$,–,$\varnothing$,K> such that **P**(K) (the power set of K) is the set of all subsets of K, and $\cap$,$\cup$,– are the standard set theoretic operations on **P**(K) (here $x \Rightarrow y =_{def} -x \vee y$) and $\varnothing$ is the empty set.

Definition 8. If M is an intensional structure, then $\mathtt{Val}_M$ relative to a sentential syntax *Syn* is called the set of ***intensional interpretations*** of *Syn*. We let **Int** range over this set.

Theorem. (Principle 2. Intensions are Compositional). Any intentional interpretation Int of a sentential syntax *Syn* relative to an intensional structure I is a homomorphism from the syntax to the intensional structure.

Definition 9. We shall call ***the reality function*** relative to an intentional interpretation Int (over an intensional structure I) and a possible world *k* of I that function ρ_k from Kx*Sen* to {T,F} such that

ρ_k(Int(*P*))=T if $k \in$Int(*P*)

ρ_k(Int(*P*))=F if $k \notin$Int(*P*)

Here, $\rho_k(\mathtt{v}(P))$ is a more perspicuous symbolism for $\rho k, \mathtt{v}(P))$. If we use $[\mathrm{Int}(P)]^{\nabla}$ to indicate the characteristic function of Int(*P*), i.e. the function that maps *k* to T if $k \in$Int(*P*) and *k* to F if $k \notin$Int(*P*), then ρ_k may be alternatively defined as:

$\rho_k(\mathrm{Int}(P))$=T if $[\mathrm{Int}(P)]^{\nabla}(k)$=F

$\rho_k(\mathrm{Int}(P))$=F if $[\mathrm{Int}(P)]^{\nabla}(k)$=F

(As we shall see, for some purposes it is even more convenient to think of this characteristic function as the proposition Int(*P*) intself.)

Theorem 12. (Principle 3. Intension Determines Extension.) $\mathtt{Val}_M$ = {Int$\circ\rho_k$ | Int is an intentional interpretation on an intensional interpretation I of *Syn*, *k* is a possible world in K of I, and ρ_k is the reality function relative to Int.}

Proof Analysis. The proof requires establishing that any function $\mathtt{v}$ on *Sen* defined as

$\mathtt{v}(P)) = \rho_k(\mathrm{Int}(P))$

qualifies for membership in the set of classical valuations $\mathtt{Val}_M$C over M_C. This is done by showing that it assigns the right truth-values to atomic sentences and then assigns truth-values to molecular sentences in a manner that conforms to the classical truth-tables.

Second, it must be shown conversely that if $\mathtt{v} \in \mathtt{Val}_M$C then there is some Int$\circ\rho$ and *k* of K such that $\mathtt{v}$= Int$\circ\rho_k$, i.e. for any *P*, $\mathtt{v}(P)) = \rho_k(\mathrm{Int}(P))$. Select that Int that assigns to each atomic *P* a set containing the world some world *k* iff $\mathtt{v}(P)$)=T. It will then follow (by induction) that for all *Q* (atomic and complex), that

$\rho(k, \mathtt{v}(Q))$=T if $k \in$Int(*Q*)).

Theorem 13. (Principle 1. Extensions are Compositional.) Any $\mathtt{v}$ in $\mathtt{Val}_M$ = {Int$\circ\rho_k$ | Int is an intentional interpretation on an intensional interpretation I of *Syn*, *k* is a possible world in K of I, and ρ_k is the reality function relative to Int.} is a homoprphism from the syntactic structure *Syn* to M_C.

We shall finish by noting in addition that in this semantics one could interpret logical inference intensionally. Classical logic is the entailment relation determined by the class of Boolean matrices determined by intensional structures in which a maximal filter is selected as distinguished elements. Likewise, according to Theorem 9 above, the ordering relation $\subseteq$ on propositions replicates entailment. That is, entailment is a kinnd of conecptual inclusion.

Definition 10. An ***intensional matrix for sentential logic*** (in the class **IMSL**) is any Boolean matrix <**P**(K),F,∩.∪.⇒, –,∅,K> relative to the Boolean structure <**P**(K),∩,∪,⇒,–,∅,K>.

Theorem14 . If $\mathcal{L}$=< *Syn*, **IMSL** is a Boolean sentential language, then $X \vdash_C P$ iff $X \models_{IMSL} P$.

Theorem 15. For any intensional matrix M and any Int in Val_M, $X \models_M P$ iff $\bigwedge\{Int(Q)\}_{Q \in X} \subseteq Int(P)$.

Corollary 16. For any intensional matrix M and any Int in Val_M,

$$\bigwedge\{Int(Q)\}_{Q \in X} \subseteq Int(P) \text{ iff } X \models_C P \text{ iff } X \vdash_C P$$

The set of possible world structures can be narrowed to a special one equally characteristic of classical entailment. This is the structure in which the worlds are themselves classical valuations over the bivalent matrix M_C. Indeed, classical valuations are "worlds" in the senset that they record a story: the set of sentences true in that world. For sentential logic, in other words, classical valuations themselves may serve adequately as the only notion needed for a Fregean intensional semantics .

Definition 11. The classical valuational structue for a sentential syntax *Syn* is

I_C=<P(Val_C),∩,∪,⇒,∅, Val_C>

Let **B M I_C** be the set of all Boolean matricies relative to I_C.

Theorems 17. For any M∈B M I_C, any Int∈Val_M, and any v∈ Int(*P*), $v(P)=\Box_v(P)$

Theorem 18. Let $\mathcal{L}$=<*Syn*, B M I_C> for a sentential syntax *Syn*. For all M∈ B M I_C and any Int∈Val_M,

$$\wedge\{Int(Q)\}_{Q \in X} \subseteq Int(P) \text{ iff } X \models_C P$$

There exists in addition a special family of I_C interpretaions. This family alone is charcteristic of classical entailment and is so independently of the choice of designated values. That is, these interpretations differ only in their choice of designated values, but each alone is equally characteristic of classical entailment and is so in a manner that does not depend on its choice of desigated values. The extra step then of introducing the matricies with its designated values in addition to I_C is for these interpretations is unnecessay. That is, we might simply identitfy them or more precisely defined a notion of intensional interpretaion directly from I_C that omits mention of designated values all together. Classsical entailment then proves to be simple set inclusion over valuations. This special interpretation, which we shall call Int_C, is that in which $Int_C(P)$ is the truth-set of *P* in classical bivalent semantics, i.e. $Val_MC(P)$.

Theorem. 14 Let Int be an intentional interpretaion of I_C relative to some M in B M I_C such that

$$Int(P) = \{ v \in Val_M C(P) \mid v(P)=T \}$$

Then,

$$\wedge\{Int(Q)\}_{Q\in X} \subseteq Int(P) \text{ iff } X \models_C P$$

Definition 12. Let ***the preferred classical interpretation*** of a sentential syntax $\mathcal{Syn}$ be that homomorphism from $\mathcal{Syn}$ to I_C, which we shall call $\mathbf{Int_C}$, such that $Int_C(P) = \{v \in Val_M C(P) \mid v(P)=T \}$.

Theorem 15. $\wedge\{Int_C(Q)\}_{Q\in X} \subseteq Int_C(P)$ iff $X \models_C P$

Exercises

1. Prove Theorem 3.
2. Prove Theorem 9.
3. Prove Theorem 13.
4. Prove Theorem 15.

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Lecture 4

Intensional Logic

1. The Idea of Intension in the History of Philosophy and Logic

In the last lecture we met briefly for the first time the main subject of this course, the concept of meaning as studied in modern logic. We saw that it was an idea introduced by Frege to explain the logic of sentences constructed from propositional attitude verbs like *believe*. Understanding what sort of problem Frege identified and what sort of theory he offers as a solution goes a long way towards explaining the dominant way in which logicians have conceived of the concept of meaning.

Meanings for Frege and the tradition that follows him are explanatory entities introduced as part of a "science" designed to explain some "data." The date in question are facts about particular logical inferences. The explanation is a general theory of inference. The theory incorporates various "laws" describing the behavior of meanings and these laws together with other parts of the theory entail the observed data.

The general shape of the over-all theory of inference is the familiar one of modern formal semantics. Though this sort of theory was only vaguely suggested in Frege's own writings it has since become standard. Validity is conceived of as some sort of truth-preserving relation among sentences in a formal syntax. The goal of the theory is to define this relation. The standard approach is to first define the notion of truth and to define truth as the correspondence of sentences with "the world." In the process the theory must met certain standards of adequacy.

Prominent among these standards is that the theory be mathematically rigorous. In practice this means that all its ideas must be well defined, and its assertions proved. The background theory usually assumed to get the process off the ground is set theory. In principle this should be some version of axiomatic set theory from which the paradoxes have be expunged, but in practice theorists use the naïve version (with an unrestricted axiom of comprehension) on the understanding that its results are "modulo axiomatization." That is, its results should properly be read as they would be written in an axiomatic version. (Proposition referring to paradoxical sets should be read as referring to "classes" or reformulated in favor of the open sentences that would naively define the set.)

A second criterion, which is responsible for much of the interest philosophers have in the theory, is that the definitions it offers of key concepts be conceptually plausible. Central among these are the concepts of "world," correspondence," and "truth." Frege's theory in addition employs "meanings" above and beyond standard entities in the world. The philosopher's ears immediate prick up. These are ideas they have been puzzling about for millennia. Any global "scientific" theory in which they play a role they find interesting indeed. A central concern in the theory is then that these key definitions conform, in the rough and ready way scientific terms always do, to previous usage, both in ordinary language and in earlier intellectual traditions.

Meanings are particularly intriguing. Philosophers have "postulated" queer entities above and beyond the common sense denizens of "the world" in order to explain the unknown. Since (at least) Plato they have done so to explain linguistic truth. Often the same entity has served multiple purposes. Plato's Forms are used in explanations in ontology, epistemology, ethics and semantics. So are Aristotle's genera and species and the universals of the Middle Ages. The ideas of the rationalists and empiricists are likewise put to various uses including semantics.

To see more clearly the link of Frege's idea of meaning to these earlier theories is necessary to be clear about the exact problems both Frege and the earlier theories were trying to solve, about the properties of the entities used to solve them.

Frege's problems and entities take a new direction. One tradition departs in a major way from Frege in that is conceives of logic as something mental and non-linguistic. The rationalists and Kant think of inference as conceptual inclusion or instances of mental laws. Even this tradition however is related to Frege. Frege is investigating the semantics of belief-sentences. These are sentences that describe "mental states." The so-called intentionalist tradition in logic is interested in part in these same mental states. Aristotle and the medievals described the mind (*animus*) as containing thoughts (*ratio*) that contained a mental content (*intentio*). The content determines the qualities ascribed to the thought's object but in such a way that the though itself does not have these objects. The ideas of the rationalists and empiricists are similar. Brentano refers to the special features of such entities *the intentional* and calls it the mark of the mental. Frege's meanings are the content of beliefs but in a linguistic fashion. They are the entities needed to be hypothesized (as referents of indirect statements in Frege's original theory) in order to explain the truth-conditions of belief-statements. Indeed he sometimes calls the senses of sentences *thoughts*. In is not a mistake then to see Frege's account as a recasting in linguistic terms of the intentionalist tradition. It is largely for this reason that Carnap coins the term *intension* (with an *s*) to refer to Frege's sentence meanings. The change in spelling indicates that the idea is recast into a new context, that of semantic theory.

It must be stressed that in taking this "linguistic turn" Frege imports a host of considerations not present in the intentionalist tradition.

One such concern is an explanation of the public inter-personal nature of linguistic communication. Frege's meanings, he says, are public in the sense that everybody understands the same meaning for the same sentence. Hence they are not part of an individual's mind. Not every intentionalist thinks intentions are private. In the Platonic tradition, for example, our mutual understanding consist of us both having a metal apprehension of one and the same "public" Idea, which is held to exist outside our minds in some public place like Platonic Heaven or God's soul. But many intentionalists are not concerted with language and conceive of intentions as parts of an individuals mind.

A second characteristic of Frege's linguistic approach is its deep link to explaining logical inference conceived of as a relation among sentences. He falls into an older tradition that includes Plato and Aristotle, the stoics, and (most) medieval logicians. The general approach has several key features:

1. views the syntactic form as determining a arguments validity,
2. its validity is conceived of as a truth-preserving relation among sentences, and
3. truth is defined as correspondence with the world.

Within the tradition individual theories differ depending on what semantic phenomenon they are tying to explain. The general strategy is see if the problem may be solved by postulating some semantic entities with special properties tied to reference. Some of these entities are quite like Frege's and some not. Universals, for example (as in the semantics of Plato's that appeals to forms or of Aristotle's that employs secondary substances and qualities), are used to explain what predicates refer to, not as in Frege to explain inferences about belief-statements. (However, as mentioned above, Plato's ideas and Aristotle's impressions of forms on the soul are used as the objects of knowledge and belief states.)

Closer to Frege are the *lecta* (literately *the-what-is-read* or *the-what-is-meant*) which the stoics postulated as the "meanings" of sentences. These lecta have parts, and the lecton of the whole sentence appears to be a function of that of its parts. These complexes with a structure that mirrors that of syntax are also the objects of knowledge and belief. A similar view was arrived at independently in the Middle Ages by Ockham who posited a level of "mental language" between spoken language and the objects words stand for. Much like Frege, Ockham explains the reference of words as working through intermediate steps: words are paired by convention with "concepts" (terms of mental language) and concepts in turn naturally determine a referent in the world, Such mental language is also the object of knowledge.

Though neither the stoics nor Ockham develop a complete theory of inference, their accounts do share an important feature that is lacking in the intentions of the rationalists and empiricists: they posit the three levels of parallel homomorphic structure.

From this introduction it is possible understand the motivation for the algebraic approach to taken in these lectures. The algebra at once exhibits the mathematical rigor required of a formal theory and does so in a conceptually perspicuous way: it displays with clarity why validity as a truth-preserving relation and how truth as correspondence to the world fall out of a more general theory of language functions as part of a three level combination of homomorphic structures.

2. Modal Operators and Cross-World Structure.

Frege observed that we cannot know from the extension of a sentence what the extensions of a belief sentence will be in which the sentence functions as an indirect statement. This general failure of "extensionality" (marked by the invalidity of the substitutivity of co-extensional parts) is a feature of other verbs that take indirect statements as complements *want, desire, hope, intend*) and of various sentential adverbs (*necessarily, possibly,*)

In this section a number of examples of intensional languages will be developed using the ideas of Carnap and Richard Montague. Montague succeed in capturing the algebraic properties of intensions by set theoretic proxies that are essential functions from possible worlds to the extensions of expressions in those worlds. The resulting theory is extremely elegant and quite abstract. Its abstractness moreover allows it to characterize the structural properties of "meanings" without making any claim about what sort of entities they might be proxies for in linguistic reality. Montague semantics, for example, has been embraced by those who think intensions are literally mental entities (parts of the brain), those who think they are abstract like mathematical entities, and those who think they are essential social phenomena.

Abstract Characterization of Fregean Intensional Semantics

Le us adopt the following set theoretic notation. If A and B are sets, then by A^B is meant the set of all functions from B into A. By 2 we mean the set of classical truth-values {T,F}={0,1}.
Let adopt the following syntactic conventions. Let Syn = $<A_1,...,A_m,f_1,...,f_n>$ be a syntax such that for some A_i =Sen. Let us use E_{Syn}, called the set of (***well-formed***) ***expressions*** of Syn, to stand for $\cup\{A_1,...,A_m\}$, and let e range over E_{Syn}. We shall let Syn range over syntaxes $<A_1,...,A_m,f_1,...,f_n>$.

Definition 1. By a ***Fregean intensional structure*** is meant a matrix interoperation $\mathcal{I}$ of Syn. That is $\mathcal{I}$=$<B_1,...,B_{m+1},h_1,...,h_n>$ such that $<B_1,...,B_m,h_1,...,h_n>$ off like character to Syn such that each h_i is a function.

Here B_{m+1} is the intended set of designated values used to define validity and each B_i is the set of possible "intensions" for expressions of category A_i. We let $\mathcal{I}$=$<B_1,...,B_{m+1},h_1,...,h_n>$ range over such structures.

Definition 2 . If $\mathcal{I}$ is an intensional structure relative syntax Syn, then its set of matrix valuations $\mathbf{Val}_{\mathcal{I}}$ is called the set of ***Fregean intensional interpretations*** of Syn. We let Int range over this set.

Theorem 1. (Compositionality of Intension: Intensions of Parts Determine Intension of the Whole). Any Int of a syntax Syn relative to $\mathcal{I}$ is a matrix homomorphism from the syntax to the intensional structure, and $\equiv_{Int}$ is an equivalence relation with the substitution property.

(In bivalent languages an interpretation is a homomorphism in the unqualified sense from Syn to $\mathcal{I}$.)

Definition 3. Let K be a non-empty set, called a set of ***possible worlds***, and let k range over K. Then, by a ***reality function relative to*** $\mathcal{I}$**, Int,** ***and*** **K** we mean any function on domain E_{Syn}xK. We let θ range over the reality functions relative to $\mathcal{I}$ and Int, and abbreviate $\theta(e,k)$ by $\theta_k(e)$.

Definition 4. By ***the extensional interpretation relative to*** $\mathcal{I}$, **Int, K** ***and*** $\boldsymbol{\theta}$ is meant that function Ext on domain E_{Syn}xK defined as follows:

$\text{Ext}_k(e) = \theta_k(\text{Int}(e))$.

We let Ext stand for the extensional interpretation relative to I, Int, K, *and* θ, and let θ_k and Ext_k stand respectively for θ and Ext relativized to *k*, i.e.

θ_k is that function *f* on E_{Syn} such that $f(e)=\theta_k(e)$.

Ext_k is that function *g* on E_{Syn} such that $g(e)=\text{Ext}_k(e)$.

Definition 5. By a ***Fregean (intensional) language*** is meant any matrix language <*Syn*,*F*> such that for each Fregean intensional structure (matrix) I of *F*, there is some non-empty set K (called ***the set of possible worlds of*** I) and some reality function θ (called ***the reality function of*** I) such that any intensional interpretation (i.e. matrix valuation) Int in $\mathbf{Val}_I$ is defined relative to I and K, and θ is defined relative to I, Int, and K.

In the remained of this lecture we shall let <*Syn*,*F*> range over Fregean intensional languages.

Theorem 2 (Sense Determines Reference). For any <*Syn*,*F*>, any I∈*F* with possible world set K and reality function θ, $\text{Ext}_k = \text{Int} \circ \theta_k$.

Definition 6. By ***the Fregean extensional structure relative to*** <*Syn*,*F*> such that ***and to*** I∈*F* ***with possible world set*** K ***and reality function*** θ is meant $E=<C_1,...,C_{m+1},R_1,...,R_n>$, such that $C_i=\{ \text{Ext}_k(e) | k\in K \text{ and } e\in A_i\}$, and $R_j=\{<\text{Ext}_k(e_1),…, \text{Ext}_k(e_n), \text{Ext}_k(e_{n+1})> | f_j(e_1,…,e_n)=e_{n+1}\}$. Let $E=<C_1,...,C_{m+1},R_1,...,R_n>$ range over the extensional structures relative to <*Syn*,*F*>, I, K, and θ.

Definition 7.

1. If $e=f_i(e_1,…,e_n)$ and there is some extensional structure of <*Syn*,*F*> such that is not a function then any occurrence of an expression within *e* is said to occur in an ***intensional context*** and *e* is said to be ***non-extensional*** and ***opaque***.
2. $E=<C_1,...,C_{m+1},R_1,...,R_n>$ (relative to <*Syn*,*F*>, I, K, and θ) is said to be ***extensional*** iff each R_j is a function.
3. I (relative to <*Syn*,*F*> K, and θ) is ***extensional*** iff each *E* relative to <*Syn*,*F*>, I, K, and θ is extensional.
4. The language <*Syn*,*F*> is ***extensional*** iff for any K and θ, and for any I∈*F* defined relative to K and θ , I is extensional.

Theorem 3. The following are equivalent:

1. $E=<C_1,...,C_{m+1},R_1,...,R_n>$ relative to <*Syn*,*F*>, I, K, and θ is extensional.
2. $E=<C_1,...,C_{m+1},R_1,...,R_n>$ is a logical matrix and $\mathbf{Val}_E$ for *Syn* is $\{ \text{Ext}_k | k\in K \}$.
3. For all k∈K, Ext_k is a matrix homomorphism from *Syn* to *E*.

Theorem 4. (Compositionality of Extension: Extension of Parts Determines Extension of the Whole). In an extensional Fregean language, every extensional interpretation Ext_k, for any $k\in K$, is a matrix homomorphism from *Syn* to *E*, and $\equiv_{\text{Ext}_k}$ is an equivalence relation with the substitution property. (When $\equiv_{\text{Ext}_k}$ is restricted to *Sen* it is called material equivalence.)

Montague's Set Theoretic Characterization of Fregean Intensions

Definition 8. By a ***Montague structure relative*** **K** is meant a Fregean intensional structure interpretation I of *Syn* relative to a world structure such that $I=<B^K_1,...,B^K_{m+1},h_1,...,h_n>$.

Here $B_{m+1}{}^{K}$ is the intended set of designated values used to define validity, and each $B_i{}^{K}$ is the set of possible "intensions" for expressions of category A_i. We let I_M range over such structures. Intuitively, K is a set of possible worlds, and we shall call its power set **P**(K), the set of ***propositions***, and let π, σ, τ range over **P**(K). We shall call the set $\{0,1\}^K$ of characteristic functions of propositions the set of ***sentential intensions***. If $f \in \{0,1\}^K$ is a characteristic function, let us use π_f to name the proposition of which it is the characteristic function. Conversely, if π is a proposition let $\pi^{\mathbf{c}}$ be its characteristic function. We let $\pi^{\mathbf{c}}, \sigma^{\mathbf{c}}, \tau^{\mathbf{c}}$ range over $\{0,1\}^K$.

Classical Logic and Classical Sentential Operators

Syntax: functions $f_{\sim}, f_{\wedge}, f_{\vee}, f_{\rightarrow}$ on signs previously defined.

Intensional Semantics. Let π and σ range over **P**(K). Relative to a non-empty set K (of ***possible worlds***) $g_{\sim}$ is the 1-place function on $\{0,1\}^K$, and $g_{\wedge}, g_{\vee}, g_{\rightarrow}$ are the 2-place functions on $\{0,1\}^K \mathrm{x} \{0,1\}^K$ such that

$$g_{\sim}(\pi^{\mathbf{c}}) = (-\pi)^{\mathbf{c}}$$
$$g_{\wedge}(\pi^{\mathbf{c}},\sigma^{\mathbf{c}}) = (\pi \cap \sigma)^{\mathbf{c}}$$
$$g_{\vee}(\pi^{\mathbf{c}},\sigma^{\mathbf{c}}) = (\pi \cup \sigma)^{\mathbf{c}}$$
$$g_{\rightarrow}(\pi^{\mathbf{c}},\sigma^{\mathbf{c}}) = (\pi \Rightarrow \sigma)^{\mathbf{c}}$$

Definition 9. L=<SL,F> is said to be ***classical for sentential Montague logic*** iff ***SL***=<$Sen, f_{\sim}, f_{\wedge}, f_{\vee}, f_{\rightarrow}$> is a sentential syntax and F is the set of all logical matrices M such that for some non-empty set K, M=<2^K,$\{K^{\mathbf{c}}\}$,$g_{\sim}, g_{\wedge}, g_{\vee}, g_{\rightarrow}$>.

Theorem 5. ***SL*** is extensional, and $X \models_{SL} P$ iff $X \models_{C} P$

(Alethic) Modal Operators

Syntax. f and $f_{\Diamond}$ are defined as the 1-place operations on signs such that $f_{\Box}(P)=\Box P$ and $f_{\Diamond}(P)=\Diamond P$.

Intensional Sematics. W=<K,≤> such that K is non-empty and ≤ is a binary relation on K(***the alternativeness relation***) is said to be

an **M** ***world structure*** iff ≤ on K is reflexive;
a **B** ***world structure*** iff ≤ on K is reflexive and symmetric;
a **S4** ***world structure*** iff ≤ on K is reflexive and transitive;
a **S5** ***world structure*** iff ≤ on K is reflexive, symmetric, and transitive.

Relative to a worlds structure W=<K,≤>, $g_{\Box}$ and $g_{\Diamond}$ are defined as 1-place operations on $\{0,1\}^K$ into $\{0,1\}^K$ such that

$g_{\Box}(\pi^{\mathbf{c}})(k)$=T if for all k' such that $k \leq k'$, $\pi^{\mathbf{c}}(k')$=T, and
$g_{\Box}(\pi^{\mathbf{c}})(k)$=F if for some k' such that $k \leq k'$, $\pi^{\mathbf{c}}(k') \neq$T.
$g_{\Diamond}(\pi^{\mathbf{c}})(k)$=T if for some k' such that $k \leq k'$, $\pi^{\mathbf{c}}(k')$=T, and
$g_{\Diamond}(\pi^{\mathbf{c}})(k)$=F if for all k' such that $k \leq k'$, $\pi^{\mathbf{c}}(k') \neq$T.

Definition 10. L=<Syn,F> is said to be a **M, B, S4,** or **S5** ***sentential modal (Montague) language*** respectively iff Syn=<$Sen, f_{\sim}, f_{\Box}, f_{\Diamond}, f_{\wedge}, f_{\vee}, f_{\rightarrow}$> and F is the set of all logical matrices M such that for some M, B, S4, or S5 world structure W=<K,≤> respectively, M=<2^K,$\{K^{\mathbf{c}}\}$,$g_{\sim}, g_{\Box}, g_{\Diamond}, g_{\wedge}, g_{\vee}, g_{\rightarrow}$>. (We shall use the letters M, B, S4, and S5 to range over such languages.

Theorem 5. If $L \in$ {M,B,S4,S5}, then L is non-extensional,

Theorem 6. $\Box P \dashv\models_L \sim\Diamond\sim P$, and $\Diamond P \dashv\models_L \sim\Box\sim P$.

Theorem 7. If $L \in$ {M,B,S4,S5}, then $P \not\models_L \Box P$ and $\Box P \models_L P$.
If $L \in$ {M,B,S4}, then $\Box(P \rightarrow Q) \models_L \Box P \rightarrow \Box Q$.
If $L \in$ {B,S5}, then $\Box P \models_L \Box\Diamond P$.
If $L \in$ {S4,S5}, then $\Box P \models_L \Box\Box P$.

Definition 11. If □ and ◊ produce valid arguments as stipulated in the consequences of the next to last previous theorem in the pattern appropriate, they are called ***duals.*** Likewise if a sentential operator *E* is replace by ☐ and *E′* by ◊ with the result that the replacements are duals then so are *E* and *E′*.

Definition 12. If ☐ and ◊ produce valid arguments as stipulated in the consequences of the previous theorem in the pattern appropriate to the languages M, B, S4,or S5, then ☐ and ◊ are called **M, B, S4,**or **S5** ***operators*** respectively. Likewise if a sentential operator *E* is replace by ☐ and *E′* by ◊ with the result that the replacements are M, B, S4,or S5 operators respectively then *E* is called an respectively an M, B, S4,or S5 ***necessity operator*** and *E′* a ***possibility operator*** for respectively M, B, S4,or S5.

Epistemic Descriptive Operators

Syntax. $f_{\mathbf{K}}$ and $f_{\mathbf{B}}$, are defined as the 1-place operations on signs such that $f_K(P)=\mathbf{K}P$, *and* $f_{\mathbf{B}}(P)=\mathbf{B}P$.

Intensional Sematics. W=<K,$\leq_K$,$\leq_{\mathbf{B}}$> is called an ***epistemic world structure*** iff K is a non-empty (of ***epistemically possible worlds***) and $\leq_{\mathbf{K}}$ and $\leq_{\mathbf{B}}$ are transitive binary relations on K (***the empistemic*** and ***doxastic alternativeness*** relations respectively) such that $\leq_K \subseteq \leq_{\mathbf{B}}$. In addition $\leq_{\mathbf{K}}$ is reflexive, and symmetric. Relative to an epistemic world structure W=<K,≤>, $g_{\mathbf{K}}$ and $g_{\mathbf{B}}$ are defined as 1-place operations on $\{0,1\}^K$ into $\{0,1\}^K$ such that

$g_{\mathbf{K}}(\pi^{\mathbf{c}})(k)$=T if for all k', $k\leq_{\mathbf{K}}k'$, $\pi^{\mathbf{c}}(k')$=T; $g_{\mathbf{K}}(\pi^{\mathbf{c}})(k)$=F if for some k', $k\leq_{\mathbf{K}}k'$, $\pi^{\mathbf{c}}(k')\neq$T.

$g_{\mathbf{B}}(\pi^{\mathbf{c}})(k)$=T if for all k', $k\leq_{\mathbf{B}}k'$, $\pi^{\mathbf{c}}(k')$=T; $g_{\mathbf{B}}(\pi^{\mathbf{c}})(k)$=F if for some k', $k\leq_{\mathbf{B}}k'$, $\pi^{\mathbf{c}}(k')\neq$T.

Definition 13. L=<Syn,F> is said to be a ***sentential epistemic (Montague) language*** iff Syn=<Sen,$f_\sim$, $f_{\mathbf{K}}$, $f_{\mathbf{B}}$,$f_\wedge$,$f_\vee$,$f_\rightarrow$> and F is the set of all logical matrices M such that for some epistemic world structure W=<K,$\leq_K$,$\leq_{\mathbf{B}}$>, M=<2^K,{$K^{\mathbf{c}}$},$g_\sim$,$g_{\mathbf{K}}$, $g_{\mathbf{B}}$,$g_\wedge$,$g_\vee$,$g_\rightarrow$>.

Theorem 8. If L is a sentential epistemic language, then

1. L is non-extensional,
2. **K** is an S5 modal operator (hence $\mathbf{K}P \models_L P$, and $\mathbf{K}P \models_L \mathbf{KK}P$),
3. $\mathbf{K}P \models_L \mathbf{B}P$
4. $\mathbf{B}P \models_L \mathbf{BB}P$

Tense Operators

Idea. We let time branch towards the future and introduce two operators **H** and **F** for future tenses (**H***P* is read "it has to be that *P*" and **F***P* is "it will be that *P*") and two operators **P** and **W** for past tenses (**P***P* is read "it has to have been that *P*" and **F***P* is "it was the case that *P*").

Syntax. $f_{\mathbf{H}}$, $f_{\mathbf{F}}$, $f_{\mathbf{P}}$, and $f_{\mathbf{W}}$ are defined as the 1-place operations on signs such that , $f_{\mathbf{H}}(P)=\mathbf{H}P$, $f_{\mathbf{F}}(\mathbf{F}P)=P$, $f_{\mathbf{P}}(P)=\mathbf{P}P$, and $f_W(P)=\mathbf{W}P$.

Intensional Sematics. W=<K,≤> is a ***temporal world structure*** iff K is a non-empty set (of ***times***) and ≤ is reflexive and transitive binary relation (of ***temporal order***) on K. Then, relative to a temporal world structure W=<K,≤>, $g_{\mathbf{H}}$, $g_{\mathbf{F}}$, $g_{\mathbf{P}}$, and $g_{\mathbf{W}}$ are defined as 1-place operations on $\{0,1\}^K$ into $\{0,1\}^K$ such that

$g_{\mathbf{H}}(\pi^{\mathbf{c}})(k)$=T if for all k', $k\leq k'$, $\pi^{\mathbf{c}}(k')$=T; $g_{\mathbf{H}}(\pi^{\mathbf{c}})(k)$=F if for some k', $k\leq k'$, $\pi^{\mathbf{c}}(k')\neq$T.

$g_{\mathbf{F}}(\pi^{\mathbf{c}})(k)$=T if for some k', $k\leq k'$, $\pi^{\mathbf{c}}(k')$=T, and $g_{\mathbf{F}}(\pi^{\mathbf{c}})(k)$=F if for all k', $k\leq k'$, $\pi^{\mathbf{c}}(k')\neq$T.

$g_{\mathbf{P}}(\pi^{\mathbf{c}})(k)$=T if for all k', $k'\leq k$, $\pi^{\mathbf{c}}(k')$=T; $g_{\mathbf{P}}(\pi^{\mathbf{c}})(k)$=F if for some k', $k'\leq k$, $\pi^{\mathbf{c}}(k')\neq$T.

$g_{\mathbf{W}}(\pi^{\mathbf{c}})(k)$=T if for some k', $k'\leq k$, $\pi^{\mathbf{c}}(k')$=T, and $g_{\mathbf{W}}(\pi^{\mathbf{c}})(k)$=F if for all k', $k'\leq k$, $\pi^{\mathbf{c}}(k')\neq$T.

Definition 14. L=<Syn,F> is said to be a ***sentential tense (Montague) language*** iff Syn=<Sen,$f_\sim$,$f_{\mathbf{H}}$,$f_{\mathbf{F}}$, $f_{\mathbf{P}}$,$f_{\mathbf{W}}$,$f_\wedge$,$f_\vee$,$f_\rightarrow$> and F is the set of all logical matrices M such that for some epistemic world structure W=<K,$\leq_K$,$\leq_{\mathbf{B}}$>, M=<2^K,{$K^{\mathbf{c}}$},$g_\sim$,$g_{\mathbf{H}}$,$g_{\mathbf{F}}$,$g_{\mathbf{P}}$,$g_{\mathbf{W}}$,$g_\wedge$,$g_\vee$,$g_\rightarrow$>.

Theorem 8. If L is a sentential tense language, then

1. L is non-extensional,
2. **H** and **F** are duals and respectively S4 necessity and possibility operators,
3. **P** and **W** are duals and respectively S4 necessity and possibility operators,
4. $\mathbf{W}P \models_L \mathbf{HW}P$ (the "necessity" of the past), but not($\mathbf{F}P \models_L \mathbf{HF}P$)

Deontic Operators

Idea. Utilitarian moral choice is a consequentialist comparison among immediate temporal alternatives "situations." Let situations be places in a temporal tree structure opening towards the future and associate with each a utility value. A utilitarian says we ought to bring about that *P* (in symbols, **O***P*) f no matter what immediate situation ~*P* is true in there is a better one in which *P* is true. Similarly, it is morally permissible to bring about that *P* if it is not the case that for any situation in which *P* is true there is a better one in which ~*P* is true.

Syntax. $f_{\mathbf{O}}$ and $f_{\mathbf{P}}$, are defined as the 1-place operations on signs such that $f_{O}(P)=\mathbf{O}P$, *and* $f_{\mathbf{Pr}}(P)=\mathbf{Pr}P$.

Intensional Sematics. W=<K,≤,U > is called a ***deontic choice structure*** iff K is a non-empty set (of ***choices***) and ≤ is binary relation (or ***temporal order***) on K such that ≤ determines an ascending tree structure with immediate successor relation << (hence ≤ is reflexive, transitive, and symmetric) such that each node has an immedicate successor, and U is a real valued function on K (called a ***utility function***). Relative to a deontic choice structure W=<K,≤,U >, $g_{\mathbf{O}}$ and $g_{\mathbf{Pr}}$ are defined as 1-place operations on $\{0,1\}^{K}$ into $\{0,1\}^{K}$ such that

> $g_{\mathbf{O}}(\pi^{\mathbf{c}})(k)$=T, if for all k' , $k<<k'$, if $g_{\sim}(\pi^{\mathbf{c}})(k')$=T, there is some k'' such that $k<<k''$, $\pi^{\mathbf{c}}(k'')$=T, and $U(k') \leq U(k'')$, and $g_{\mathbf{O}}(\pi^{\mathbf{c}})(k)$=F otherwise.
>
> $g_{\mathbf{Pr}}(\pi^{\mathbf{c}})(k)$=T, if for some k' , $k<<k'$, $\pi^{\mathbf{c}}(k')$=T, there is no k'' such that $k<<k''$, $g_{\sim}(\pi^{\mathbf{c}})(k'')$=T, and $U(k') \leq U(k'')$, and $g_{\mathbf{P}}(\pi^{\mathbf{c}})(k)$=F otherwise.

Definition 15. L=<*Syn*,*F*> is said to be a ***deontic sentential (Montague) language*** iff *Syn*=<*Sen*,$f_{\sim}$, $f_{\mathbf{O}}$, $f_{\mathbf{Pr}}$,$f_{\wedge}$,$f_{\vee}$,$f_{\rightarrow}$> and *F* is the set of all logical matrices M such that for some deontic world structure W=<K,≤,U >, M=<2^{K},{$K^{\mathbf{c}}$},$g_{\sim}$,$g_{\mathbf{O}}$, $g_{\mathbf{Pr}}$,$g_{\wedge}$,$g_{\vee}$,$g_{\rightarrow}$>.

Theorem 9. If L is a deontic sentential language, then

1. L is non-extensional,
2. **O** and **Pr** are duals.

The Languages Combined

Definition 16. <K,≤,$\leq_{\mathbf{T}}$,$\leq_{K}$,$\leq_{\mathbf{B}}$,U > is a ***global sentential world structure*** iff K is a non-empty, <K,≤> is an S5 world structure; <K,$\leq_{K}$,$\leq_{\mathbf{B}}$> is an epistemic worlds structure such that $\leq_{K} \subseteq \leq$; <K,$\leq_{\mathbf{T}}$> is a temporal world structure such that $\leq_{\mathbf{T}} \subseteq \leq$; <K,$\leq_{\mathbf{T}}$,$U$ > is a deontic world structure. Let g , $g_{\Diamond}$ be defined relative to <K,≤>; let $g_{\mathbf{K}}$, $g_{\mathbf{B}}$ be defined relative to <K,$\leq_{K}$,$\leq_{\mathbf{B}}$>; let $g_{\mathbf{H}}$,$g_{\mathbf{F}}$,$g_{\mathbf{P}}$,$g_{\mathbf{W}}$ be defined relative to <K,$\leq_{\mathbf{T}}$>; and let $g_{\mathbf{O}}$, $g_{\mathbf{Pr}}$ be defined relative to <K,$\leq_{\mathbf{T}}$,U >.

Definition 17. L=<*Syn*,*F*> is said to be a ***global sentential (Montague) language*** iff *Syn*=<*Sen*,$f_{\sim}$,f , $f_{\Diamond}$,$f_{\mathbf{O}}$,$f_{\mathbf{K}}$ $f_{\mathbf{B}}$,$f_{\mathbf{H}}$,$f_{\mathbf{F}}$,$f_{\mathbf{P}}$,$f_{\mathbf{W}}$,$f_{\mathbf{Pr}}$,$f_{\wedge}$,$f_{\vee}$,$f_{\rightarrow}$> and *F* is the set of all logical matrices M such that for some global sentential world structure <K,≤,$\leq_{\mathbf{T}}$,$\leq_{K}$,$\leq_{\mathbf{B}}$,U >,
M=<2^{K},{$K^{\mathbf{c}}$},$g_{\sim}$,g ,$g_{\Diamond}$,$g_{\mathbf{O}}$,$g_{\mathbf{K}}$,$g_{\mathbf{B}}$,$g_{\mathbf{H}}$,$g_{\mathbf{F}}$,$g_{\mathbf{P}}$,$g_{\mathbf{W}}$,$g_{\mathbf{O}}$,$g_{\mathbf{Pr}}$,$g_{\wedge}$,$g_{\vee}$,
$g_{\rightarrow}$>.

Theorem 10. If L is a global sentential language, then

1. L is non-extensional,
2. $\Box P \models_{L} \mathbf{K}P \models_{L} P \models_{L} \Diamond P$, but not $\models_{L} \mathbf{B}P$,
3. $\Box P \models_{L} \mathbf{O}P \models_{L} \Diamond \mathbf{F}P \models_{L} \Diamond P$ ("ought" implies "can")
4. not($_{L}\mathbf{Pr}P \models_{L} \Diamond \mathbf{F}P$)

Exercises

1. Prove Theroems 1, 2, and 3 using the definitions and facts previously proven about homomorphisms and matricx interpretations.
2. Prove Theorem, 7, part 4.
3. Prove Theorem 10, part 3.

Lecture 5

Tarski's Boolean Algebra of Satisfaction Sequences

1. Classical First-Order Logic

In this lecture we will explore how Boolean algebras may be applied to develop the statement of the semantics of classical first-order logic and of the intensional logics built upon it. The challenge in doing so rests in negotiating the semantic complexity of expressions with variable binding operators. The Fregean paradigm of homomorphic syntax and semantics is vitiated even at the extensionsal level for such operators. In the standard classical semantics sentences, both open and closed, are evaluated in two stages. At the first stage they are assigned a truth-value first relative to a model and an interpretation. Then using this interpretation the second stage is developed in which they are assigned a truth-value relative to a model alone. But since variables only have interpretations relative to a variable interpretation, it is only at the first stage that all the parts of a quantified sentences have interpretations. Hence it is only at this stage that the homomorphic mirroring of syntax by semantics could take place. But the stage one semantics is straightforwardly non-extensional. It is easy to find examples of open sentences *P* and *Q*, a model $\mathbb{A}$ and an interpretation *s* of the variables such both *P* and *Q* are assigned T relative to model $\mathbb{A}$ and *s*, yet the universal closure $\forall vP$ is T while $\forall vQ$ is F.

This standard two stage semantics was first presented by Alfred Tarski in his early papers[1] (in the 1930's) in which he first explored the recursive definition of truth for formal languages. In the early 1950's, however, he turned again to formal semantics and used set theory to develop the theory more rigorously. In doing so he incorporated the now standard definition of validly (a relation that preserves truth across all logically possible interpretations) that Rudolf Carnap[2] and others had somewhat informally be trying to combine with Tarski original semantics. In his new work, which was called ***model theory***, Tarski defines the modern notion of a model for the first time.[3] He defines truth-relative-to-a-model and then defines validity as a truth-preserving relation across models. Moreover, being one of the leading algebrist of his day, Tarski understood the perspicuity of the homomorphic framework that had been sketched by Frege and incorporated into the matrix semantics of sentential logic by Lukasiewicz and others since the 1920's. Tarski employs this framework in his model theory and achieved an elegant statement of first-order semantics. In it he abandons the two stages of his earlier accounts and provides in a single stage a fully extensional semantics in which semantic structure is perfectly homomorphic to first-order syntax. The sentential connectives are moreover interpreted by the standard Boolean operators over sets. Tarski's "trick" lies in his choice for the semantic value of

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[1] Alfred Tarski, "The Concept of Truth in Formalized Languages," [1931] in *Logic, Semantics, Metamathematics* (Oxford: Clarendon Press, 1956).
[2] Rudolf Carnap, *Meaning and Necessity* (Chicago: Univ. of Chicago Press, 1947).
[3] Alfred Tarski, "Contributions to the Theory of Models," *Indagationes Mathematicae* 16 (1954), 572-588.

sentences. Relative to a model, a sentence stands for the sets of variable assignments that make the sentence true (that, to use his words, "satisfy" the sentence). Given this choice, all closed sentences are bivalent in the sense that they either stand for the entire set of variable assignments (this set is identified with "the true" or T) or the set of no variable assignments (the empty set, identified with "the false" or F). In addition, some open sentences are simply true or false in this sense. The remaining open sentences are true relative to some assignments but false relative to others, as intuition requires. Unlike the more common two stage theory found in most elementary logic texts -- presumable because it preserves the more naïve idea that all sentences, both open and closed, are all either "true" or "false" -- the semantics in terms of sets of "satisfaction sequences" is briefly stated.

Syntax. Syntax is the standard one found in elementary texts. For fist-order logic we first need the syntactic operations $f_{\sim}$,$f_{\wedge}$,$f_{\vee}$,$f_{\rightarrow}$ for the sentential connectives. These will be as previously defined. In addition, two new operations are needed, one for basic (atomic) sentences and one for universally quantified sentences. The formation operation f_{BS} for basic expressions generates a basic sentence from and *n* singular terms and an *n*-place predicate by concatenating the terms, in order, preceded by the predicate. The operation $f_{\forall}$ for universal sentences maps a variable and sentence into it's the universal closure for that variable. Various sets of basic expressions are also required: an at most denumerable set $\mathcal{C}$ of constants, an at most denumerable set $\mathcal{V}$ of variables, and various at most denumerable sets $\mathcal{P}^n$, for 1≤n, of predicates of degree n. In terms of the rules and the basic expressions, the set of sentences is defined by induction.

First-Order Syntax[4]

Definition 1. A $\mathcal{FOSyn}$ ***(first-order syntax)*** is any $<\mathcal{C},\mathcal{V},\mathcal{P}^1,\ldots,\mathcal{P}^n,\ldots,\mathcal{Sen}, f_{BS}, f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow},f_{\forall}>$ such that $\mathcal{C},\mathcal{V},\mathcal{P}^1,\ldots,\mathcal{P}^n,\ldots,$ are disjoint at most denumerable subsets of Σ^*, and f_{BS}, $f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow},f_{\forall}$ are operations on Σ^* defined as follows:

$f_{BS}(e_1,\ldots,e_n,e_{n+1})= e_{n+1}e_1\ldots e_n$
$f_{\sim}(x)= \sim x$
$f_{\wedge}(x,y)= (x\wedge y)$
$f_{\vee}(x,y)= (x\vee y)$
$f_{\rightarrow}(x,y)= (x\rightarrow y)$
$f_{\forall}(e_1,e_2)=\forall e_1e_2$,

A $\mathcal{MFOSyn}$ ***(modal first-order syntax)*** is any $<\mathcal{C},\mathcal{V},\mathcal{P}^1,\ldots,\mathcal{P}^n,\ldots,\mathcal{Sen}, f_{BS},f_{\Box},f_{\Diamond},f_{\sim}, f_{\wedge},f_{\vee},f_{\rightarrow},f_{\forall}>$ such that $f_{\Box}$ and $f_{\Diamond}$ are as defined in Lecture 4 and $<\mathcal{C},\mathcal{V},\mathcal{P}^1,\ldots,\mathcal{P}^n,\ldots,\mathcal{Sen}, f_{BS},f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow},f_{\forall}>$ is a sifrst-order syntax.

For either syntax $\mathcal{ASen}$={ $f_{BS}(e_{n+1},\ldots,e_n)$ | $e_{n+1}\in \mathcal{P}^n$ & $e_1,\ldots,e_n\in\mathcal{C}\cup\mathcal{V}$}, and $\mathcal{Sen}$ is the least set including $\mathcal{ASen}$ and closed under $f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow}, f_{\forall}$. We let c range over $\mathcal{C}$,v over $\mathcal{V}$, and P^n_i over $\mathcal{P}^n$. We introduce the existential quantifier by eliminative definition: $\exists vP$ means $\sim\forall v\sim P$.

Theorem 1. If $\mathcal{SL}_{\mathcal{FOSyn}}$ is the least set including $\mathcal{ASen}$ and closed under $f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow}$, then $\mathcal{SL}_{\mathcal{FOSyn}}$ is a sentential syntax.

Semantics. The semantics develops the idea that the interoperation of a sentence, either open or closed, should be the set of those assignments of variables to objects that satisfy the sentence.

In order to talk about sequences over the domain $\mathcal{D}$, we adopt some notation. An ***(infinite) sequence*** is defined as a function s from the set ω of the natural numbers into $\mathcal{D}$. We let d range over $\mathcal{D}$, i over ω, and s over $\mathcal{D}^{\omega}$. Also, we abbreviate s(i) as s_i.

Tarski observed that interpretations of the variables, called ***variable assignments***, may in fact be identified with the elements of $\mathcal{D}^{\omega}$. This is so because we know that for each variable there is an integer index i from ω (e.g. v's rank in the syntactic list of variables). Let us now understand that relative to a sequence s in $\mathcal{D}^{\omega}$, v stands for s(i). Accordingly, since there are as many possible pairings of i-th sequence positions with elements as there are elements of a domain $\mathcal{D}$, it will follow that for every $d\in\mathcal{D}$, there is some sequence s in $\mathcal{D}^N$ such that $s_i=d$. For any variable v, let us therefore choose some index i and define its (extensional) "interpretation" as the function ϕ on sequences "characteristic" of it as follows. Let $\mathcal{D}^{(\mathcal{D}^{\omega})}$ stand for the set of all functions from $\mathcal{D}^{\omega}$ into $\mathcal{D}$. Then, for some i, v stands for that ϕ in $\mathcal{D}^{(\mathcal{D}^{\omega})}$ such that $\phi(s)=s_i$.[5]

[4] As developed here, first-order syntax lacks functors. This omission makes the syntax no weaker than otherwise because n-place functors may be introduced to the syntax as defined here by eliminative definition in terms of an equal number of n+1 place predicates:

$$P[f(t_1,\ldots,t_n)] =_{def} \exists v(R(t_1,\ldots,t_n,v)\wedge P[v]\wedge\forall v'(R(t_1,\ldots,t_n,v')\rightarrow v=v'))$$

[5] Though there may be some interpretations Int on an intensional matrix M such that two variables are "synonymous" relative to Int in the sense that they have the same index and are

> **Extensions for Varaibles**.
>
> **Definition.** If $\phi \in \mathcal{D}^{(\mathcal{D}^{\omega})}$, then i is said to be an ***index for*** ϕ iff for for all $s \in \mathcal{D}^{\omega}$, $\phi(i)=s_i$.
>
> **Definition.** $\mathcal{D}^{(\mathcal{D}^{\omega})}_{\text{Index}}$ is $\{\phi \in \mathcal{D}^{(\mathcal{D}^{\omega})} \mid$ there is some index i for $\phi\}$.

Since variables are interpreted by functions from $\mathcal{D}^{\omega}$ into $\mathcal{D}$, and since variables and constants have similar semantic functions as linguistic representatives of individuals, we should assign to them the same sort of semantic interpretation. That is, constants too should stand for functions from $\mathcal{D}^{\bullet}$ into $\mathcal{D}$. Fortunately there is a simple and easy way to pair elements of the domain (the entities that constants intuitively stand for) with functions of sequences of this type. Tarski observed that constants have the happy property that they stand for the same thing regardless of what assignment happens to be interpreting to the variables. Suppose there is an object *d* in the domain that we want to adopt as the extension of a constant *c*. Then no matter what *s* happens to be interpreting the variables, *c* is supposed to stand for *d.* There is then a constant function ♎ on sequences: in symbols, ♎ in $\mathcal{D}^{(\mathcal{D}^{\omega})}$) and is "constant" in the sense that for all $s \in \mathcal{D}^{\bullet}$, ♎($s$)=$d$. Accordingly, we adopt a constant function ϕ from $\mathcal{D}^{\omega}$ into $\mathcal{D}$ as the referent (extension) of *c.*

> **Extensions of Constants**
>
> **Definition.** Let A^{B}_{con} be the set of all constant functions from B into A. Then, $\mathcal{D}^{(\mathcal{D}^{\bullet})}_{\text{con}}$ is the set of all constant functions from $\mathcal{D}^{\omega}$ into $\mathcal{D}$.

A basic (atomic) sentence can then be understood as standing for the set of all the sequences *s* in $\mathcal{D}^{\omega}$ that "satisfy" it. That is, the sentence $P^n t_1 \ldots t_n$ stands for the set of all sequences *s* such that the referents $\phi_1(s),\ldots,\phi_n(s)$ of the terms $t_1,\ldots,t_n$ fall, in that order, under the relation *R* picked out by the predicate P^n. In the semantics a function on extensions is defined that corresponds to the formation rule for atomic expressions. It will take as an inputs the semantic values $R,\phi_1,\ldots,\phi_n$ of the predicate P^n and terms $t_1,\ldots,t_n$, and yield as output the semantic value of $P^n t_1 \ldots t_n$, namely the set of all sequences that satisfy.

The particular elegance of Tarski's interpretation lies in the fact that it allows for the interpretation of sentential connectives by Boolean operations on sets of sequences, and for the interpretation of the universal quantifier by a simple and clear rule: if a sentence is satisfied by every sequence (i.e. interpretation of the variables), then its universal closure is also satisfied by every sequence, but if it is falsified by some sequence (i.e. interpretation of the variables), then its universal closure is false.

---

hence mapped by Int onto the same function ♎ in $\mathcal{D}^{(\mathcal{D}^{\bullet})}$), there will be other interpretations Int′ over M such that this is not the case. There will then be no M-entailments that turn on synonymy relative to an interpretation.

Definition 2. A ***classical first-order matrix*** is any

$M=<\mathcal{D}_{con}^{(\mathcal{D}^{\omega})},\mathcal{D}_{Index}^{(\mathcal{D}^{\omega})},\mathcal{D}^1,\ldots,\mathcal{D}^n,\ldots,P(\mathcal{D}^{\omega}),\{\mathcal{D}^{\omega}\},g_{BS},-,\cap,\cup,\Rightarrow,g_{\forall}>$ such that

1. $\mathcal{D}$ is non-empty,
2. the functions $-,\cap,\cup,\Rightarrow$ are the set theoretic Boolean operations defined on subsets of $\mathcal{D}^{\blacklozenge}$, and g_{BS} and $g_{\forall}$ are defined as follows:

$g_{BS}(R,\phi_1,\ldots,\phi_n,)=\{ s\in\mathcal{D}^{\blacklozenge} \mid <\phi_1(s),\ldots,\phi_n(s)>\in R\}$

$g_{\forall}(\underline{\Omega},X)=\{ s\in\mathcal{D}^{\omega} \mid i$ is an index for ϕ, and for all $d\in\mathcal{D}$, $(s-\{<i,s_i>\})\cup\{<i,d>\}\in X\}$

(In alternative notation, let s' be an ***i-variant*** of s iff for any $j\neq i$, $s'_j=s_j$. Then

$g_{\forall}(\phi,X)=\{ s\in\mathcal{D}^{\omega} \mid {}^{\omega}\mid i$ is an index for ϕ, and every i-variant of s is in $X\}$.)

It is customary to identity $\mathcal{D}^{\blacklozenge}$ with T, and $\varnothing$ with F.

Theorem. 2 $<P(\mathcal{D}^{\omega}),-,\cap,\cup,\Rightarrow$ is a Boolean algebra, $<P(\mathcal{D}^{\blacklozenge}),\{\mathcal{D}^{\blacklozenge}\},-,\cap,\cup,\Rightarrow>$ is a sentential matrix, and $<\{\varnothing,\mathcal{D}^{\omega}\},\{\mathcal{D}^{\omega}\}-,\cap,\cup,\Rightarrow>$ the bivalent sentential matrix homomorphic to it.

Definition 3. $\mathcal{L}=<\mathcal{FOLS}yn,\mathcal{F}>$ is said to be a ***(classical) first-order language*** iff $\mathcal{FOLS}yn$ is a first order syntax and $\mathcal{F}$ is the set of all firs-order matrices of the same character as $\mathcal{FOLS}yn$.

Theorem 3. If $\mathcal{L}=<\mathcal{SL}_{\mathcal{FOS}yn},\mathcal{F}>$ such that $\mathcal{F}$ is the set of all Boolean algebras $<P(\mathcal{D}^{\omega}),-,\cap,\cup,\Rightarrow>$ for a non-empty $\mathcal{D}$, and if $\mathcal{L}_B=<\mathcal{SL}_{\mathcal{FOS}yn},\mathcal{F}>$ such that $\mathcal{F}_B$ is the set containing the unique bivalent sentential matrix $<\{\varnothing,\mathcal{D}^{\omega}\},\{\mathcal{D}^{\omega}\}-,\cap,\cup,\Rightarrow>$, then $X\models_{\mathcal{L}}P$ iff $X\models_{\mathcal{L}_B}P$ iff $X\models_C P$.

Proof Theory. This syntax and semantics is appropriate to the standard natural deduction proof theory for classical first-order logic. In this section we state the theory, mainly because we shall be contrasting it with intuitionistic logic later. Since the statement of the proof rules turns on substitution, we must begin by defining the sort of substitution relevant to quantificational contexts in which we want to substitute without introducing new variables that are become bound in occurrences in which the variable they replace were free.

Substitutions

Definition 4

- If an occurrence of a term t in P said to be ***free*** in P if
  it is a variable v then its occurrence is not part of some formula $\forall vQ$ or $\exists vQ$ in P, or
  it contains an occurrence of a variable v, then that occurrence of v is not part of some formula $\forall vQ$ or $\exists vQ$ in P;

otherwise the occurrence of t is said to be ***bound***.

- A term or literal that does not contain a variable is said to be ***grounded***.
- An occurrence of a term t' is ***free for a term*** t in P iff the occurrence of t' in P is not part of some formula $\forall vQ$ or $\exists vQ$ in P for any variable v that in t,.
- A formula without free variables is customarily called a ***sentence***.
- A formula P is called ***general*** if all its quantifiers occur on the outside (leftmost side) of P in the sense that P is some $E_1v_1...E_1v_nQ$ such that $\{E_1,...,E_n\}\subseteq\{\forall,\exists\}$ and Q is some truth-functional formula in which neither $\forall$ nor $\exists$ occur. Such a general formula $E_1v_1...E_nv_nQ$ is said to be ***universal*** if all E_i are $\forall$.

σ ***substitution function for terms*** in $\mathbf{F}_{\mathcal{FOL}}$ a total function ϕ from some subset of $\mathbf{Trms}_{\mathcal{FOL}}$ into $\mathbf{Trms}_{\mathcal{FOL}}$) defined as follows: If t is some variable v, then $\phi(t)\in\mathbf{Trms}_{\mathcal{FOL}}$, The substitution function is extended to formulas in three ways.

Substitution for all Free Terms

The extension σ of a substitution function σ* ***to sentences, relative to all free terms***, is defined inductively as follows:

Atomic Case. $\sigma(P^n t_1 \ldots t_n) = P^n \sigma^*(t_1) \ldots \sigma^*(t_n)$

Molecular Case. $\sigma(\sim P) = \sim\sigma(P)$

$\sigma(P \wedge Q) = \sigma(P) \wedge \sigma(Q)$

$\sigma(P \vee Q) = \sigma(P) \vee \sigma(Q)$

$\sigma(P \rightarrow Q) = \sigma(P) \rightarrow \sigma(Q)$

$\sigma(\forall vP) = \forall\sigma\, v(P)$ if

1. the t such that $\sigma(t)=v$ is free in $(\forall vP)$, and
2. for all t in $(\forall vP)$, $\sigma(t)$ is free for t in $(\forall vP)$.

We introduce now some common notation that makes using substitution easier.

$\boldsymbol{P[t'_1,\ldots,t'_n/t_1 \ldots t_n]}$**,** read "the result of substituting in P for all free occurrences of $t_1 \ldots t_n$ respectively by of $t'_1,\ldots,t'_n$" is a partial function from $\mathbf{F}_{FOL}$ to $\mathbf{F}_{FOL}$ defined as $P_\bullet$ where σ is substitution function for $\mathbf{F}_{FOL}$ and $\sigma(t'_i)=t_i$.

Sometimes rather than list all terms being substituted, it is more convenient to refer first to a sentence ***P*** using the notation $\boldsymbol{P[t_1 \ldots t_n]}$ and then later to $\boldsymbol{P[t'_1,\ldots,t'_n/t_1 \ldots t_n]}$ simply by the simpler notation $\boldsymbol{P[t'_1,\ldots,t'_n]}$. Thus, the earlier mention of $P[t_1 \ldots t_n]$ simply means that we are dealing with the formula P and that we are selecting the terms $t_1 \ldots t_n$ which may or may not be in P for special consideration. When we later refer to $P[t'_1,\ldots,t'_n]$ we are then referring to the result of substituting $t'_1,\ldots,t'_n$ for $t_1 \ldots t_n$ in P, i.e. to $P[t'_1,\ldots,t'_n/t_1 \ldots t_n]$.

Examples. If a t_i does not occur in $P^n{}_m t_1 \ldots t_n$, then its replacement by t'_i makes no change in $P^n{}_m t_1 \ldots t_n$. In the extreme case in which there are no occurrences of any $t_1 \ldots t_n$ in P, then $P^n{}_m[t'_1,\ldots,t'_n/t_1 \ldots t_n] = P^n{}_m t_1 \ldots t_n$, Moreover, substitution is only a partial operation on terms and formulas. If there is even one occurrence of t_i that is not free in P or is not free for $\bullet(t_i)$, then $P^n{}_m[t'_1,\ldots,t'_n/t_1 \ldots t_n]$ is undefined.

$Fx[y/x]$	=	Fy	
$Fz[y/x]$	=	Fz	x does not occur in Fz.
$Fx \wedge Gy[y/x]$	=	$Fy \wedge Gy$	
$(\forall xFx)[y/x]$		undefined	The occurrence of x in $\forall xFx$ is not free.)
$(\forall y(Fy \wedge Gx))[y/x]$		undefined	The occurrence of x in $\forall y(Fy \wedge Gx)$ is not fee for y.
$(Fx \wedge (\forall y(Fy \wedge Gx))[y/x]$		undefined	One occurrence of x in $Fx \wedge \forall y(Fy \wedge Gx)$ is not fee for y.

Substitution for some Free Terms

An extension σ ***of a substitution function*** σ^* ***to sentences, relative to some free terms***, is defined a partial function from $\mathbf{F}_{\mathcal{FOL}}$ to $\mathbf{F}_{\mathcal{FOL}}$ inductively as follows:

Atomic Case. $\sigma(P^n t_1 ... t_n) = P^n t'_1 ... t'_n$, such that for some $i \leq n$, $\sigma^*(t_i) = t'_i$

Molecular Case. $\sigma(\sim P) = \sim\sigma(P)$

$\sigma(P \wedge Q) = \sigma(P) \wedge \sigma(Q)$

$\sigma(P \vee Q) = \sigma(P) \vee \sigma(Q)$

$\sigma(P \rightarrow Q) = \sigma(P) \rightarrow \sigma(Q)$

$\sigma(\forall vP) = \forall\sigma\, v(P)$ if the t such that $\sigma(t)=v$ is free in $(\forall vP)$, and for all t in $(\forall vP)$, $\sigma(t)$ is free for t in $(\forall vP)$.

$\boldsymbol{P[t'_1,...,t'_n//t_1...t_n]}$, read "a result of substituting in P for some free occurrences of $t_1...t_n$ by of receptively some of $t'_1,...,t'_n$" is defined as P_σ where σ is substitution function for $\mathbf{F}_{\mathcal{FOL}}$ and $\sigma(t'_i)=t_i$.

Substitution for all Terms: Alphabetic Variance

The full extension σ ***of a substitution function*** σ^* ***to sentences, relative to all terms***, is a partial function from $\mathbf{F}_{\mathcal{FOL}}$ to $\mathbf{F}_{\mathcal{FOL}}$ defined inductively as follows:

Atomic Case. $\sigma(P^n t_1 ... t_n) = P^n \sigma^*(t_1) ... \sigma^*(t_n)$

Molecular Case. $\sigma(\sim P) = \sim\sigma(P)$

$\sigma(P \wedge Q) = \sigma(P) \wedge \sigma(Q)$

$\sigma(P \vee Q) = \sigma(P) \vee \sigma(Q)$

$\sigma(P \rightarrow Q) = \sigma(P) \rightarrow \sigma(Q)$

$\sigma(\forall vP) = \forall\sigma(t)\sigma(P)$ if $(t)=v$, and for all t' in $(\forall vP)$, $\sigma^*(t')$ is free for t in $(\forall vP)$.

P is an ***alphabetic variant*** of Q, briefly $P \equiv Q$, iff there is some 1-1 substitution function σ for $\mathbf{F}_{\mathcal{FOL}}$ such that $Q = P_\sigma$.

Theorem 4. If $\sigma(P)$ is well defined, $P \equiv \sigma(P)$.

Examples of Alphabetic Variants. Let σ be a full extension substitution function for all terms.

$Fx[y/x]=Fy$

$(\forall x(Fx\wedge Gx))[y/x] = (\forall y(Fx\wedge Gx)[y/x] = \forall yFx[y/x]\wedge Gx)[y/x] = \forall y(Fy\wedge Gy)$

$(\forall xFx\wedge Gx)[y/x] = (\forall xFx)[y/x]\wedge Fx[y/x] = \forall xFx\wedge Fy$

$(\forall xFx\wedge Gx)[x/x] = (\forall xFx)[x/x]\wedge Fx[x/x] = \forall xFx\wedge Fx$

$(\forall y(\forall xFx\wedge Gy))[z/x] = \forall y((\forall xFx\wedge Gy)[z/x]) = \forall y((\forall xFx)[z/x]\wedge Gy[z/x]) = \forall y(\forall zFz\wedge Gy)$

$(\forall y(\forall xFx\wedge Gy))[x/x] = \forall y(\forall xFx\wedge Gy[x/x]) = \forall y((\forall xFx)[x/x]\wedge Gy[x/x])) = \forall y(\forall xFx\wedge Gy)$

$(\forall y(Hy\wedge\forall x(Fx\wedge Gy)))[y/x]$ = undefined, y is not free for x in $\forall y(Hy\wedge\forall x(Fx\wedge Gy))$

$(\forall z(Hz\wedge\forall x(Fx\wedge Gy)))[y/x] = \forall z((Hz[y/x]\wedge\forall x(Fx\wedge Gy)[y/x])) = \forall z(Hz\wedge\forall x(Fx\wedge Gy)[y/x])$=undefined, y is not free for x in $\forall x(Fx\wedge Gy)$

A Natural Deduction Systems QC for the Classical First-Order Logic

QC=$<BD_{QC},\vdash_{QC},R_{\bot+},R_{\bot-},R_{\sim+},R_{\sim-},R_{\wedge+},R_{\wedge-},R_{\vee+}, R_{\vee-},R_{\rightarrow+},R_{\rightarrow-},R_{Th}>$ is the inductive system such that

1. Let $<X,P>$ be a ***deduction*** iff $X\subseteq\mathcal{Sen}$ and $P\in\mathcal{Sen}$. We adopt these abbreviations:

$X\vdash_{QC}P$	for	$<X,P>$ is in $\vdash_{QC}$;
$X,Y\vdash_{QC}P$	for	$X\cup Y\vdash_{QC}P$;
$X,P\vdash_{QC}Q$	for	$X\cup\{P\}\vdash_{QC}Q$;
$P_1,\ldots,P_n\vdash_{QC}Q$	for	$\{P_1,\ldots,P_n\}\vdash_{QC}Q$;
$\vdash_{QC}P$	for	$\varnothing\vdash_{QC}P$.
$\bot$	for	$P_1\wedge\sim P_1$ (Here P_1 is the 1^{st} atomic sentence.)

2. BD_{QC} is the set of all deductions $<X,P>$ such that $P\in X$.
3. The rules in $\{R_{\bot+},R_{\bot-},R_{\sim+},R_{\sim-},R_{\wedge+},R_{\wedge-},R_{\vee+}, R_{\vee-},R_{\rightarrow+},R_{\rightarrow-}, R_{\forall+},R_{\forall-}, R_{Th}\}$ are defined as follows:

Introduction (+) Rules ***Elimination (-) Rules***

$\bot$

$$\frac{X\vdash_{QC}P \quad Y\vdash_{QC}\sim P}{X,Y\vdash_{QC}\bot} \qquad\qquad \frac{X\vdash_{QC}\bot}{X\text{-}\{\sim P\}\vdash_{QC}P}\ (\text{for} P\neq\sim Q)$$

$\sim$

$$\frac{X\vdash_{QC}\bot}{X-\{P\}\vdash_{QC}\sim P} \qquad\qquad \frac{X\vdash\sim\sim P}{X\vdash_{QC}P}$$

$\wedge$

$$\frac{X\vdash_{QC}P \quad Y\vdash Q}{X\vdash_{QC}P\wedge Q} \qquad\qquad \frac{X\vdash_{QC}P\wedge Q}{X\vdash_{QC}P} \qquad \frac{X\vdash_{QC}P\wedge Q}{X\vdash_{QC}Q}$$

$\vee$

$$\frac{X\vdash_{QC}P}{X\vdash_{QC}P\vee Q} \qquad \frac{X\vdash_{QC}P}{X\vdash_{QC}P\vee Q} \qquad\qquad \frac{X\vdash_{QC}P\vee Q \quad Y\vdash_{QC}R \quad Z\vdash_{QC}R}{X,Y-\{P\},Z-\{Q\}\vdash_{QC}R}$$

$\rightarrow$

$$\frac{X\vdash_{QC}P}{X-\{Q\}\vdash_{QC}Q\rightarrow P} \qquad\qquad \frac{X\vdash_{QC}P \quad X\vdash_{QC}P\rightarrow Q}{X\vdash_{QC}Q}$$

$\forall$

$$\frac{X\vdash_{QC}P[t/v]}{X\vdash_{QC}\forall vP} \qquad\qquad \frac{X\vdash_{QC}\forall vP[}{X\vdash_{QC}P[t/v]}$$

where v is not free in any $P\in X$

Thinning
$$\frac{X\vdash_{QC}P}{X,Y\vdash_{QC}P}$$

In addition we add two redundant rules for the existential quantifier:

$\exists$

$$\frac{X\vdash_{QC}P[t//v]}{X\vdash_{QC}\exists v'P} \qquad\qquad \frac{X\vdash_{QC}\exists v'P \quad Y,P[t/v]\vdash_{QC}Q}{X,Y\vdash_{QC}Q}$$ (if t is not free in $X,Y, \exists v'P$ or Q)

We extend the notion of deduction to possibly infinite sets of premises X by saying $X\vdash_{QC}Q$ relative to $\vdash_{QC}$ iff, there is some finite subset $\{P_1,\ldots,P_n\}$ of X such that $P_1,\ldots,P_n\vdash_{QC} Q$.

Theorem 5. The relation $\vdash_{QC}$ is a deducibility relation.

Theorem 6 (Soundness and Completeness). If $\mathcal{L}=<\mathcal{FOLSyn},\mathcal{F}>$ is a (classical) first-order language, $X\models_{\mathcal{L}}P$ iff $X\vdash_{QC}P$.

2. Intensional Semantics for First-Order Logic

Just as for sentential logic, the logic of non-extensional expressions may explained by interposing Boolean matrix of intensions between an extended first-order syntax and its extensional structure. The straightforward way to do so is simply to make intensions into functions from a set K of possible worlds to the first-order extensions. [6] Since the Tarski matrix for extensions is defined in such a way that all interpretations over the matrix have the same domain, the straightforward adaptation of that matrix to intensions has the property that all intensional interpretations end over a matrix interpret expressions over the same domain. Since in the semantics of modal logic the notion of interpretation relative to an alternative world is defined relative to the interpretation of the original world within that matrix, alternative to a worlds end us having the same domain. That is, the domain over possible worlds accessible to one another have the same domain. Existence then becomes necessary: anything that exists in one world exists in every alternative to it. This is not a happy consequence, although it was a feature of the early systems of modal semantics investigated by Carnap and others in the 1940's. We first state this simple flawed theory and then the revisions that avoid unwanted entailments.

If φ is an n-place Boolean operation on subsets of a set *A* then let φ^C be corresponding operation on the characteristic functions of sets: φ^{Int} $(x_1^C,\ldots, x_n^C)= \nearrow(x_1,\ldots, x_n)^C$. Accordingly, $-^C,\cap^C,\cup^C,\Rightarrow^C$ are the operations on characteristic functions of the Boolean operations $-,\cap,\cup,\Rightarrow$.

Definition 5. An ***"robust" intensional first-order matrix relative to a non-empty set*** **K** is any $M=\langle (\mathcal{D}_{con}^{(\mathcal{D}^{\omega})})^K,(\mathcal{D}_{Index}^{(\mathcal{D}^{\omega})})^K,(\mathcal{D}^1)^K,\ldots,(\mathcal{D}^n)^K,\ldots,(P(\mathcal{D}^{\omega}))^K,\{(\mathcal{D}^{\bullet})^K\},g^{Int}{}_{BS},-^{Int},\cap^{Int},\cup^{Int},\Rightarrow^{Int},g^{Int}{}_{\forall}\rangle$ such that

1. $\mathcal{D}$ is non-empty,
2. the functions $-,\cap,\cup,\Rightarrow$ are the set theoretic Boolean operations defined on subsets of $\mathcal{D}^{\omega}$, and $g^{Int}{}_{BS}$ and $g^{Int}{}_{\forall}$ are functions defined as follows:

   $g^{Int}{}_{BS}$ is a function from $(\mathcal{D}^n)^K x(\mathcal{D}^{(\mathcal{D}^{\omega})})^K)_1 x\ldots x(\mathcal{D}^{(\mathcal{D}^{\omega})})^K)_n$ such that

   $g^*{}_{BS}(x,y_1,\ldots, y_n)(k)=\{ s\in\mathcal{D}^{\bullet} \mid \langle y_1(k)(s),\ldots,y_n(k)(s)\rangle\in x(k)\}$

   $g^{Int}{}_{\forall}$ is a function from $(P(\mathcal{D}^{\bullet}))^K$ into $(P(\mathcal{D}^{\bullet}))^K$

   $g^*{}_{\forall}(x,y)(k)=\mathcal{D}^{\bullet}$ if $y(k)=\mathcal{D}^{\bullet}$, and $g^*{}_{\forall}(x,y)(k)=\varnothing$ otherwise.

   (Here *x* makes no contribution.)

   It is customary to identity $\mathcal{D}^{\omega}$ with T, and $\varnothing$ with F.

An ***"robust" modal M,B,S4,*** or ***S5 intensional first-order matrix relative*** is any $M=\langle (\mathcal{D}_{con}^{(\mathcal{D}^{\omega})})^K,(\mathcal{D}_{Index}^{(\mathcal{D}^{\omega})})^K,(\mathcal{D}^1)^K,\ldots,(\mathcal{D}^n)^K,\ldots,(P(\mathcal{D}^{\omega}))^K,\{(\mathcal{D}^{\omega})^K\}, g^{Int}{}_{BS},-^{Int},\cap^{Int},\cup^{Int},\Rightarrow^{Int},g^{Int}{}_{\forall}\rangle$ such that for some $\langle K,\leq\rangle$, $M=\langle (\mathcal{D}_{con}^{(\mathcal{D}^{\bullet})})^K,(\mathcal{D}_{Index}^{(\mathcal{D}^{\bullet})})^K,(\mathcal{D}^1)^K,\ldots,(\mathcal{D}^n)^K,\ldots,(P(\mathcal{D}^{\bullet}))^K,\{(\mathcal{D}^{\bullet})^K\}$, $g^*{}_{BS},-^C,\cap^C,\cup^C,\Rightarrow^C,g^*{}_{\forall}\rangle$ is a "robust" intensional first-order matrix relative to a K ; $\langle K,\leq\rangle$ is a a M-, B-, S4-, or S5-world structure; and $g_{\Box}$ and $g_{\Diamond}$ are as defined in Lecture 4 relative to $\langle K,\leq\rangle$.

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[6] The initial early work by Carnap (see *Meaning and Necessity*) was formalized in the now standard form by Richard Montague. See his papers on intensional logic in Richmond Thomason, ed., *Formal Philosophy* (New Haven; Yale Univ. Press, 1974).

Definition 6. $\mathcal{L}=<\mathcal{FOLS}yn,\mathcal{F}>$ is said to be a ***robust (classical) first-order intensional language*** iff $\mathcal{FOLS}yn$ is a first order syntax and $\mathcal{F}$ is the set of all robust intensional first-order matrices of the same character as $\mathcal{FOLS}yn$.
$\mathcal{L}=<\mathcal{MFOLS}yn,\mathcal{F}>$ is said to be a ***robust modal M,B,S4,*** or ***S5 (classical) first-order intensional language*** iff $\mathcal{MFOLS}yn$ is a first order syntax and $\mathcal{F}$ is the set of all robust modal M-, B-, S4-, or S5-intensional first-order of the same character as $\mathcal{MFOLS}yn$.

Theorem 7. If $\mathcal{L}=<\mathcal{FOLS}yn,\mathcal{F}>$ is a robust (classical) first-order intensional language,
If $\exists!v =_{def} \exists v(Fv\vee{\sim}Fv)$, where F is the 1st predicate of degree 1
($\exists!$ is the ***existence predicate*** and reads "exists"), then $\exists!v \models_{\mathcal{L}} \Box\exists!v$
If $\mathcal{L}=<\mathcal{MOOLS}yn,\mathcal{F}>$ is a robust modal S5 (classical) first-order intensional language,
$\forall v\Box P \models_{\mathcal{L}} \Box\forall vP$ (Equivalently: $\models_{\mathcal{L}} \forall v\Box P\rightarrow\Box\forall vP$ (the ***Barcan formula***)
$\Box\forall vP \models_{\mathcal{L}} \forall v\Box P$ (i.e. $\models_{\mathcal{L}} \forall v\Box P\rightarrow\Box\forall vP$ (the ***converse Barcan formula***)

The standard way to avoid the Barcan formula and its converse in modal logic is to allow the domains of alternative worlds to have different domains. To do so relative to a single matrix homomorphic to syntax requires that there be some way to vary the domain across interpretations. The simplest and most natural way to do this is to incorporate some expression in the syntax that "names" a domain relative to a world. If you think about the universal quantifier, it really serves two functions, First it indicates the domain and then it indicates "how much" of it is being considered. Since in standard first-order logic there is no need to vary the domain from world to world relative to a matrix, there is no need to make explicit that the quantifier indicates the domain. Moreover since only one domain is at issue relative to a matrix, there is also no need to specify that the constants, variables and predicates are all restricted to the elements of this domain.

In modal logic there is, however, a need to vary the domain across worlds in a matrix. There is also another more fundamental application of first-order logic in which it is important to have an expression that indicates the domain of quantification. This is so-called ***(existential presupposition) free logic***.[7] In this version of first-order logic, names need not stand for objects in the domain of quantification, not do predicates need to embrace only objects that "exist" (i.e. are in the domain). For example, *Santa Clause is fat* is true, but Santa Clause does not exist. In free logic *fat* embraces objects outside the domain and *Santa Clause* refers to one of these. Hence existential generalization fails: *Santa Clause is fat* does not entail *There exists and x such that x is fat*.

The easiest way to show how to incorporate domain variation in intensional semantics is to first state the standard extensional semantics for free logic. We begin by adding to the syntax the universal quantifier $\forall$ in its own category $\{\forall\}$ of basic descriptive terms. This we shall list first in the syntax. The grammatical rule $g_{\forall}$ then is expanded to a three argument function. It produces the universal v-closure $\forall vP$ from three items: (1) the descriptive expression $\forall$

[7] See the survey article by Ermanno Bencivenga, "Free Logic", in D. Gabby et al,. *Handbook of Philosophical Logic, vol. III* (Dordrecht: Reidel, 1986).

that stands for a domain, (2) the variable *v* whose extension is irrelevant in this rule, and (3) the sentence *P* which stands for a set of variable assignments.

Definition 7. A $\mathcal{DFOSyn}$ ***(first-order syntax) with descriptive a descriptive quantifier*** any $<\{\forall\},\mathcal{C},\mathcal{V},\mathcal{P}^1,\ldots,\mathcal{P}^n,\ldots,\mathcal{Sen}, f_{BS}, f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow},f_{\forall}>$ such that $\mathcal{C},\mathcal{V},\mathcal{P}^1,\ldots,\mathcal{P}^n,\ldots,$ are disjoint at most denumerable subsets of Σ^*, and $f_{BS}, f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow},f_{\forall}$ are operations on * defined as follows:

$f_{BS}(e_1,\ldots,e_n,e_{n+1})= e_{n+1}e_1\ldots e_n$
$f_{\sim}(x)= \sim x$
$f_{\wedge}(x,y)= (x\wedge y)$
$f_{\vee}(x,y)= (x\vee y)$
$f_{\rightarrow}(x,y)= (x\rightarrow y)$
$f_{\forall}(\forall,e_1,e_2)=\forall e_1e_2,$

A $\mathcal{MDFOSyn}$ is any $<\{\forall\},\mathcal{C},\mathcal{V},\mathcal{P}^1,\ldots,\mathcal{P}^n,\ldots,\mathcal{Sen}, f_{BS}, f_{\Box},f_{\Diamond},f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow},f_{\forall}>$ such that $<\{\forall\},\mathcal{C},\mathcal{V},\mathcal{P}^1,\ldots,\mathcal{P}^n,\ldots,\mathcal{Sen}, f_{BS}, f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow},f_{\forall}>$. For both languages $\mathcal{ASen}=\{ f_{BS}(e_{n+1},\ldots,e_n) \mid e_{n+1}\in \mathcal{P}^n$ & $e_1,\ldots,e_n\in\mathcal{C}\cup\mathcal{V}\}$, and $\mathcal{Sen}$ is the least set including $\mathcal{ASen}$ and closed under $f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow}, f_{\forall}$. We let *c* range over $\mathcal{C}$,*v* over $\mathcal{V}$, and P_i^n over $\mathcal{P}^n$. We introduce the existential quantifier by eliminative definition: $\exists vP$ means $\sim\forall v\sim P$.

Definition 8. A ***free first-order matix*** is any

$M=< P(\mathcal{D}), \mathcal{D}_{con}^{(\mathcal{D}^\omega)}, \mathcal{D}_{Index}^{(\mathcal{D}^\omega)},\mathcal{D}^1,\ldots,\mathcal{D}^n,\ldots,P(\mathcal{D}^\omega),\{\mathcal{D}^\omega\},g_{BS},-,\cap,\cup,\Rightarrow,g_{\forall}>$ such that

1. $\mathcal{D}$ is non-empty (often dropped in free logic),
2. the functions $-,\cap,\cup,\Rightarrow$ are the set theoretic Boolean operations defined on subsets of $\mathcal{D}^\omega$, and g_{BS} and $g_{\forall}$ are defined as follows:

$g_{BS}(R,\varphi_1,\ldots,\varphi_n,)=\{ s\in\mathcal{D}^\omega \mid <\varphi_1(s),\ldots,\varphi_n(s)>\in R\}$
$g_{\forall}(X,v,Y)=\mathcal{D}^\omega$ if $Y=X^\omega$, and $g_{\forall}(X,v,Y)=\varnothing$ otherwise. (Here *v* makes no contribution.) It is customary to identity $\mathcal{D}^\omega$ with T, and $\varnothing$ with F.

Definition 9. $\mathcal{L}=<\mathcal{DFOLSyn},\mathcal{F}>$ is said to be a ***free first-order language*** iff $\mathcal{DFOLSyn}$ is a first order syntax and $\mathcal{F}$ is the set of all free first-order matrices of the same character as $\mathcal{DFOLSyn}$.

Theorems 8. If $\mathcal{L}=<\mathcal{DFOLSyn},\mathcal{F}>$ is a free first-order intensional language,

It is ***not*** the case that if $X\models_{\mathcal{L}} P[t//v]$ then $X\models_{\mathcal{L}} \exists v'P$, but
It is the case that $X\models_{\mathcal{L}} P[t//v]$ then $X,\exists!v'\models_{\mathcal{L}} \exists v'P$

The construction of the intensional matrix to stand between the syntax and extensional structure is then perfectly parallel to the earlier cases.

Definition 10. An ***(existential presupposition) free intensional first-order matix relative to a non-empty set*** **K** is any

$M=<(\mathcal{D}^{\omega})^K,(\mathcal{D}_{con}^{(\mathcal{D}^{\omega})})^K,(\mathcal{D}_{Index}^{(\mathcal{D}^{\omega})})^K,(\mathcal{D}^1)^K,\ldots,(\mathcal{D}^n)^K,\ldots,(P(\mathcal{D}^{\omega}))^K,\{(\mathcal{D}^{\omega})^K\}, g^{Int}{}_{BS},g^{Int}{}_{\Box},g^{Int}{}_{\Diamond}, -^{Int},\cap^{Int},\cup^{Int},\Rightarrow^{Int}, g^{Int}{}_{\forall}>$ such that

1. $\mathcal{D}$ is non-empty (often omitted in free logic),
2. the functions $-,\cap,\cup,\Rightarrow$ are the set theoretic Boolean operations defined on subsets of $\mathcal{D}^{\bullet}$, and $g^{Int}{}_{BS}$ and $g^{Int}{}_{\forall}$ are functions defined as follows:

   $g^{Int}{}_{BS}$ is a function from $(\mathcal{D}^n)^K x(\mathcal{D}^{(\mathcal{D}^{\omega})})^K)_1 x\ldots x(\mathcal{D}^{(\mathcal{D}^{\omega})})^K)_n$ such that

   $g^{*}{}_{BS}(x,y_1,\ldots,y_n)(k)=\{s\in\mathcal{D}^{\omega} \mid <y_1(k)(s),\ldots,y_n(k)(s)>\in x(k)\}$

   $g^{Int}{}_{\forall}$ is a function from $(P(\mathcal{D}^{\bullet}))^K$ into $(P(\mathcal{D}^{\bullet}))^K$

   $g^{Int}{}_{\forall}(x,y,z)(k)=\mathcal{D}^{\omega}$ if $z(k)=x(k)$, and $g^{*}{}_{\forall}(x,y,z)(k)=\varnothing$ otherwise.

   (Here y makes no contribution.)

It is customary to identity $\mathcal{D}^{\bullet}$ with T, and $\varnothing$ with F.

A ***modal M,B,S4,*** or ***S5 (existential presupposition) free intensional first-order matix relative to a non-empty set*** **K** is any $M=<(\mathcal{D}^{\omega})^K,(\mathcal{D}_{con}^{(\mathcal{D}^{\omega})})^K,(\mathcal{D}_{Index}^{(\mathcal{D}^{\omega})})^K,(\mathcal{D}^1)^K,\ldots,(\mathcal{D}^n)^K,\ldots,(P(\mathcal{D}^{\omega}))^K,\{(\mathcal{D}^{\omega})^K\}, g^{Int}{}_{BS},g^{Int}{}_{\Box},g^{Int}{}_{\Diamond}, -^{Int},\cap^{Int},\cup^{Int},\Rightarrow^{Int}, g^{Int}{}_{\forall}>$
such that for some $<K,\leq>$, $<(\mathcal{D}^{\bullet})^K,(\mathcal{D}_{con}^{(\mathcal{D}^{\omega})})^K,(\mathcal{D}_{Index}^{(\mathcal{D}^{\bullet})})^K,(\mathcal{D}^1)^K,\ldots,(\mathcal{D}^n)^K,\ldots,(P(\mathcal{D}^{\bullet}))^K,\{(\mathcal{D}^{\omega})^K\}, g^{Int}{}_{BS}, -^{Int},\cap^{Int},\cup^{Int},\Rightarrow^{Int}, g^{Int}{}_{\forall}>$ is a free intensional first-order matrix relative to K; $<K,\leq>$ is a a M-, B-, S4-, or S5-world structure; and $g_{\Box}$ and $g_{\Diamond}$ are defined relative to $<K,\leq>$.

Definition 11. $\mathcal{L}$=<*DFOLSyn*,$\mathcal{F}$> is said to be ***free first-order intensional language*** iff *DFOLSyn* is a first order syntax and $\mathcal{F}$ is the set of all free intensional first-order matrices of the same character as *DFOLSyn*. $\mathcal{L}$=<*MDFOLSyn*,$\mathcal{F}$> is said to be ***free modal first-order intensional language*** iff $\mathcal{F}$ is the set of all modal M, B, S4, or S5 free intensional first-order matrices of the same character as *MDFOLSyn*.

Theorem 9. If $\mathcal{L}$=<*DFOLSyn*,$\mathcal{F}$> is a free first-order intensional language,
It is ***not*** the case that if $X\models_{\mathcal{L}} P[t//v]$ then $X\models_{\mathcal{L}} \exists v'P$, (existential generalization is invalid) , but

if $X\models_{\mathcal{L}} P[t//v]$ then $X,\exists!v'\models_{\mathcal{L}} \exists v'P$.

If $\mathcal{L}$=<*MDOLSyn*,$\mathcal{F}$> is a modal S5 free first-order intensional language,

not($\forall v\Box P\models_{\mathcal{L}} \Box\forall vP$)

(Equivalently: not$\models_{\mathcal{L}} \forall v\Box P\rightarrow\Box\forall vP$ (the Barcan formula is invalid.)

not($\Box\forall vP\models_{\mathcal{L}} \forall v\Box P$)

(i.e. not$\models_{\mathcal{L}} \forall v\Box P\rightarrow\Box\forall vP$ (the converse Barcan formula is invalid.)

The retaining within intensional logic of all the validities of classical first-order logic, including existential generalization requires a qualification on the matrix format. The desired entailment relation is not one defined over ***all*** valuations relative to ***every*** matrix. Rather it is defined relative to ***a special subset*** of valuations on ***every*** matix, the so called "classically acceptable" valuations of the matrix. These are the valuations (intensional interpretations Int) that obey the classical restriction that the singular terms and predicate extensions are all interpreted within the domain of quantification.

Definition 12. The set $\mathrm{Val}_{\mathcal{I}\text{-}\mathcal{FOL}}$ of ***classically acceptable first-order intensional valuations*** is the set of all Int such that Int is a valuation of some free intensional first-order matix M relative to a non-empty set K such that

$t \in \mathcal{C} \cup \mathcal{V}$, $\mathrm{Int}(t)(k) \in \mathrm{Int}(\forall)$

for any $P^n \in \mathcal{P}^n$, $\mathrm{Int}(P^n)(k) \subseteq \mathcal{D}^n$.

Let $X \models_{\mathcal{I}\text{-}\mathcal{FOL}} P$ iff for all $\mathrm{Int} \in \mathrm{Val}_{\mathcal{I}\text{-}\mathcal{FOL}}$, if for all $Q \in X$, Int$Q$)=T, then Int(P)=T.

Theorem 10. If $\mathcal{L}=<\mathcal{FOLSyn},\mathcal{F}>$ is a free first-order intensional language and $X \models_{\mathcal{L}} P$, then $X \models_{\mathcal{I}\text{-}\mathcal{FOL}} P$.

If $X \models_{\mathcal{I}\text{-}\mathcal{FOL}} P[t//v]$ then $X \models_{\mathcal{I}\text{-}\mathcal{FOL}} \exists v'P$, (existential generalization is valid)

Exercises

1. Prove Theorem 7, Part 1, and that the Barcan inference (or the Barcan formula) is valid.

2. Prove Theorem 8.

3. Prove Theorem 9, Part 1, and that the Barcan inference (or the Barcan formula) is invalid.

Lecture 6

Heyting Lattices and Intuitionistic Logic

1. Lattices

Order and Lattices

Definition 1. By ***a partially ordered structure*** (or a ***partial ordering***) is meant any $<C,\leq>$ such that $\leq$ is a reflexive, symmetric and transitive binary relation on C.

Definition 2. Relative to a partial ordering $<C,\leq>$, $A\subseteq C$, and a,b,c,$\in$C, we say
1. c is an ***upper bound*** of A iff $\forall x\in A$, $X\leq c$;
2. c is ***lower bound*** of A iff $\forall x\in A$, $c\leq x$;
3. c is a ***least upper bound*** (and ***lub***) of A iff c is an upper bound of A and $\forall x$ (if x is an upper bound of A then $c\leq x$);
4. c is a ***greatest lower bound*** (***glb***) of A iff c is a lower bound of A and $\forall x$ (if x is a lower bound of A, then $x\leq c$);
5. $a\wedge b$ (called the ***meet*** of a and b) is the glb of {a,b};
6. $a\vee b$ (called the ***join*** of a and b) is lub of {a,b};
7. $<C,\leq>$ is ***$\wedge$-complete*** iff for $\forall A\subseteq C$, glb of A exists;
8. $<C,\leq>$ is ***$\vee$-complete*** iff for $\forall A\subseteq C$, lub of A exists.

Definition 3
. Relative to a partial ordering $<C,\leq>$,
1. $<C,\wedge>$ is called a ***meet semi-lattice*** iff for any $x,y\in C$, $x\wedge y$ exists;
2. $<C,\vee>$ is called a ***join semi-lattice*** iff for any $x,y\in C$, $x\vee y$ exists;
2. $<C,\wedge,\vee>$ is called a ***lattice*** iff $<C,\wedge>$ is a meet semi-lattice and $<C,\vee>$ is a join semi-lattice.

Theorem 1. $<C,\wedge,\vee>$ is called a lattice iff $\wedge$ and $\vee$ are binary operation on C under which C is closed, and are idempotent, commutative, and associative. That is, for any x, y and z in C,
1. $\wedge,\vee\subseteq C\times C$;
2. $x\wedge y\in C$ and $x\vee y\in C$ (closure);
3. $x\wedge x=x\vee x=x$ (idempotence);
4. $x\wedge y=y\wedge x$, and $x\vee y=y\vee x$ (commutation);
5. $x\wedge(y\wedge z)=(x\wedge y)\wedge z$, and $x\vee(y\vee z)=(x\vee y)\vee z$ (association).

Definition 4. If $<C,\wedge>$/ $<C,\vee>$/ $<C,\wedge,\vee>$ is a meet semi-lattice/join semi-lattice/lattice, then the order relation $x\leq y$ defined as $x\wedge y=x/x\vee y=y/x\wedge y=x$ or $x\vee y=y$ is ***the ordering determined by*** the lattice.

Theorem 2. If $<C,\wedge>$/ $<C,\vee>$/ $<C,\wedge,\vee>$ is a meet semi-lattice/join semi-lattice/lattice and $\leq$ is an ordering determined by the lattice, then $<C,\leq>$ is a partial ordering.

Congruence Relations on Lattices

Definition 5. If $<C,\leq>$ and $<C,\leq>$ are partial orderings, then φ is an ***order*** or ***isotonic homomorphism*** from $<C,\leq>$ into $<C,\leq>$ iff φ is a function from C into C′ and for any x,y, in C, $x\leq y$ iff $\varphi(x)\leq(y)$, and is called ***inverse*** or ***antitonic*** iff any x,y, in C, $x\leq y$ iff $\varphi(y)\leq\varphi(x)$.

Definition 6. Relative to a partial ordering $<C,\leq>$ and an equivalence relation $\equiv$ on C,
1. $\equiv$ is said to be an ***order congruence*** on $<C,\leq>$ iff if $x\equiv x'$ and $y\equiv y'$ and $x\leq y$, then $x'\leq y'$.
2. $\equiv$ is a lattice ***meet congruence relation*** on $<C,\wedge>$ iff $\equiv$ is a binary relation on C, and for any x, x′, y, and y′ in C, if $x\equiv x'$ and $y\equiv y'\in C$, then $x\wedge y\equiv x'\wedge y'$.
3. $\equiv$ is a lattice ***join congruence relation*** on $<C,\vee>$ iff $\equiv$ is a binary relation on C, and for any x, x′, y, and y′ in C, if $x\equiv x'$ and $y\equiv y'\in C$, then $x\vee y\equiv x'\vee y'$.
4. $\equiv$ is a ***lattice congruence relation*** on $<C,\wedge,\vee>$ iff $\equiv$ is a meet congruence relation on $<C,\wedge>$ and a join congruence on $<C,\vee>$.

Theorem 3. A homomorphism from one semi-lattice/lattice to another determines a semi-lattice/lattice congruence relation.
(This follows from an earlier theorem about homomorphism and congruence relations on algebra in general.)

Theorem 4. Relative to a partial ordering $<C,\leq>$, there is some meet/ join/lattice congruence on $<C,\wedge>$/ $<C,\vee>$/ $<C,\wedge,\vee>$ that is not an order congruence on $<C,\leq>$.

(The proof may use the earlier example of a congruence on Boolean operations that is not an order congruence on a Boolean algebra. It suffices since every Boolean algebra is a lattice.)

Filters and Ideals on Lattices

Definition 7. Relative to a partial ordering <C,≤>,
1. **[a]↑={**x | a≤x} and **[a]↓={**x | x≤a};
2. a is an ***upper unit element*** of C iff for any x∈C, x≤a;
3. a is a ***lower unit element*** of C iff for any x∈C, a≤x;
4. if an upper unit element of C exists, it is called **1**;
5. if a lower unit element of C exists, it is called **0**.

Definition 8. If <C,∧>/<C,∨>/<C,∧,∨> is a meet semi-lattice/join semi-lattice/lattice, and x≤y iff x∧y=x/ x≤y iff x∨y=y/ x≤y iff x∧y=x∨y=y, then 1/0 exists for <C,≤>, then 1 is the ***upper unit element*** of <C,∧> and <C,∧,∨>/ 0 is ***the lower unit element*** of <C,∨> and <C,∧,∨>.

Theorem 5. Any finite meet/join semi-lattice has a 1/0 element; and any finite lattice has 1 and 0 elements.

We let <C,∧,1>/<C,∧,∨,1> range over meet semi-lattices/lattices with upper unit element 1; <C,∨,0>/<C,∧,∨,0> range over join semi-lattices/lattices with lower unit element 0; and <C,∧,∨,1, 0> over lattices with upper unit element 1 and lower unit element 0.

Definition 9.
1. If <C,∧,∨,1> is a lattice with upper unit 1 and A is a non-empty subset of C, then
 A is a (***lattice***) ***filter*** on <C,∧,∨,1> iff
 a. for any x,y∈C, if (x∈A & y∈C), then x∨y∈A, and
 b for any x,y∈A, x∧y∈A.
2. If <C,∧,∨,0> is a lattice with lower unit 0 and A is a non-empty subset of C, then
 A is a (***lattice***) ***ideal*** on <C,∧,∨,0> iff
 a. for any x,y∈C, if (x∈A & y∈C), then x∧y∈A, and
 b. for any x,y∈A, x∨y∈A.
3. If <C,∧,∨,1> is a lattice with upper unit 1, <C′,∧′,∨′,1′> is a lattice of with upper unit 1′ is of the same character as <C,∧,∨,1>, and φ is a homomorphism from <C,∧,∨,1> onto <C′,∧′,∨′,1′>, then the ***kernel*** of φ is φ(1).

We let <C,∧,∨,1> ranger over lattices

Theorem 6. If <C,∧,∨,1> is a lattice with upper unit 1, <C′,∧′,∨′,1′> is a lattice of with upper unit 1′ is of the same character as <C,∧,∨,1>, and φ is a homomorphism from <C,∧,∨,1> onto <C′,∧′,∨′,1′>, then the kernel of φ is a filter on <C,∧,∨,1>.

Theorem 7.
1. If <C,∧,∨,1> is a lattice with upper unit 1and ≡ is a lattice congruence relation on C, then {x∈C|x≡1} is a filter on <C,∧,∨,1>.
2. If <C,∧,∨,0> is a lattice with lower unit 0 and ≡ is a lattice congruence relation on C, then {x∈C|x≡0} is an ideal on <C,∧,∨,0>.

Absorption and Distribution

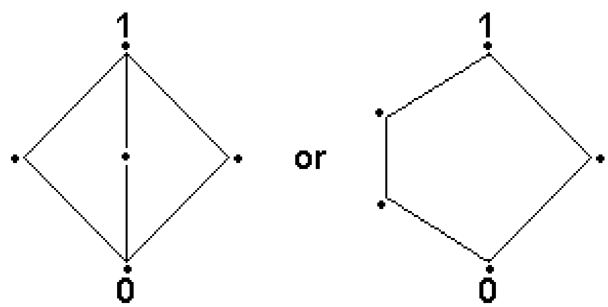
Definition 10. Let $<C,\wedge,\vee>$ be a lattice.

1. The lattice satisfies absorption iff, for all $x,y\in C$, $x\vee(x\wedge y)=x\wedge(x\vee y)=x$,
2. The lattice is distributive iff, for all $x,y,z\in C$, $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$ and $x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$.

Theorem 8. Let $<C,\wedge,\vee>$ be any lattice.

1. The lattice satisfies absorption.
2. The lattice satisfies the distributive inequalities: for any $x,y,z\in C$
   $x\wedge(y\vee z)\leq(x\wedge y)\vee(x\wedge z)$ and
   $(x\vee y)\wedge(x\vee z)\leq x\vee(y\wedge z)$.
3. The lattice is distributive iff any one of the three following (equivalent) conditions holds:
   a. for any $x,y,z\in C$, $(x\wedge y)\vee(x\wedge z)\leq x\wedge(y\vee z)$, and $x\vee(y\wedge z)\leq(x\vee y)\wedge(x\vee z)$,
   b. for any $x,y,z\in C$, $[(x\wedge y)\vee(y\wedge z)\vee(x\wedge z)]=[(x\vee y)\wedge(y\vee z)\wedge(x\vee z)]$,
   c. the lattice contains a sublattice isomorphic to one of those below.

Examples



Paradigm Non-distributive Lattices

Theorem 9. If $<C,\wedge,\vee>$ is a distributive lattice and C is inductively defined in terms of some finite set D and the operations $\wedge$ and $\vee$ (i.e. if for some finite D, $C=\bigcap\{X|\ D\subseteq X$ and for any $x,y\in X$, $x\wedge y,x\vee y\in X\}$), then C is finite.

(This theorem turns on the identities generated by distribution within a lattice and is somewhat tedious to prove. We shall merely assume it here without proof.)

2. Heyting Lattices

Heyting Lattices

Definition 11. If $\langle C,\wedge,\vee\rangle$ be a lattice and $a,b\in C$, then ***relative pseudo-complement*** relation is defined as follows: for $a,b\in C$, $a\Rightarrow b$ (read *"the pseudo-complement of* a *relative to* b") obeys either of the following equivalent conditions:

1. $a\Rightarrow b$ is the largest $x\in C$ such that $a\wedge x\leq b$, or
2. $a\Rightarrow b=x$ iff $\forall y\in C\ (a\wedge y\leq b$ iff $y\leq x)$ and (uniqueness) $\forall z[(\forall y\in C\ (a\Rightarrow y\leq b$ iff $y\leq z)]\rightarrow z=x)$

Definition 12.
If $\langle C,\wedge,\vee,1\rangle$ be a lattice with upper unit elements 1 and φ is a unary operation on C, then φ is an ***upper semi-complement*** operation on the lattice iff $\forall x\in C$, $x\vee\varphi(x)=1$,
If $\langle C,\wedge,\vee,0\rangle$ be a lattice with lower unit elements 0 and $\varphi$is a unary operation on C, then φ is ***a lower semi-complement*** operation on the lattice iff $\forall x\in C$, $x\wedge\varphi(x)=0$,
A lattice $\langle C,\wedge,\vee,0,1\rangle$ with unit elements 0 and 1 is ***complemented*** iff φ is an upper and lower semi-complement of the lattice.

Definition 13. ***A Heyting lattice*** (or ***algebra***) is any $\langle C,-,\wedge,\vee,\Rightarrow,0\rangle$ such that

1. $\langle C,\wedge,\vee,0\rangle$ is a lattice with lower unit element 0,
2. C is closed under relative pseudo-complement $\Rightarrow$, i.e. $\forall x,y\in C\ (x\Rightarrow y\in C)$,
3. – is a unary operation on C, defined as follows: $\forall x\in C$, $\bar{x}$ x□ $=x\Rightarrow 0$. (We usually abbreviate $-x$ as $\bar{x}$, and $--x$ as $\bar{\bar{x}}$).

Theorem 10. If $\langle C,-,\wedge,\vee,\Rightarrow,0\rangle$ is a Heyting lattice,

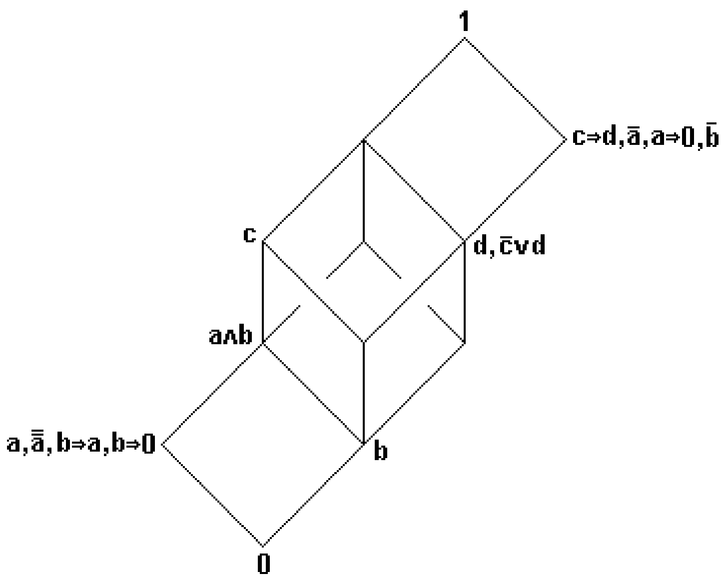
1. There are some Heyting lattices that are not complemented because they do no have an upper semi-complement.
2. x[illegible] is the largest y such that $x\wedge y=0$,
3. C is closed under – (i.e.$\forall x\in C$, $\bar{x}\in C$),
4. – is a lower semi-complement of $\langle C,\wedge,\vee,0\rangle$ (i.e. $x\in C$, $x\wedge\bar{x}=0$).
5. $\langle C,\wedge,\vee\rangle$ is distributive.

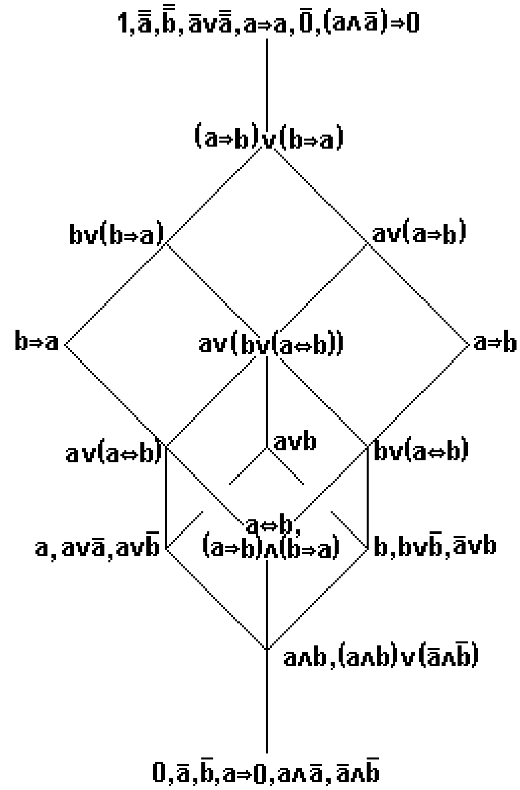
Theorem 11. Any **finite** distributive lattice $\langle C,\wedge,\vee,0,1\rangle$ is a Heyting algebra.

Theorem 12. Let $\langle C,-,\wedge,\vee,\Rightarrow,0\rangle$ be a Heyting lattice and $x,y,z\in C$,

1. $(x\wedge y\leq z)\rightarrow(x\leq y\Rightarrow z)$
2. $y\leq x\Rightarrow y$ (because $x\wedge y\leq y$),
3. $0=\bar{\bar{0}}$
4. $x\wedge\bar{x}=0$,
5. $x\leq -0$, and if C is finite, $-0=1$ and $x\leq 1$.
6. $\bar{x}\vee y\leq x\Rightarrow y$ (because $x\vee(\bar{x}\vee y)=(x\vee\bar{x}$. $\vee y=0\vee y=y$ and $y\leq x\Rightarrow y$),
7. If 1 is an upper unit element of $\langle C,-,\wedge,\vee\rangle$, then $x\wedge y$ iff $x\Rightarrow y=1$,
8. $x\leq y\rightarrow(x\wedge z\leq y\wedge z)$,
9. $x\leq\bar{\bar{x}}$ (because $x\wedge\bar{x}=0$, hence $x\leq\bar{x}\Rightarrow 0$),
10. $x\leq y\rightarrow x\wedge\bar{y}=0$ (because $x\leq y\rightarrow x\wedge\bar{y}\leq y\wedge\bar{y}=0$),
11. $x\leq y\rightarrow\bar{y}\leq\bar{x}$ (because $x\leq y\rightarrow x\wedge\bar{y}=0\rightarrow\rightarrow\bar{y}\leq x\Rightarrow 0=\bar{x}$) .
12. $((x\Rightarrow y)\wedge x\leq y$ (by the definition of $\Rightarrow$),
13. $((x\Rightarrow y)\wedge z)\leq w\rightarrow(x\Rightarrow y)\leq(z\Rightarrow w)$ (by definition of $\Rightarrow$),
14. If C is finite, $1=-0$ and $0=-1$ (because1 is the largest x s.t. $0\wedge x\leq 0$ and 0 is the largest x s.t. $1\wedge x\leq 0$).
15. $x\leq y$ iff $x\Rightarrow y=1$ (If $x\leq y$ and $x\wedge 1=x$, then $x\wedge 1\leq y$ and 1 is the largest such. Hence $x\Rightarrow y=1$. If $x\Rightarrow y=1$, then by def. of $\Rightarrow$, $x\wedge 1=y$. But $x\wedge 1=x$. Hence $x\leq y$.)

Examples of Heyting Lattices





Finite non-complemented Heyting Lattices

2. Intuitionistic Natural Deduction: The Systems IC and N

Definition 14. Prawitz' Intuitionistic Natural Deduction Systems: IC for Sentential Logic and N for First-Order Logic

Relative to a syntax $\mathcal{Syn}$, let $<X,P>$ be a ***deduction*** iff $X\subseteq\mathcal{Sen}$ and $P\in\mathcal{Sen}$. We adopt these abbreviations:

$X\vdash P$	for	$<X,P>$ is in $\vdash$;
$X,Y\vdash P$	for	$X\cup Y\vdash P$;
$X,P\vdash Q$	for	$X\cup\{P\}\vdash Q$;
$P_1,\ldots,P\vdash Q$	for	$\{P_1,\ldots,P\}\vdash Q$;
$\vdash P$	for	$\varnothing\vdash P$.
$P\dashv\vdash Q$	for	$P\vdash Q$ and $Q\vdash P$
$\sim P$	for	$P\rightarrow\bot$
$\exists vP$	for	$\sim\forall v\sim P$
$BD_{\mathcal{Syn}}$	for	$\{<X,P>\mid <X,P>$ is a deduction of $\mathcal{Syn}$ and $P\in X\}$

Various rules are defined as follows:

	Introduction (+) Rules	***Elimination (-) Rules***
$\bot$	$\dfrac{X\vdash_N P \quad Y\vdash_N \sim P}{X,Y\vdash_N\bot}$ (derivable)	$\dfrac{X\vdash_N\bot}{X-\{\sim P\}\vdash_N P}$ (for $P\neq\sim Q$)
$\wedge$	$\dfrac{X\vdash_N P \quad Y\vdash Q}{X\vdash_N P\wedge Q}$	$\dfrac{X\vdash_N P\wedge Q}{X\vdash_N P}$ $\quad$ $\dfrac{X\vdash_N P\wedge Q}{X\vdash_N Q}$
$\vee$	$\dfrac{X\vdash_N P}{X\vdash_N P\vee Q}$ $\quad$ $\dfrac{X\vdash_N P}{X\vdash_N P\vee Q}$	$\dfrac{X\vdash_N P\vee Q \quad Y\vdash_N R \quad Z\vdash_N R}{X,Y-\{P\},Z-\{Q\}\vdash_N R}$
$\rightarrow$	$\dfrac{X\vdash_N P}{X-\{Q\}\vdash_N Q\rightarrow P}$	$\dfrac{X\vdash_N P \quad X\vdash_N P\rightarrow Q}{X\vdash_N Q}$
$\forall$	$\dfrac{X\vdash_N P[t/v]}{X\vdash_N \forall vP}$ where v is not free in any $P\in X$	$\dfrac{X\vdash_N \forall vP[}{X\vdash_N P[t/v]}$

Thinning $\quad \dfrac{X\vdash_N P}{X,Y\vdash_N P}$

In addition we derivable rules for $\sim$ and $\exists$:

$\sim$	$\dfrac{X\vdash_N\bot}{X-\{P\}\vdash_N\sim P}$	$\dfrac{X\vdash_N P \quad Y\vdash_N\sim P}{X,Y\vdash_N\bot}$
$\exists$	$\dfrac{X\vdash_N P[t//v]}{X\vdash_N\exists v'P}$	$\dfrac{X\vdash_N\exists v'P \quad Y,P[t/v]\vdash_N Q}{X,Y\vdash_N Q}$ (if t is not free in $X,Y,\exists v'P$ or Q)

I$=<BD_{\mathcal{SL}\text{-}\mathcal{Sen}},\vdash_{IC},R_{\bot+},R_{\bot-},R_{\sim+},R_{\sim-},R_{\wedge+},R_{\wedge-},R_{\vee+},R_{\vee-},R_{\rightarrow+},R_{\rightarrow-},R_{Th}>$

N$=<BD_{\mathcal{FOL}\text{-}\mathcal{Sen}},\vdash_{N},R_{\bot+},R_{\bot-},R_{\sim+},R_{\sim-},R_{\wedge+},R_{\wedge-},R_{\vee+},R_{\vee-},R_{\rightarrow+},R_{\rightarrow-},R_{\forall+},R_{\forall-},R_{Th}>$

We extend the notion of deduction to possibly infinite sets of premises X by saying $X\vdash Q$ relative to $\vdash$ iff, there is some finite subset $\{P_1,\ldots,P_n\}$ of X such that $P_1,\ldots,P_n\vdash Q$.

Below we shall follow our past practice of providing metatheoretic proofs that are classically valid, using classical logic as that appropriate to metatheory. The proofs in this section however do in fact meet the stronger conditions required of intuitionistic metatheory inasmuch as they are intuitionistically constructive. In particular each claim made that about intuitionistically provable deductions are decidable. This fact is one of a number of very interesting proof theoretic results about intuitionistic logic that we do not have time to prove here. Here are some of the important ones:

Proof Theoretic Results Summarized

By proof theoretic means (which we shall not pursue here) the following are provable.

Theorem 13 The relations $\vdash_I$ and $\vdash_N$ are deducibility relations. (trivial.)

Theorem 14. The relations $\vdash_I$ and $\vdash_N$ are consistent (there is some *P*, such that not $\vdash_I P$ and not $\vdash_N P$; or equivalently not $\vdash_I \bot$ and not $\vdash_N \bot$).

Theorem 15. The relation $\vdash_N$ (and hence $\vdash_I$) is decidable: its characteristic function is calculable.

We turn now to see to what extent matrix semantics may be used to characterize intuitionistic entailment. We shall limit ourselves in this lecture to consideration of only sentential logic and the system I.

3. Argument Soundness and Completeness for Provability in System I.

The first characterization if I-entailment that we review is quite weak. It is a simple application of the general technique of characterizing the provability relation in terms of Lindenbaum algebras.

Review of Facts about Lindenbaum Algebras. (Lecture 2.)

Definition 15. If $\dashv\vdash$ is a congruence relation on M-$\mathcal{Syn}$=<$\mathcal{Sen}$,$\mathbf{Th}_{\vdash}$,~,∧,∨,→>, then the quotient algebra determined by $\dashv\vdash$, namely

$$\mathcal{LM}_{\vdash}=<\{ [P]_{\vdash} \mid P\in\mathcal{Sen}\},\{\mathbf{Th}_{\vdash}\},\sim,\wedge,\vee,\rightarrow>,$$

exists and is called the ***Lindenbaum matrix*** or ***algebra*** for M-$\mathcal{Syn}$.

Theorem 16. If $\dashv\vdash$ is a congruence relation on M-$\mathcal{Syn}$, then the mapping $[\ ]_{\vdash}$ is a strict homomorphism from M-$\mathcal{Syn}$ to $\mathcal{LM}_{\vdash}$..

Theorem 17. If $\mathcal{LM}_{\vdash}$ exists, then $\mathtt{Val}_{\mathcal{M}\vdash} = \{ \bullet\circ[\ ]_{\vdash} \mid \bullet\in \mathbf{Sub}\mathcal{Sen}\}$

Theorem 18. If $\mathcal{LM}_{\vdash}$ exists, then $X \Vdash P$ iff $X \models_{\mathcal{LM}\vdash} P$

Theorem 19.

1. The relation $\dashv\vdash_{I}$ is a congruence relation on the structure M-Syn$_{SL}$=<$\mathcal{Sen}$,**Th**$_{\vdash I}$,~,∧,∨,→>, and hence
2. *2.* the Lindenbaum algebra $\mathcal{LM}_{\vdash I}$ =<{ $[P]_{\vdash I}$ | $P\in\mathcal{Sen}$},{**Th**$_{\vdash I}$},−,∧,∨,⇒> for I exists, and

Theorem 20. Argument Soundness and Completeness for Provability.

$$X \Vdash_{I} P \text{ iff } X \models_{\mathcal{LM}_{\vdash I}} P.$$

Note on Notation: Below, relative to a syntax, we abbreviate $\models_{\mathcal{LM}_{\vdash I}}$ as $\models_{\mathcal{LM}-I}$, and the intuitionistic Lindenbaum matrix $\mathcal{LM}_{\vdash I}$ as $\mathcal{LM}_{I}$, and its entailment relation $\models_{\mathcal{LM}_{\vdash I}}$ as $\models_{\mathcal{LM}-I}$. Notice that ∧, and ∨ are being used to stand for both the syntactic operations of conjunction and disjunction in the syntactic structure and the corresponding operations in the quotient (Lindenbaum) algebra. Below we see that they are genuine meet and join operations, that – is a lower semi-complementation operation, and ⇒ a relative pseudo-complement.

Since the provability relation however is weaker than entailment this characterization is only of minimal interest. We now consider a quite satisfactory argument-soundness result for entailment itself in terms of Lindenbaum algebras.

3. Argument Soundness Relative to Lindenbaum Algebras.

Properties of the Intuitionistic Lindenbaum Algebra

Theorem 21. If $\mathcal{LM}_I$ =<{ $[P]_{\vdash I}$ | $P \in \mathcal{Sen}$},{$\mathbf{Th}_{\vdash I}$},−,∧,∨,⇒> is the intuitionistic Lindenbaum relative to a syntax, then the structure <{ $[P]_{\vdash I}$ | $P \in \mathcal{Sen}$},{$\mathbf{Th}_{\vdash I}$},−,∧,∨,⇒> is such that

1. it is a lattice;
2. for any $[P]_I$, $[Q]_I$ in $[P]_{\vdash I}$ | $P \in \mathcal{Sen}$}, $[P]_I \leq [Q]_I$ iff $P \vdash_I Q$
3. for any $[P]_I$, $[Q]_I$ in $[P]_{\vdash I}$ | $P \in \mathcal{Sen}$}, $[P]_I = [Q]_I$ iff $P \dashv\vdash_I Q$.

(These results follows from the facts about the provability of conjunctions and disjunctions under $\vdash_I$, and the fact that $[\ ]_{\vdash I}$ determines a homomorphism onto its quotient algebra.)

Definition 16. If $\mathcal{LM}_I$ =<{ $[P]_{\vdash I}$ | $P \in \mathcal{Sen}$},{$\mathbf{Th}_{\vdash I}$},−,∧,∨,⇒> is the intuitionistic Lindenbaum algebra relative to a syntax, then H_I is defined as <{ $[P]_{\vdash I}$ | $P \in \mathcal{Sen}$},−,∧,∨,⇒,0> such that $0 = -\mathbf{Th}_{\vdash I}$

Theorem 22. H_I=<{ $[P]_{\vdash I}$ | $P \in \mathcal{Sen}$},−,∧,∨,⇒,0> such that $0 = -\mathbf{Th}_{\vdash I}$ is a Heyting algebra.
(The result follows from the previous theorem, facts about $\vdash_I$, and the definitions of the operations in H_I.)

Theorem 23. In H_I ,

1. $-[P]_I = ([P]_I \Rightarrow 0)$
2. $-0 = 1 = [P \rightarrow P]_I = \mathbf{Th}_{\vdash I}$

Theorem 24. In H_I , for any x,y,z∈$[P]_{\vdash I}$ | $P \in \mathcal{Sen}$},

1. $x \Rightarrow (y \Rightarrow x) = \mathbf{Th}_{\vdash I}$,
2. $x \Rightarrow (y \Rightarrow (x \wedge y)) = \mathbf{Th}_{\vdash I}$,
3. $(x \wedge y) \Rightarrow x = \mathbf{Th}_{\vdash I}$,
4. $(x \wedge y) \Rightarrow y = \mathbf{Th}_{\vdash I}$,
5. $x \Rightarrow (x \vee y) = \mathbf{Th}_{\vdash I}$,
6. $y \Rightarrow (x \vee y) = \mathbf{Th}_{\vdash I}$,
7. $(x \vee y) \Rightarrow ((x \Rightarrow z) \Rightarrow ((y \Rightarrow z) \Rightarrow z)) = \mathbf{Th}_{\vdash I}$,
8. $(x \Rightarrow y) \Rightarrow ((x(y \Rightarrow z)) \Rightarrow (x \Rightarrow z)) = \mathbf{Th}_{\vdash I}$,
9. $(x \Rightarrow y) \Rightarrow ((x \Rightarrow -x) \Rightarrow -x = \mathbf{Th}_{\vdash I}$,
10. $x \Rightarrow (-x \Rightarrow y) = \mathbf{Th}_{\vdash I}$,
11. if $x = \mathbf{Th}_{\vdash I}$ and $x \Rightarrow y = \mathbf{Th}_{\vdash I}$, then $y = \mathbf{Th}_{\vdash I}$, .

Theorem 25. Argument Soundness. Relative to some $Syn_{\mathcal{SL}}$, $\mathcal{LM}_I$ is argument sound for $\vdash_I$. That is,

$$X \vdash_I P \text{ only if } X \models_{\mathcal{LM}-I} P.$$

Proof Sketch. The poof is by induction and follows the usual strategy for soundness theorems. First the basic deductions are shown (trivially) to be valid in $\mathcal{LM}_I$, and then assuming as the inductive hypothesis that the arguments for natural deduction inference rules are valid, showing that the value for the rule is valid. There is an equivalent axiomatic formulation (given a deduction theorem) and an inductive soundness result may be shown for it by showing that each of the axioms of the system (the sentences corresponding to 1-10 in the last theorem) is valid (always designated), and assuming the inputs for modus pones is valid, so is its output (as in item 11 of the last theorem).

Unfortunately Lindenbaum algebras of this sort do not yield a completeness result -- the converse of the above theorem is false. We turn therefore to a characterization in terms of Heyting algebras. It is then possible to obtain a soundness and completeness result for statements only, relative to the large family of matrices associated with finite Heyting algebras. That these algebras is finite is significant for an appreciation of what sort of restrictions on "possible interpretation" is implicit in intuitionistic inference.

5. Statement Soundness and Completeness Relative to the Family of Matrices Associates with Finite Heyting Lattices.

Lemma 1. $H=\langle C,-,\wedge,\vee,\Rightarrow,0\rangle$ is a Heyting lattice iff the logical matrix $M=\langle C,\{1\},-,\wedge,\vee,\Rightarrow\rangle$ with $1=-0$ is argument sound for $\vdash_I$.

Proof. <u>If-Part.</u> We assume for conditional proof that $H=\langle C,-,\wedge,\vee,\Rightarrow,0\rangle$ is a Heyting lattice. We show by induction that the logical matrix $M=\langle C,\{1\},-,\wedge,\vee,\Rightarrow\rangle$ with $1=-0$ is argument sound for $\vdash_I$, that is that for any *X,P,* if
$X\vdash_I P$ then $X\models_M P$. In other words, we show that for any proof in the natural deduction system I, if the proof terminates with the deduction $\langle X,P\rangle$ at its final (root) node, then $X\models_M P$. This last formulation we show by the usual strategy in a soundness proof: we do an induction on the length of proof in I. We show first (in the basis step) that if $\langle X,P\rangle$ is a basic deduction then $X\models_M P$. In the inductive step, we assume as the hypothesis of induction that all deductions at the line immediately above $\langle X,P\rangle$ are M-valid, and then show that $\langle X,P\rangle$ must be M-valid. In other words we show (as in the usual soundness proof) that the rules preserve validity. <u>Basis Step</u>. Trivial: since $P\in X$, for any $\mathtt{v}\in \mathtt{Val}_M$, if for all $Q\in X$, $\mathtt{v}(Q)=1$, then $\mathtt{v}(P)=1$. <u>Inductive Step.</u> The argument for one two rule will be given for illustration. The others are more direct. <u>The Rule $\rightarrow$+ Introduction.</u> Assume $X\models_M P$, and if for all $S\in X-\{Q\}$, $\mathtt{v}(S)=1$. There are two cases. Case 1: $Q\notin X$. $\mathtt{v}(P)\leq\mathtt{v}(Q)\Rightarrow\mathtt{v}(P)$. Hence, $\mathtt{v}(Q)\Rightarrow\mathtt{v}(P)=1=\mathtt{v}(Q\rightarrow P)$. Case 2: $Q\in X$. Let $X=\{S_1,\ldots,S_n,Q\}$. Hence $\mathtt{v}(S_i)=1$ and $\mathtt{v}(P)\wedge\mathtt{v}(S_1)\wedge\ldots\wedge\mathtt{v}(S_n)=\mathtt{v}(P)\leq\mathtt{v}(Q)\Rightarrow\mathtt{v}(P)=\mathtt{v}(Q\rightarrow P)$. Moreover, since $\mathtt{v}(P)\leq\mathtt{v}(Q)\Rightarrow\mathtt{v}(P)$, $\mathtt{v}(Q)\Rightarrow\mathtt{v}(P)=1$. Hence $\mathtt{v}(Q\rightarrow P)=1$.
<u>Then-Part.</u> Assume the logical matrix $M=\langle C,\{1\},-,\wedge,\vee,\Rightarrow\rangle$ with $1=-0$ is argument sound for $\vdash_I$. It must now be shown that $H=\langle C,-,\wedge,\vee,\Rightarrow,0\rangle$ has the defining features of a Heyting lattice. The details are straightforward.

QED

Corollary 26. Argument Soundness. If $X\vdash_I P$, then for any associated matrix $M=\langle C,\{1\},-,\wedge,\vee,\Rightarrow\rangle$ with $1=-0$ of a Heyting lattice $H=\langle C,-,\wedge,\vee,\Rightarrow,0\rangle$, $X\models_M P$.

Corollary 27. Statement Soundness. If $\vdash_I P$, then for any finite Heyting lattice $H=\langle C,-,\wedge,\vee,\Rightarrow,0\rangle$, its associated matrix $M=\langle C,\{1\},-,\wedge,\vee,\Rightarrow\rangle$ with $1=-0$ is such that $\models_M P$.

Lemma 2. If not($\vdash_I P$), then there is some finite Heyting lattice H=<C,–,∧,∨,⇒,0> with an associated logical matrix M=<C,{1},–,∧,∨,⇒> with 0=–1 such that not($\models_M P$).

Proof. Suppose not($\vdash_I P$). Consider the Lindenbaum algebra H_I=<{ $[P]_{\vdash_I}$ | $P \in \mathcal{Sen}$},–,∧,∨,⇒,0> such that 0=–$\mathbf{Th}_{\vdash_I}$ (which is a Heyting algebra), and its associated logical matrix M_I=<$[P]_{\vdash_I}$ | $P \in \mathcal{Sen}$},{ $\mathbf{Th}_{\vdash_I}$ },–,∧,∨,⇒>. Consider further $\mathtt{v} \in \mathtt{Val}_{M_I}$ such that for any P, $\mathtt{v}(P)=[P]_I$. Let P, be a sentence and consider now the restriction $H_I(P)$ if H_I. That is $H_I(P)$=< {C_P,–,∧,∨,⇒*,0> such that C_P is defined inductively as the closure of the set of subformulas of P (i.e. as the closure of $[Q]_{\vdash_I}$ | $Q \in \mathcal{Sen}$ and Q is a subformula of P}) under the operations ∧ and ∨, and ⇒* is defined as the relative pseudo-complement defined in terms of ∧ and ∨ in C_P. This structure is a Heyting lattice. Since it is distributive and the set of P's subformulas is finite, C_P is also finite. It is also straightforward to show:

For any x,y∈C_P, x⇒*y = x⇒y.

If f is a n-placed function and A is a set, let $f|A$ (read "***the restriction of f to A***" be <$x_1,\ldots,x_{n+1}$> such that <$x_1,\ldots,x_n$>∈A. Let $M_{I(P)}$=< C_P,{ $\mathbf{Th}_{\vdash_I} \cap C_P$,},–,∧,∨,⇒*>.

Then, by induction on the length of formulas, exploiting the identity of the operations over both algebras, it may be shown that

$\mathtt{v}|C_P \in \mathtt{Val}_{M_{I(P)}}$, and

for any $\mathtt{v} \in \mathtt{Val}_{M_I}$,any P, and any subformula Q of P, $\mathtt{v}|C_P(Q)=\mathtt{v}(Q)$.

Now by the original assumption, not($\vdash_I P$). Hence for some $\mathtt{v} \in \mathtt{Val}_{M_I}$, $\mathtt{v}(P) \neq 1$, and therefore $\mathtt{v}|C_P \in \mathtt{Val}_{M_{I(P)}}$ is such that $\mathtt{v}|C_P(P) \neq 1$. Hence, not($\models_{M_{I(P)}} P$). Hence for some finite Heyting lattice its associated matrix M is such that not($\models_M P$).

QED

Corollary 28. Statement Completeness. If $\mathcal{F}_{FHL}$ be the set of all logical matrices associated with any finite Heyting Lattices, $\mathcal{Syn}_{SL}$ is a sentential syntax, and $\mathcal{L}_{FHL}$ is the language <$\mathcal{Syn}_{SL}$, $\mathcal{F}_{FHL}$>,then

$$\models_{\mathcal{L}_{FHL}} P \text{ only if } \vdash_I P$$

(The corollary is the contrapositive of the previous lemma.)

Theorem 29. (Statement Soundness and Completeness). If $\mathcal{F}_{FHL}$ be the set of all logical matrices associated with any finite Heyting Lattices, $\mathcal{Syn}_{SL}$ is a sentential syntax, and $\mathcal{L}_{FHL}$ is the language <$\mathcal{Syn}_{SL}$, $\mathcal{F}_{FHL}$>,then

$$\vdash_I P \text{ only if } \models_{\mathcal{L}_{FHL}} P$$

Theorem 30. (Finite Argument Soundness and Completeness). If $\mathcal{F}_{FHL}$ be the set of all logical matrices associated with any finite Heyting Lattices, $\mathcal{Syn}_{SL}$ is a sentential syntax, $\mathcal{L}_{FHL}$ is the language <$\mathcal{Syn}_{SL}$, $\mathcal{F}_{FHL}$>, and X is finite, then

$$X \vdash_I P \text{ only if } X \models_{\mathcal{L}_{FHL}} P$$

(The result follows from the previous theorem and the deduction theorem for intuitionistic logic (if $X=\{P_1,\ldots,P_n\}$ is finite, then $X \vdash_I Q$ iff $\vdash_I P_1 \wedge \ldots \wedge P_n \rightarrow Q$, which is a straightforward result in the proof theory.)

Exercises. Prove the following

1. Theorem 10:2
2. Theorem 12: 3, 8, 13.
3. Theorem 19:1
4. Theorem 22:2
5. Theorem 23:1
6. Theorem 24: 6
7. Lemma 1.
    a. If-Part: the case for ~ Introduction.
    b. Then-Part: Show ⇒ is a relative pseudo-complement.

Lecture 7

The Representation of Finite Heyting Lattices in Topological Spaces

In the previous lecture intuitionistic logic was weakly characterized by finite Heyting lattices in the sense that it was shown to be statement sound and complete for the family of such lattices. It this lecture this result will be strengthened.[1]

A special subset of Heyting lattices will be singled out and shown to adequate for the characterization. This result is much like that of an earlier lecture in which the general characterization of classical logic by the family of all Boolean algebras of sets was strengthened by limiting this set first of the family of powerset algebras, and then to the single classical algebra over {0,1}. The special Heyting algebras sufficient for intuitionistic logic are the topological Heyting algebras generated by partially ordered sets. As in the classical case, the result is shown in the form of a representation theorem: every finite Heyting lattice will be shown to be isomorphic to some finite topological Heyting lattice. The statement soundness and completeness result previously proven for the wider family then carry over to the narrower.

To introduce the structure used in the representation, we must begin with some ideas from topology.

1. Topological Ideas.

The original motivation for topology was to abstract from key ideas in real analysis like those of limit and continuity, in an attempt to characterize these notions without an appeal to the ideas of metric (distance) or order (the less than relation). In place of the open interval found in the traditional definition, topology uses the idea of an open set. Since it is open sets that form the elements of the Heyting algebras we shall be discussing, it is worth pausing to gain some understand the idea.

It is customary to introduce the ideas of open set indirectly, much as the notion of open interval presupposes prior notions of order.

Definition. A ***topological space*** is any structure $<H,\vartheta>$ such that
1. $\vartheta\subseteq P(H)$
2. ϑ is closed under finite intersection, i.e. $\forall x,y\in\vartheta$, $x\cap y\in\vartheta$,
3. ϑ is closed under infinite union, i.e. $\cup X_{X\subseteq\vartheta}\in\vartheta$;

and $X\in\vartheta$ is said to be an ***open set*** in $<H,\vartheta>$.

[1] The ideas here and in Lecture 8 are developed in Michael Dummett, *Elements of Intuitionism* (Oxford: Clarendon Press, 1977).

The closed interval [n-a,n+a], or the set of points that are within the distance n from d (i.e. $\{n| \ |n-a|\leq d \}$ (in two dimensions), and the ball with center a and the ball with radius d abstract to what are called "neighborhoods."

Definition. if $X\subseteq H$ and $\langle H,\vartheta\rangle$ is a topological space, then we say X is ***a neighborhood of*** a (briefly, $X=\mathbf{N}a$) iff $\exists Y\subseteq X\ (Y\in\vartheta\ \&\ a\in Y)$.

In the following theorem the metric concepts of open interval (in two dimensions) or ball boundary (in three dimensions) is abstracted to that of an open set.

Theorem. In a topological space $\langle H,\vartheta\rangle$, X is open iff $\forall y\in X, \exists\mathbf{N}y\subseteq X$.

What the theorem says in abstract terms is that X is "open" iff you can construct a "ball" around each point in X.

Definition X is closed in a topological space $\langle H,\vartheta\rangle$ iff $H-X$ is open.

Theorem. The set of closed sets in $\langle H,\vartheta\rangle$ is the family of subsets of H that is closed under infinite intersection and finite union.

Theorem. The following are equivalent relative to a topological space $\langle H,\vartheta\rangle$:

1. X is closed in $\langle H,\vartheta\rangle$
2. $-X$ is open in $\langle H,\vartheta\rangle$
3. $\forall y\in -X, \exists\mathbf{N}y\subseteq -X$
4. $\forall y\in -X, \exists\mathbf{N}y\cap X=\varnothing$
5. $\forall y\in H, (\forall\mathbf{N}y, \mathbf{N}y\cap X=\varnothing \rightarrow y\in X)$

We that a point a is a limit point of a set X , in less abstract terms, when every ball with radius a intersects X.

Definition. If $\langle H,\vartheta\rangle$ is a topological space and $X\subseteq H$, then a is limit point of X iff $\forall\mathbf{N}a, \exists y\in X\ (y\neq a)$

Theorem. If $\langle H,\vartheta\rangle$ is a topological space and $X\subseteq H$, then X is closed iff $\{y|\ y$ is a limit point of $X\}\subseteq X$.

Definition. An operation c from P(H) into P(H) is called a ***closure operation*** (on H) iff for any X,Y⊆H

1. $\varnothing^c = \varnothing$,
2. $X \subseteq X^c$,
3. $(X \cup Y)^c = X^c \cup Y^c$.

Theorem. If c is a closure operation on H and $\vartheta = \{-Y \mid Y^c = X\}$, then $\langle H, \vartheta \rangle$ is a topological space and for any Y⊆H, Y^c is a closed in $\langle H, \vartheta \rangle$.

We now define the "interior" operation I on subsets X of H.. Intuitively I(X) it draws into a set all the points y in X that have some "ball" with "center" y inside of X.

Definition. If $\langle H, \vartheta \rangle$ is a topological space and X⊆H, then a is ***an interior point*** to X iff X=**N**a, and I is an ***interior operation*** on P(H) defined as follows: ∀X⊆H,

$$I(X) = \{y \mid y \text{ is an interior point of } X\} = \{y \mid X = \mathbf{N}y\}$$

Theorem. If $\langle H, \vartheta \rangle$ is a topological space and let X⊆H, then

1. I(X) is an open set'
2. $I(X) = \bigcap\{Y \mid Y \text{ is open and } Y \subseteq X\}$
3. X is open iff X=I(X),
4. $I(X) = \{y \in X \mid y \text{ is not a limit point of } -X\}$
5. $(-X)^c = I(X)$.
6. $-I(X) = (-X)^c$
7. $I(X) = -((-X)^c)$
8. $X^c = -I(-X)$

Tough we do not have time here to illustrate the applications of these ideas , e.g. in the definition of the limit of a function and of a continuous function, this introduction is more than enough for the purposes of the representation theorem for Heyting lattices. (See the Appendix to the lecture for an discussion of the definition of limit.)

2. Topological Spaces of Partially Ordered Sets and their Heyting Lattices.

The ordering relation on a partially ordered set may be used to define and interior operation. This in turn generates a topological space: the set together with the open sets generated by the interior operation. The topological space in turn yields a Heyting algebra: ∧,∨ and 0 are just set theoretic ∩,∪ and ∅ on the open sets, and there is a simple way to define a pseudo-complementation operation ⇒. It is such topological algebras that will be shown to represent arbitrary Heyting lattices.

Definition. If $<H,\leq>$ is a pre/partial ordering and $A\subseteq H$, then $I^H_{\leq}(A)=\{x\in A|\ (\forall y\leq x)(y\in A)\}$

Note: Since $\leq$ is reflexive, $I^H_{\leq}(A)=\{x|\forall y\leq x,\ y\in A\}$

Theorem. $I^H_{\leq}$ is an interior operation.

Theorem. $<H,\vartheta_{\leq}>$ such that $\vartheta_{\leq} = \{A\subseteq H|\ I^H_{\leq}=A\}$ is a topological space.

Definition. If 1. $<H,\leq>$ is a pre/partial ordering,

2. $I^H_{\leq}(A)=\{x|\forall y\leq x,\ y\in A\}$, and

3. $\vartheta_{\leq} = \{A\subseteq H|\ I^H_{\leq}=A\}$,

then $<H,\vartheta_{\leq}>$ is called ***the topological space determined by*** $<H,\leq>$.

Definition. If $<H,\vartheta_{\leq}>$ is the topological space determined by $<H,\leq>$, ***the*** I***-relative pseudo complement*** (***relative to is*** $<H,\vartheta_{\leq}>$) is the binary operation $\Rightarrow$ on $\vartheta_{\leq}$ defined as follows: $A\Rightarrow B = I^H_{\leq}\{x|$ if $x\in A$, then $y\in A\}$.

Theorem. If $<H,\vartheta_{\leq}>$ is the topological space determined by $<H,\leq>$, then the I-relative pseudo complement $\Rightarrow$ on $\vartheta_{\leq}$ (relative to is $<H,\vartheta_{\leq}>$) is a relative pseudo-complementation operation under which $\vartheta_{\leq}$ is closed.

Proof. The result is shown in two steps. Note that the partial ordering on $\vartheta_{\leq}$ of L^{I}_{C} is the subset relation $\subseteq$.

Lemma1. For any x, $x\in A\Rightarrow B$ iff $\forall y$(if $y\leq x$ & $y\in A$, then $y\in B$)

Proof. $x\in A\Rightarrow B$	iff	$x\in I^C_{\leq}(\{z\|\ z\in A$ only if $z\in B\})$	(by def of $\Rightarrow$)
	iff	$x\in\{w\|\ \forall y\leq w(y\in \{z\|\ z\in A$ only if $z\in B\}\ )\ \}$	(by def of $I^C_{\leq}$)
	iff	$x\in\{w\|\ \forall y\leq w\ (y\in A$ only if $y\in B)\}$	(by abstraction)
	iff	$x\in\{w\|\ \forall y$ (if $y\leq w$ & $y\in A$, then $y\in B)\}$	(by logic)
	iff	$\forall y$(if $y\leq x$ & $y\in A$, then $y\in B$)	(by abstraction)

Lemma 2. $\Rightarrow$ is a relative pseudo-complementation operation.

Proof. That $A\cap A\Rightarrow B\leq B$ follows from the fact that $x\in A$ and $\forall y$(if $y\leq x$ & $y\in A$, then $y\in B$) entails $x\in B$, given that $x\leq x$. In addition, for any C, $A\cap C\subseteq B$ entails that $C\leq A\Rightarrow B$. Let assume $A\cap C\subseteq B$. Let $x\in C$ and for arbitrary y let $y\leq x$ & $y\in A$. Since $x\in C$, $y\leq x$, and C is closed downward under $\leq$, $y\in C$. But then $y\in A\cap C$, and by the hypothesis $y\in B$. Hence, if $x\in C$, then $\forall y$(if $y\leq x$ &$y\in A$, then $y\in B$) and by lemma 1, $x\in A\Rightarrow B$. **QED**

Definition. If $<C,\leq>$ is a partial ordering and is a Heyting Lattice and $<C,\vartheta_{\leq}>$ is the topological space determined by $<C,\leq>$, then the ***topological Heyting Lattice determined by* L** is $L^{I}_{C} = <\vartheta_{\leq}, \cap,\cup,\varnothing,\Rightarrow>$ such that $\cap$ and $\cup$ are set theoretic intersection and union on $\vartheta_{\leq}$, $\varnothing$ is the empty set and $\Rightarrow$ is and the I-relative pseudo complement relative to is $<C,\vartheta_{\leq}>$.

Theorem. If <C,≤> is a partial ordering and L^{I}_{C} = <$\vartheta_{\leq}$, ∩,∪,∅,⇛> is the topological Heyting Lattice determined by L then

1. $I^{C}_{\leq}$ is an interior operation,
2. <C,$\vartheta_{\leq}$> is a topological space, and
3. L^{I}_{C} = <$\vartheta_{\leq}$, ∩,∪,∅,$\Rrightarrow_{\vee}$> is a Heyting algebra.

(The result is immediate from previous theorems and definitions.)

3. The Representation Theorem.

Given an arbitrary Heyting lattice <C,∧,∨,0,⇒> we may recover its partial ordering <C,.≤> and then apply this theorem to generate a topological Heyting lattice of open sets<$\vartheta_{\leq}$, ∩,∪,∅,⇛>. This lattice however will not be homomorphic to <C,∧,∨,0,⇒> because $\vartheta_{\leq}$ has in general a greater cardinality than C. We can however reduce C to a subset C′ so that <C′,.≤> generates a topological Heyting algebra isomorphic to <C,∧,∨,0,⇒>. We do so by restricting attention to subset $C_{\vee}$ of C that contains what are called the ∨-irreducible elements of C. These are the C-elements *e* such that in the graph of C, there is at most one ascending path join *e* from below.

Definition. If <C,∧,∨,0,⇒> is a Heyting Lattice, then **$C_{\vee}$,** the set of ∨ ***(join) irreducible elements of* C** is defined as follows:

$C_{\vee}$ = {x∈C| x≠0 & (∀y,z∈C)(if x=y∨z then either x=y or x=z}

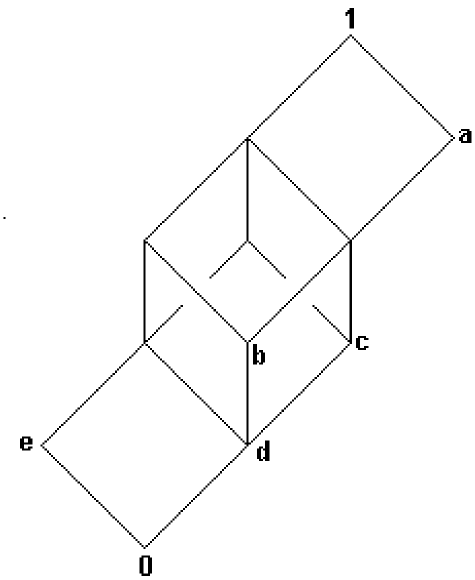
Definition. If L=<C,∧,∨,0,⇒> is a Heyting Lattice, its **v-*irreducible restriction* $L_{\vee}$** is <$C_{\vee}$,∧|$C_{\vee}$, ∨|$C_{\vee}$>.

Theorem. If L=<C,∧,∨,0,⇒> is a Heyting Lattice, its v-irreducible restriction $L_{\vee}$ = <$C_{\vee}$,∧|$C_{\vee}$, ∨|$C_{\vee}$> is a lattice with partial ordering $\leq_{\vee}$ = ≤|$C_{\vee}$.

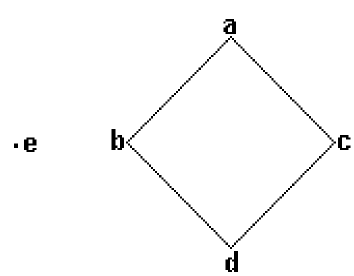
Definition. If L=<C,∧,∨,0,⇒> is a Heyting Lattice and $L_{\vee}$ = <$C_{\vee}$,∧|$C_{\vee}$, ∨|$C_{\vee}$> is its v-restriction, then ***the* v-*restricted partial ordering determined by* L** is <$C_{\vee}$,$\leq_{\vee}$ > such that ≤ is the partial ordering determiend by $L_{\vee}$..

Theorem. If L=<C,∧,∨,0,⇒> is a Heyting Lattice, $L_{\vee}$ = <$C_{\vee}$,∧|$C_{\vee}$, ∨|$C_{\vee}$> is the v-restriction, and <$C_{\vee}$,$\leq_{\vee}$ > is the v-restricted partial ordering determined by L, then $L^{I}_{C_{\vee}}$ = <$\vartheta_{\leq\vee}$, ∩,∪,∅,⇛>, the ∨ (join) restrictive topological Heyting Lattice determined by L, is is is well defined.

Example. In the following Heyting Lattice L, the elements **a,b,c,d,** and **e** of its domain C are v-irreducible, and $C_\vee$={**a,b,c,d,e**}.



Example. The partially ordered structure $\langle C_\vee, \leq_\vee \rangle$, with $C_\vee$={**a,b,c,d,e**}, determined by the lattice L of the previous example is:



Note that not all elements of this structure are connected -- the element **e** is not "≤-comparable" to any of the other elements.

We now apply the previous results to the restricted partial ordering $<C_\vee,\leq_\vee>$ to generate a topological Heyting lattice.

Theorem. If $L=<C,\wedge,\vee,0,\Rightarrow>$ is a Heyting Lattice and $L^{I}_{C} = <\vartheta_{\leq\vee}, \cap,\cup,\varnothing,\Rrightarrow>$ is its $\vee$ (join) restrictive topological Heyting Lattice determined by L, i.e.

1. For any $A\subseteq C$, $I^{C_\vee}_{\leq_\vee}(A) = \{x|\ x\in A\ \&\ (\forall y\leq_\vee x)(y\in A)\}$,
2. $\vartheta_{\leq\vee} = (A|\ A\subseteq C_\vee\ \&\ I^{C_\vee}_{\leq_\vee}(A)=A\}$,
3. $\cap,\cup$, and $\varnothing$ are set theoretic intersection, union and the empty set on $\vartheta_{\leq\vee}$,
4. $A\Rrightarrow B = I^{C_\vee}_{\leq_\vee}(x|$ if $x\in A$ then $x\in B\}$, a relative pseudo complementation operation.

then

1. $I^{C_\vee}_{\leq_\vee}$ is an interior operation,
2. $<C_\vee,\vartheta_{\leq\vee}>$ is a topological space, and
3. $L^{I}_{C_\vee} = <\vartheta_{\leq\vee}, \cap,\cup,\varnothing,\Rrightarrow>$ is a Heyting algebra.

(The theorem follows directly from earlier results.)

Theorem. Representation of Finite Heyting Lattices in Lattices of Topological Spaces. Any finite Heyting lattice $L=<C,\wedge,\vee,0,\Rightarrow>$ is isomorphic to its join restrictive topological Heyting lattice $L^{I}_{C} = <\vartheta_{\leq\vee}, \cap,\cup,\varnothing,\Rrightarrow>$.

Proof. Assume that $L=<C,\wedge,\vee,0,\Rightarrow>$ is a Heyting lattice and that $L^{I}_{C} = <\vartheta_{\leq\vee}, \cap,\cup,\varnothing,\Rrightarrow>$ is its join restrictive topological Heyting lattice. We define a mapping φ from C into $\vartheta_{\leq\vee}$:

$$\varphi(x) = \{y|\ y\in C_\vee\ \&\ y\leq x\}$$

We show that is an isomorphism. Note that by definition if $\in C_\vee$, then $y\in\varphi(x)$ iff $y\leq x$.

<u>Claim 1.</u> $\text{Range}(\varphi)\subseteq\vartheta_{\leq\vee}$. Assume $A\in \text{Range}(\varphi)$, i.e. that for some a, $\varphi(a)=A$. We show $A\in\vartheta_{\leq\vee}$, i.e that (1) $A\subseteq C_\vee$, and (2) $I^{C_\vee}_{\leq_\vee}(A) =A$. For (1) assume that $x\in A$. Hence $x\in\varphi(a)$, and by def., $x\in\{y|\ y\in C_\vee\ \&\ y\leq a\}$. Hence $x\in C_\vee$. For (2) observe first the trivial fact that for any x, $(x\in C_\vee\ \&\ x\leq a)$ iff $[x\in C_\vee\ \&\ x\leq a\ \&\ (\forall w\leq_\vee x)(w\in C_\vee\ \&\ w\leq a)]$. We show by extensionality that $I^{C_\vee}_{\leq_\vee}(A)=A$. Recall that $A=\varphi(a)= \{y|\ y\in C_\vee\ \&\ y\leq a\}$.

$x\in A$	iff	$x\in\{y\|\ y\in C_\vee\ \&\ y\leq a\}$,
	iff	$x\in C_\vee\ \&\ x\leq a$,
	iff	$x\in C_\vee\ \&\ x\leq a\ \&\ (\forall w\leq_\vee x)(w\in C_\vee\ \&\ w\leq a\})\}$,
	iff	$x\in\{z\|\ z\in C_\vee\ \&\ z\leq a\ \&\ (\forall w\leq_\vee z)(w\in C_\vee\ \&\ w\leq a\})\}$,
	iff	$x\in\{z\|\ z\in\{y\|\ y\in C_\vee\ \&\ y\leq a\}\ \&\ (\forall w\leq_\vee z)(w\in\{y\|\ y\in C_\vee\ \&\ y\leq a\})\ \}$,
	iff	$x\in I^{C_\vee}_{\leq_\vee}(\{y\|\ y\in C_\vee\ \&\ y\leq a\})$,
	iff	$x\in I^{C_\vee}_{\leq_\vee}(A)$.

<u>Claim 2.</u> For any $a,b\in C_\vee$, $\varphi(a\wedge b)= \varphi(a)\cap \varphi(b)$. Now, $\varphi(a\wedge b) = \{x|\ x\in C_\vee\ \&\ x\leq a\wedge b\} = \{x|\ x\in C_\vee\ \&\ x\leq a\ \&\ x\leq b\} = \{x|\ x\in C_\vee\ \&\ x\leq a\}\cap\{x|\ x\in C_\vee\ \&\ x\leq b\} = \varphi(a)\cap \varphi(b)$. <u>Claim 3.</u> $\varphi(a\vee b)= \varphi(a)\cup \varphi(b)$. Since $\leq$ is a partial ordering, ($x\leq a$ or $x\leq b$) entails $x\leq a\vee b$. But the converse does not generally hold for partial orderings. But the definition of $C_\vee$, however, bridges the gap. If $x\in\varphi(a\vee b)$, then $x\in C_\vee$ &

$x \leq a \vee b$, and $x \vee (a \vee b) = x = (x \vee a) \vee b$. Since $x \in C_\vee$, x is v-irreducible, and thus ($x = x \vee a$ or $x = b$) But the former entails $x \leq a$ and the latter $x \leq b$. Hence, $x \in C_\vee$ & $x \leq a$ or $x \leq b$. That is, ($x \in C_\vee$ & $x \leq a$) or ($x \in C_\vee$ & $x \leq b$). Hence, $x \in \varphi(a) \cap \varphi(b)$. The reverse entailment is trivial. Claim 4. $\varphi(a \Rightarrow b) = \varphi(a) \Rrightarrow \varphi(b)$. Proof is by extensionality. If-Part: Assume $x \in \varphi(a \Rightarrow b)$. Hence, $x \leq a \Rightarrow b$ and $a \wedge a \Rightarrow b \leq b$. For arbitrary y assume for CP that $y \leq x$ and $y \in \varphi(a)$. Since $y \leq x$ and $x \leq a \Rightarrow b$, $y \leq a \Rightarrow b$. Since $y \in \varphi(a)$, $y \in C_\vee$ and $y \leq a$. Thus, $y \leq a$ & $y \leq a \Rightarrow b$. Hence $y \leq a \wedge a \Rightarrow b$ and $a \wedge a \Rightarrow b \leq b$. Thus, $y \in C_\vee$ and $y \leq b$, and $y \in \varphi(b)$. Since y was arbitrary, by CP, $\forall y$(if $y \leq x$ & $y \in \varphi(a)$, then $y \in \varphi(b)$), that is (by an earlier lemma) $x \in \varphi(a) \Rrightarrow \varphi(b)$. Then-Part: Assume $x \in \varphi(a) \Rrightarrow \varphi(b)$. Hence, $x \in I^{C_\vee}_{\leq_\vee}(\{w |$ if $w \in \varphi(a)$ then $w \in \varphi(b)\})$, and therefore $x \in I^{C_\vee}_{\leq_\vee}(\{w |$ if $w \leq a$ then $w \leq b\})$. Thus, $x \in \{z | \forall y \leq z (y \in \{w |$ if $w \leq a$ then $w \leq b\}$, $x \in \{z | \forall y$(if $y \leq z$ & $y \leq a$ then $y \leq b)\}$, $\forall y$(if $y \leq x$ & $y \leq a$ then $y \leq b$), and $\forall y$(if $y \leq x \wedge a$ then $y \leq b$). Hence $x \wedge a \leq b$ and $x \leq a \Rightarrow b$. That is, $x \in \varphi(a \Rightarrow b)$. **QED**

Theorem. (Finite Argument Soundness and Completeness). If $\mathcal{F}_{FTHL}$ be the set of all logical matrices associated with any finite topological Heyting Lattices, $\mathcal{Syn}_{SL}$ is a sentential syntax, $\mathcal{L}_{FHTL}$ is the language <$\mathcal{Syn}_{SL}$, $\mathcal{F}_{FTHL}$>, and *X* is finite, then

$$X \vdash_I P \text{ iff } X \models_{\mathcal{L}_{FTHL}} P.$$

(The result follows from the previous theorem and the finite argument soundnesss and completeness theorem of the last lecture.)

Appendix to Lecture 7: The Topological Definition of Limit

The traditional definition *the limit of a function* f *as* x *approaches* a is the following:

$\lim_{x\to a} f = \lambda$ iff for any distance δ, there is some subdomain of f within a distance ε of a such that the range of f restricted to this set is all within a distance δ of λ.
iff $\forall\delta\exists\varepsilon\forall x(|x\text{-}a|<\varepsilon\to|f(x)-\lambda|<\delta)$

The topological definition uses the notion of neighborhood.

Definition.

$\lim_{x\to a} f = \lambda$ iff $\forall \mathbf{N}\lambda,\exists Na$, Range(f|**N**a)=N$\lambda$.

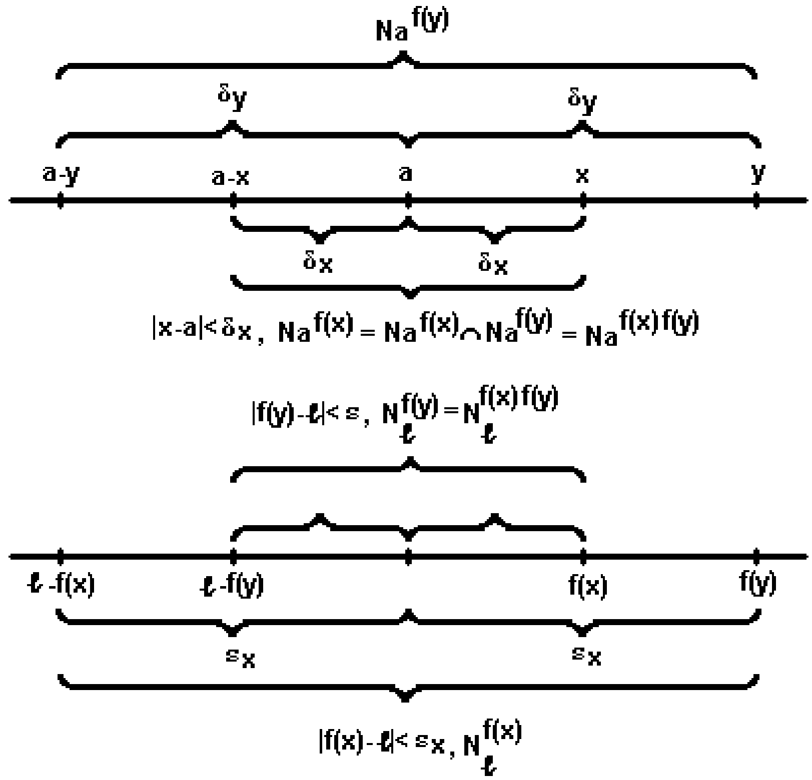
To understand the motivation for the definition, we shall look at an example. Let x,y be in the domain of a function f and let |x-a|<|y-a|. Consider not the neighborhoods:

$$\mathbf{N}_\lambda^{f(x)} = \{z|\ |z-\lim_{x\to a}f|<|f(x)-\lim_{x\to a}f|\}, \text{ and}$$
$$\mathbf{N}_\lambda^{f(y)} = \{z|\ |z-\lim_{x\to a}f|<|f(y)-\lim_{x\to a}f|\}.$$

Clearly both $\mathbf{N}_\lambda^{f(x)}$ and $\mathbf{N}_\lambda^{f(y)}$ are neighborhoods (contain open intervals) of λ.
By the topological definition of limit there exists $\mathbf{N}_a^{f(x)}$ and $\mathbf{N}_a^{f(y)}$ such that

Range(f| $\mathbf{N}_a^{f(x)}$) = $\mathbf{N}_\lambda^{f(x)}$
Range(f| $\mathbf{N}_a^{f(y)}$) = $\mathbf{N}_\lambda^{f(y)}$.

Moreover $\mathbf{N}_a^{f(x)}$ and $\mathbf{N}_a^{f(y)}$ intersect because a is a member of both. In addition, $\mathbf{N}_a^{f(x)} \cap \mathbf{N}_a^{f(y)}$ is the neighborhood $\mathbf{N}_a^{f(x)f(y)}$ of a such that Range(f| $\mathbf{N}_a^{f(x)f(y)}$)$\subseteq \mathbf{N}_\lambda^{f(x)} \cap \mathbf{N}_\lambda^{f(y)}$ which is a neighborhood of λ. If f(x)=δ_x and f(y)=δ_y where $\delta_y<\delta_x$, it will not follow that ε_y (the distance from y to a) is less than ε_x (the distance from x to a). But there will be ***some*** neighborhood (a set containing an open interval of a) within the distance ε_y that takes f-values at least as close to λ as do values within ε_x. The situation is illustrated below:


Na f(y)
δy
δy
a-y
a-x
a
x
y
δx
δx
|x-a|< δx , Na f(x) = Na f(x) ∩ Na f(y) = Na f(x) f(y)
|f(y)-ℓ|< ε, N ℓ f(y) = N ℓ f(x) f(y)
ℓ-f(x)
ℓ-f(y)
f(x)
f(y)
εx
εx
|f(x)-ℓ|< εx, N ℓ f(x)

Lecture 8

Possible Worlds for Intuitionistic Semantics

1. Information Structures

In this lecture we shall explore a reformulation of the topological of Heyting algebras which we previously used in giving a weak completeness result for intuitionistic logic. The characterization in terms of pure Heyting algebras is already mathematically interesting. It is elegant both in its simplicity and it has a full complement of formal properties analogous to those of Boolean algebras but which differ from them in ways governed by simple laws. As a semantic theory, however, it leaves something to be desired. Heyting Lattices as quite abstract, and topological Heyting lattices are only slightly less so. As semantic structures that are supposed to represent the structure of "the world," they lack flesh and blood. Though they be said to preserve some of the structural properties of traditional semantics formulated in terms of truth and falsity, they do so only is a vary abstract way indeed. Certainly there is little of the structure of ontology as understood in traditional philosophy. Are the formal features captured sufficient to justify a claim that the theory is semantic, or that it explains the relation of language to the world?

Soundness and completeness proofs are only as interesting as the concepts that appear on either side of the biconditional. On one side is the deducibility (or provability) relation. This is a proof theoretic idea and usually has little interest beyond its syntactic constructibility.[1] Flanking the biconditional on the other side, however, is the concept of valid argument. On its usual analysis this is a semantic idea. It is defined is terms of the concept of "truth in a world." But definitions are only as plausible as the concepts they employ. Such a theory bears the heavy burden; its notion of truth must be "conceptually adequate." Adequacy here may be spelled out. Definitions must conform to ordinary or pre-analytic usage, and departures from these must be well motivated. If "truth in a world" is in turn defined, its *definientes* must in turn be conceptually adequate. The resulting theory then generally contains a large number of explanatory terms that have a long history in logic and philosophy, and their conceptual justification is a major exercise in conceptual analysis.

In this lecture we shall explore an interpretation of topological Heyting algebras that goes some way towards linking its concepts to those of more traditional semantics. It does so moreover in a way that is true to the peculiar slant intuitionists give to the notion of truth. The key idea is to read a point in a topological Heyting lattice as a possible worlds in an intensional world structure. Moreover, these "worlds" are not to be thought of in the traditional realistic way of

[1] In intuitionistic proof theory more is usually claimed for proof theoretic ideas. The statement of proofs in terms of introduction and elimination rules that lend themselves to reformulation into what are called "normal proofs" (proofs in the deduction at a nodes is composed only of formulas that are subformulas of deductions on prior nodes) is said to capture the "use" of the connectives. Therefore, the proof rules independently of any formal semantics or model theory is said to provide a theory of *meaning as use* for the connectives.

classical semantics. They are not to be understood as entities external to language and the mind to which the expressions of language either correspond or not. Rather, they are the formal proxies of "states of information." The information states are those appropriate to the intuitionistic conception of truth as provability. At a given point in epistemic space we have proofs for some propositions and refutations of others. This epistemic state is a "point of information."

Definition. ***A (partial order sentential) world structure for intuitionistic logic*** is any finite partial $<K,\leq>$. We let w range over K.

Theorem. If $<K,\leq>$ is a (partial order sentential) world structure for intuitionistic logic, then
I. if $X\subseteq K$,

1. $<K,\vartheta_{\leq}>$ is the topological space
2. $I^{K}_{\leq}(A)=\{x|\forall y\leq x, y\in A\}$, and
3. $\vartheta_{\leq} = \{A\subseteq H| I^{H}_{\leq}=A\}$,
4. $I^{K}_{\leq}(X) = \{y|$ y is an interior point of $X\}$ $= \{y| X=\mathbf{N}y\}$
5. $I^{K}_{\leq}(X)$ is an open set'
6. $I^{K}_{\leq}(X) =\bigcap\{Y|$ Y is open and $Y\subseteq X\}$
7. X is open iff $X=I^{K}_{\leq}(X)$,
8. $I^{K}_{\leq}(X) = \{y\in X|$ y is not a limit point of $-X\}$
9. $(-X)^{c} = I^{K}_{\leq}(X)$, where $X^{c} = -I^{K}_{\leq}(-X)$.
10. $-I^{K}_{\leq}(X) = (-X)^{c}$
11. $I^{K}_{\leq}(X) = -((-X)^{c})$

II. $L=<\vartheta_{\leq},\cap,\cup,\varnothing,\Rightarrow>$ is its topological Heyting Lattice determined by $<K,\leq>$, then

1. For any $A\subseteq K$, $I^{K}_{\leq}(A) = \{x| x\in A \;\&\; (\forall y\leq x)(y\in A)\}$,
2. $\vartheta_{\leq} = (A| A\subseteq K \;\&\; I^{K}_{\leq}(A)=A\}$,
3. $\cap,\cup$, and $\varnothing$ are set theoretic intersection, union and the empty set on $\vartheta_{\leq}$,
4. $A\Rightarrow B = I^{K}_{\leq}\{x|$ if $x\in A$ then $x\in B\}$, a relative pseudo complementation operation,

III. the both logical matrix $M_{L}=<\vartheta_{\leq},\{1\},-,,\cap,\cup,\varnothing,\Rightarrow>$ associated with L (where $0=\varnothing$, $-A=A\Rightarrow\varnothing$ and $1=-0$) and its set of valuations Val_{ML} are well defined, and for any $A\in\vartheta_{\leq}$, and any $w,w'\in K$, $w\leq w'$ and $w'\in A$, then $w\in A$.

Proof. All but the latter part of III are restatements of earlier theorems. Assume $w'\in A$. Then $w'\in I^{K}_{\leq}(A) = \{x| x\in A \;\&\; (\forall y\leq x)(y\in A)\}$. Hence $\forall y\leq w'$, $y\in A$. Hence if $w\leq w'$. Therefore, $w\in A$. **QED.**

As in classical semantics the semantic theory explains how to assign "truth-values" relative to such a "world." Th relevant logical matrix is that associated a topological Heyting lattice on a topology of information points. To each atomic sentence is assigned an intension: the set of points at which it is true. The operations on intensions then determine the intensions of the complex expressions in the manner of intensional matrix semantics.

There are several features of the Heyting structure that correspond to "properties" of information or the epistemic states associated with the notion of truth as provability. The ordering relation ≤ on information states represents the an increase in information or progress proving or refuting sentences. (The ordering is read downwards: if w≤w′ then w is "later than" and w is an improvement on w′ (or at least equal to it). Characteristic then of intuitionistic reasoning is the fact that information is accumulative: if a sentence *P* is proven at point w, then at any "later" point.

A second interesting feature of intuitionistic logic is the semantics of the connectives. Conjunction and Disjunction are treated classically. A conjunction is proven iff both conjuncts are, and a disjunction is, if either or both disjuncts are. Negation and the conditional are non-classical, and are in fact non-extensional "modal operators" in the sense the extension of the whole is not functionally determined by that of its parts.

Negation. Asserting ~*P* is understood as the same as saying that it is not possible to prove *P*, or in terms of "points of information:" ~*P* is "true" at w iff *P* is not "true" at any point w′ lower than w. This semantics identifies the epistemic state of provability with truth at point or "world" w, possibility with provable at some world w, and impossibility with not provable at any accessible world. Accordingly the semantics conforms to a variety of the "principle of plenitude." Aristotle's original version linked possibility with time: any possible world will turn at some point in actual history. The version appropriate to intuitionistic semantics is that to be a possible information state amounts to being among the further developments of information. If *P* then does not turn up proven at any world "accessible" to w, then it is literally impossible to prove *P.* This fact is itself an epistemic that removes *P* from the dubious category of sentences into that we are justified in denying. A similar intuition underlies the empiricist positive rejection of the empirically unverifiable. If in principle *P* is not empirically verifiable (e.g. *God exists* or *An invisible rhinoceros exists on the first planet that evolved after the big bang*), then this is just the sort of proposition that we are justified in rejecting. According to strict verificationists it is rational to positively reject belief in the unverifiable. We reserve disbelief for the sensible (empirically meaningful) propositions about which there is empirical evidence in principle but which is lacking in for them. Intuitionistic semantics takes a step beyond empiricism by identifying the epistemic and semantic statuses of a sentence. It then follows that a sentence is false if not provable in principle.

Thought the semantics is bivalent (every sentence being either 0 or 1), the rationale behind the assignment of the value 0 to the negated whole is in a sense "three-valued." A similar idea is used in the semantics of Frege and Bochvar. Both allow that a sentence *P* might be in one of three semantic states: true, false, or "other." (The other for Frege included *meaninglessness*, and *unfilled presuppositions*; Bochvar used the third status for paradoxical sentences.) Frege then imposes a bivalent semantics by joining together into one category of "non-truth" both falsity and "other". Frege used the horizontal bar —— for negation, with the following truth-conditions:

Frege's Horizontal. ——*P* is true at w if *P* is non-true (false or "other") at w , and ——*P* is false at w otherwise.

For Frege and Bochvar, then, the law of bivalence holds for the horizontal:

Law of Bivalence: Every *P* is either true or non-true (false or "other")
Excluded Middle: for any *P* and any w, $P\vee$——P is true at w.

Informational semantics relates negation in a different way to the a similar three-fold semantic division.

Three-fold Semantic Classification in Informational Semantics

Truth:	*P* is proven	*P* is 1 at w, $\sim P$ is 0 at w
Falsity Type 1:	"Unknown": *P* not proven but provable	*P* is 0 at w, $\sim P$ is 0 at w
Falsity Type 2:	Refuted: *P* not provable	*P* is 0 at w, $\sim P$ is 1 at w.

The truth conditions for informational negation then take Truth (1) and Unknown (a variety of 0) to 0, and Refuted (a variety of 0) to 1. The connective is therefore non-truth-functional. Moreover, the assignment of 0 to $\sim P$ is also ambiguous. If $\sim P$ is 0 because *P* is 1, then *P* always will be 1 and $\sim P$ is 0 because it is itself refuted. On the other hand, if $\sim P$ is 0 at w because *P* is provable, then the "future" informational developments branch at some point, into a subtree headed by a point at which *P* is proven, and $\sim P$ is refuted, and into a subtree with points at which both *P* and $\sim P$ are 0. Hence the assignment of 0 to $\sim P$ at w in this case means that $\sim P$ is "unknown".

The Conditional. In a similar way the conditional is non-truth-functional. It is a kind of "strict implication": $P\rightarrow Q$ is "true" at w iff at every point w′ lower than w, the material conditional of P and Q would be true, i.e. if *P* is "true" at w′, then *Q* is "true" at w′. The term ***strict implication*** was introduced in the early days of modal logic by C.I. Lewis for the necessity of the material conditional: $P\rightarrow Q=_{def}\Box(P\supset Q)$, and therefore semantically $Int_w(P\rightarrow Q)=1$ iff $(\forall w'\leq w$, $Int_w(P\supset Q)=1)$ iff$(\forall w'\leq w$, if $Int_w(P)=1$ then $Int_w(Q)=1)$.

Definitions.

1. If $M_L=<\vartheta_{\leq},\{1\},-,,\cap,\cup,\varnothing,\Rightarrow>$ is associated with the Heyting lattice $L=<\vartheta_{\leq},\cap,\cup,\varnothing,\Rightarrow>$ (where $0=\varnothing$, $-A=A\Rightarrow\varnothing$ and $1=-0$) determined by an intuitionistic world structure $<K,\leq>$, then we shall refer to Val_{M_L} are as the set of ***intensional interpretations*** of M_L, and let **Int** range over Val_{M_L}.

2. If $Int\in Val_{M_L}$, then ***the extensional interpretation determined by*** **Int** is that function **Ext** mapping *Sen*xK into {0,1} such that Ext_w is the characteristic function of Int(*P*), i.e. for nay *P*,
$Ext_w(P)=1$ if $w\in Int(P)$, and

$Ext_w(P)=0$ if $w \notin Int(P)$.

3. By an ***abstract informational language for intuitionistic sentential logic*** is meant any $\langle \mathcal{S}yn_{SL},\mathcal{F}\rangle$ such that $\mathcal{F}$ is the family of all topological Heyting lattices determined by finite partially ordered structure.

The first of the following theorems shows that the intended idea that information is never lost but only accrues. The second shows that the extensions of molecular sentences at a world w are a function off the extensions of their parts. Conjunction and disjunction are extensional in the sense that the extension of the whole at w is a function of the extension of the parts at the same world w. Negation and the conditional are "modal" and non-extensional in the sense that the extension of the whole at w is a function of the extension of the parts at worlds other than w.

Theorem. Accumulation of Information. For any intensional interpretation Int relative to a world structure $\langle K,\leq\rangle$ and is associated logical matrix $L=\langle \vartheta_{\leq},\cap,\cup,\varnothing,\Rightarrow\rangle$ of an abstract informational language, and any $w,w'\in K$,

if $w\leq w'$ and $Int_{w'}(P)=1$, then $Int_w(P)=1$.

Theorem. For any Int relative to a world structure $\langle K,\leq\rangle$ and is associated logical matrix $L=\langle \vartheta_{\leq},\cap,\cup,\varnothing,\Rightarrow\rangle$ of an abstract informational language, and any $w,w'\in K$,

1. $Int_w(\sim P)=1$ iff $\forall w'\leq w$, $Int_w(\sim P)=0$.

Proof: $Int_w(\sim P)=1$ iff $w\in Int(\sim P)$
iff $w\in -Int(P)$
iff $w\in\{z|\ not\exists w'\leq z(w'\in Int(P))\}$
iff $not\exists w'\leq w(w'\in Int(P))$
iff $not\exists w'\leq w(Int_y(P)=1)$
iff $\forall w'\leq w(w'\in Int_y(P)=0)$

2. $Int_w(P\wedge Q)=1$ iff $Int_w(P)= Int_w(P)=1$.

Proof: $Int_w(P\wedge Q)=1$ iff $w\in Int(P)$
iff $w\in Int(P)\cap Int(Q)$
iff $w\in Int(P)$ & $w\in Int(Q)$
iff $Int_w(P)=1$ & $Int_w(P)=1$.

3. $Int_w(P\vee Q)=1$ iff $Int_w(P)=1$ or $Int_w(P)=1$. (Similar to case 3.)

4. $Int_w(P\rightarrow Q)=1$ iff $Int_w(P)= Int_w(P)=1$ iff $\forall w'\leq w$, if $Int_{w'}(P)=1$, then $Int_{w'}(Q)=1$.

Proof: $Int_w(P\rightarrow Q)=1$ iff $w\in Int(P\rightarrow Q)$
iff $w\in Int(P)\Rightarrow Int(Q)$
iff $w\in\{z|\forall w'\leq z(\text{if } y\in Int(P) \text{ then } w'\in Int(Q)\}$
iff $\forall w'\leq w(\text{if } w'\in Int(P) \text{ then } w'\in Int(Q)$
iff $\forall w'\leq w(\text{if } w'\in Int(P) \text{ then } w'\in Int(Q)$
iff $\forall w'\leq w(\text{if } Int_{w'}(P)=1 \text{ then } Int_{w'}(Q)=1$

Theorem. Ext_w is an extensional interpretation, i.e. world structure $\langle K,\leq\rangle$, some Int relative to $\langle K,\leq\rangle$ with associated matrix $L=\langle \vartheta_{\leq},\cap,\cup,\varnothing,\Rightarrow\rangle$ of an abstract informational language, and some sentence P, $Ext_w(P)=Int(w)$, iff

For any atomic P , $Ext_w(P)\in(0,1\}$, and

1. $Int_w(\sim P)=1$ iff $\forall w'\leq w$, $Int_w(\sim P)=0$,
2. $Int_w(P\wedge Q)=1$ iff $Int_w(P)= Int_w(P)=1$,
3. $Int_w(P\vee Q)=1$ iff $Int_w(P)=1$ or $Int_w(P)=1$,
4. $Int_w(P\rightarrow Q)=1$ iff, $\forall w'\leq w$, if $Int_{w'}(P)=1$ then $Int_{w'}(Q)=1$.

2. Kripke Trees

Partially ordered structures of information states can be made more intuitive yet by recasting them as tree structures. Tree structures of "possible worlds" are quite natural for both tense and denote logic because they capture both the idea of progress and the intuition that the "past is necessary" in the sense that the structure of prior worlds does not branch. Intuitively informational structures have the same property. Once information is accumulated it is not possible for it to be lost. This fact would seem to mean that information states form a tree structures. This intuition in fact harmonizes well with the partial order semantics we have just explored in as much as trees are a special case of partial orderings. It is in fact a straightforward matter to show that our earlier characterization of intuitionistic logic in terms of the matrices generated from the topological Heyting algebras of arbitrary partial orderings may be further tightened to just those generated by partial orderings that are trees. Historically, this was the approach of Saul Kripke to the semantics of intuitionistic sentential logic. Kripke was in fact able to strengthen the characterization to a strong soundness and completeness result. Using a slightly different semantics over informational tree structures E.W. Beth was able to extend this completeness result to intuitionistic first-order logic. In this section we shall explore the reformulation into Kripke trees. We begin with the relevant notions about trees.

Definition. If $<C,\leq>$ is a partial ordering, x is ***the immediate predecessor of*** y, briefly $x<<y$, iff $x\leq y$ and for z, if $x\leq z\leq y$ then x=z or y=z. Further, if $y_1,\ldots,y_n\in C$, then $<y_n,\ldots,y_1>$ such that for any i, $y_{i+1}<<y_i$, is called ***a (finite) chain ending from*** $\mathbf{y_1}$ ***to*** $\mathbf{y_n}$ **.** We also call a ***≤-maximal element of*** $<C,\leq>$ iff ,$x\in C$ and $(\forall y\in C, x\leq y\rightarrow x=y)$

Definition. $<C,\leq>$ is a tree iff

1. $<C,\leq>$ is a partial ordering
2. there is a unique maximal element 1 in C, i.e. $\forall x\in C$, $x\leq 1$,
3. $\forall x\in C$, there is a unique $<y_n,\ldots,y_1>$ such that $y_n=x$, $y_1=1$, & for any i, $y_{i+1}<<y_i$.

Definition. If $T=<C,\leq>$ is a tree, then by a ***branch of*** **T** is meant any sieres *s* of elements of T (function from ◆ onto some subset of T) such that $s_1=1$ and for any s_i, $s_{I+1}<<s_i$. By ***the finite branch of*** **T** ***ending with*** $\mathbf{y_n}$ is meant any branch with range {1,…,n}, (equivalenly, any finite chain $<s_n,\ldots,s_1>$) such that $s_n=x$, $s_1=1$, and for any i, $s_{I+1}<<s_i$. Further if *s* and *s′* are branches then *s* is said to ***contain*** *s′* and *s′* is a ***sub-branch*** of *s* iff for any i in the domain of *s′*, $s'_i=s_i$.

Definition. If $T=<C,\leq>$ is a tree, $D\subseteq C$, and $x\in C$, then D is said to ***bar*** x iff any branch b of T that contains some finiite branch b′ of T ending with x also containing a finite sub-branch b′′ which in trun contains b′ and which also ends in some element b''_j of D $b''_j<<x$.

Theorem. If D bars x in a tree T, then there is some finite chain $<y_n,\ldots,y_1>$ such that $y_n\in D$ & $y_1=x$.

We now introduce the concepts that will show how to reformulate informational semantics in terms of partial orderings into a semantics of trees. We first define a tree <C*,≤*> that corresponds to a partial ordering <C,≤>. The idea is that a path down the ordered set <C,≤> to a node x is a kind of a "branch." If these branches are themselves taken as points. These branches are by their nature organized by a containment relation. This containment relation proves to be a partial ordering on the branches that moreover organizes them into a tree. The original partial ordering ≤ is easily shown to be order homomorphic to the containment relation on branches. We shall see in addition that the full logical matrix associated with the topological Heyting lattice determined by the original partially ordered set is "homomorphic into" the logical matrix determined by the topological Heyting lattice of the tree structure.

Definition. If $\langle C,\leq\rangle$ is a partial ordering, then C* is such that

$C^*=\{\langle x_n,\ldots,x_1\rangle \mid \langle x_n,\ldots,x_1\rangle$ is a chain in $\langle C,\leq\rangle$ and x_1 is a $\leq$-maximal element of $\langle C,\leq\rangle\}$.

Theorem. There is a unique partition of C* into subsets each of which is a tree.

Definition. If $\langle C,\leq\rangle$ is a partial ordering, then $\leq^*$ is the union of containment relation of each tree in the partition of C* into trees.

Definition. If $\langle K,\leq\rangle$ is a world system, $L=\langle\vartheta_\leq,\cap,\cup,\varnothing,\Rightarrow\rangle$ is its associated topological Heyting lattice and is its associated logical matrix $M_L=\langle\vartheta_\leq,\{1\},-,,\cap,\cup,\varnothing,\Rightarrow\rangle$, then $L=\langle\vartheta^*_\leq,\cap,\cup,\varnothing,\Rightarrow^*\rangle$ is the topological Heyting lattice associated with $\langle K^*,\leq^*\rangle$ and $M_{L^*}=\langle\vartheta^*_\leq,\{1\},-*,,\cap,\cup,\varnothing,\Rightarrow^*\rangle$ it its associated logical matrix.

Theorem. If $\langle K,\leq\rangle$ is a world system, $L=\langle\vartheta_\leq,\cap,\cup,\varnothing,\Rightarrow\rangle$ is its associated topological Heyting lattice and is its associated logical matrix If $\langle K,\leq\rangle$ is a world system, $L=\langle\vartheta_\leq,\cap,\cup,\varnothing,\Rightarrow\rangle$ is its associated topological Heyting lattice and is its associated logical matrix $M_L=\langle\vartheta_\leq,\{1\},-,,\cap,\cup,\varnothing,\Rightarrow\rangle$, then there is a homomorphism from $M_L=\langle\vartheta_\leq,\{1\},-,,\cap,\cup,\varnothing,\Rightarrow\rangle$ into $M_{L^*}=\langle\vartheta^*_\leq,\{1\},-*,,\cap,\cup,\varnothing,\Rightarrow^*\rangle$ that preserves designation and non-designation.

Proof. Define ϕ from $\vartheta_\leq$ into $\vartheta^*_\leq$: for any $A\in\vartheta_\leq$, $\phi(A)=\{\langle x_n,\ldots,x_1\rangle \mid x_n\in A\}$.

1. Clearly $\phi(1)=1$ and if $x\notin\{1\}$, $\phi(x)\notin\{1\}$. Hence ϕ preserves designation and non-designation.
2. ϕ is a homomorphism. Two lemmas that follow immediately from the definitions are useful:

   Lemma 1. $x_n\in A$ iff $\langle x_n,\ldots,x_1\rangle\in\phi(A)$.

   Lemma 2. $x_n\leq x_m$ iff $\phi(x_n)\leq\ \phi(x_m)$.

We demonstrate the homomorphism for $\cap$ and $\Rightarrow$. The other cases are similar.

Case for $\cap$:

$\langle x_n,\ldots,x_1\rangle\in\phi(A\cap B)$	iff	$\langle x_n,\ldots,x_1\rangle\in\{\langle y_n,\ldots,y_1\rangle \mid y_n\in A\cap B\}$
	iff	$\langle x_n,\ldots,x_1\rangle\in\{\langle y_n,\ldots,y_1\rangle \mid y_n\in A\ \&\ y_n\in B\}$
	iff	$\langle x_n,\ldots,x_1\rangle\in\{\langle y_n,\ldots,y_1\rangle \mid \langle y_n,\ldots,y_1\rangle\in\phi(A)\ \&\langle y_n,\ldots,y_1\rangle\in$ ❑$(B)\}$
	iff	$\langle x_n,\ldots,x_1\rangle\in\{\langle y_n,\ldots,y_1\rangle \mid \langle y_n,\ldots,y_1\rangle\in\phi(A)\cap^*\phi(B)\}$
	iff	$\langle x_n,\ldots,x_1\rangle\in$ ❑$(A)\cap^*\phi(B)$

Case for $\Rightarrow$:

$\langle x_n,\ldots,x_1\rangle\in\phi(A\Rightarrow B)$	iff	$\langle x_n,\ldots,x_1\rangle\in\{\langle y_n,\ldots,y_1\rangle \mid y_n\in A\Rightarrow B\}$
	iff	$x_n\in A\Rightarrow B$
	iff	$x_n\in I^K_\leq\{w \mid \text{if } w\in A \text{ then } w\in B\}$
	iff	$x_n\in\{y \mid \forall z_m\leq y,\ y\in\{w \mid \text{if } w\in A \text{ then } w\in B\}\}$
	iff	$\forall z_m\leq x_n,\ x_n\in\{w \mid \text{if } w\in A \text{ then } w\in B\}$
	iff	$\forall z_m\leq x_n$, if $x_n\in A$ then $x_n\in B$
	iff	$\forall\langle z_m,\ldots,z_1\rangle\leq\langle x_n,\ldots,x_1\rangle$, if $x_n\in A$ then $x_n\in B$
	iff	$\langle x_n,\ldots,x_1\rangle\in\{\langle y_n,\ldots,y_1\rangle \mid \forall\langle z_m,\ldots,z_1\rangle\leq\langle y_n,\ldots,y_1\rangle$, if $y_n\in A$ then $y_n\in B\}$
	iff	$\langle x_n,\ldots,x_1\rangle\in I^{K^*}_{\leq^*}\{\langle y_n,\ldots,y_1\rangle \mid \text{if } y_n\in A \text{ then } y_n\in B\}$
	iff	$\langle x_n,\ldots,x_1\rangle\in I^{K^*}_{\leq^*}\{\langle y_n,\ldots,y_1\rangle \mid \text{if } \langle y_n,\ldots,y_1\rangle\in\phi(A)$ then $\langle y_n,\ldots,y_1\rangle\in\phi(B)\}$
	iff	$\langle x_n,\ldots,x_1\rangle\in\phi(A)\Rightarrow^*\phi\ (B)$

Definition.

1. Let ***Kripke world structure*** be any finite tree <K,≤>.
2. Let $\mathcal{F}_{FKTHL}$ be the set of all logical matrices associated with any finite topological Heyting Lattices determined by any Kripke world structure <K,≤>.
3. Let a ***Kripke language for sentential intuitionistic logic*** be any $\mathcal{L}_K$ = <$\mathcal{Syn}_{SL}$, $\mathcal{F}_{FKTHL}$>,

Theorem. If $\mathcal{L}_K$ is a Kripke language for sentential intuitionistic logic and X is finite, then

$$X \vdash_I P \text{ iff } X \models_{\mathcal{L}_K} P.$$

Proof. We already know (from Lecture 7) that if $\mathcal{F}_{FTHL}$ be the set of all logical matrices associated with any finite topological Heyting Lattices, $\mathcal{Syn}_{SL}$ is a sentential syntax, $\mathcal{L}_{FHTL}$ is the language <$\mathcal{Syn}_{SL}$, $\mathcal{F}_{FTHL}$>, and X is finite, then

$$X \vdash_I P \text{ iff } X \models_{\mathcal{L}_{FTHL}} P.$$

We show that $X \models_{\mathcal{L}_{FTHL}} P$ iff $X \models_{\mathcal{L}_K} P$. That $X \models_{\mathcal{L}_{FTHL}} P$ entails $X \models_{\mathcal{L}_K} P$ follows from the fact that every Kripke world structure, being a tree, is automatically a partial ordering. The converse that $X \models_{\mathcal{L}_K} P$ entails $X \models_{\mathcal{L}_{FTHL}} P$ follows from the previous "homomorphism into" theorem and the theorems of Lecture 2 that insure that in such case the entailment of a language associated with the target matrix is a subset of that off the source matrix. **QED**

Using Henkin methods, Kripke proves a stronger versions of this result, an argument soundness and completeness theorem for sets of premises of arbitrary size. [2]

Definition.

1. A rule R defined below

$$\frac{X_1 \vdash P_1>, \quad .... \quad X_k \vdash P_k}{Y \vdash Q}$$

is an ***atomic rule*** iff, $X_1,\ldots,X_n$, and Y are all empty and $\varnothing$ and $P_1,\ldots,P_n,Q$ are all atomic. Let $\mathcal{R}$ range over sets of atomic rules for some $\mathcal{Syn}_{SL}$.

2. If $\mathcal{R}$ is a set of atomic rules for $\mathcal{Syn}_{SL}$, then

$$\mathbf{I{+}\mathcal{R}} = <BD_{\mathcal{SL}\text{-}\mathcal{Sen}}, \vdash_{I+\mathcal{R}}, \{R_{\bot+},R_{\bot-},R_{\sim+},R_{\sim-},R_{\wedge+},R_{\wedge-},R_{\vee+}, R_{\vee-},R_{\to+},R_{\to-},R_{Th}\} \cup \mathcal{R}>$$

If $\mathcal{R}$ is a unit set {R}, we identify the two.

4. Let $\mathcal{Syn}_{SL}$ be a sentential language and <K.≤> be a Kripke world structure.

a. The set $\mathtt{Val}_{<K,\leq>}$ of ***Kripke valuations*** relative <K,≤> is the set of all $\mathtt{v}$ on $\mathcal{Sen}_{SL}$ such that

For any atomic P , $\mathtt{v}_w(P)\in(0,1\}$, and $\mathtt{v}_w(P)=1$ only if $\forall w'\leq w$, $\mathtt{v}_{w'}(P)=1$.

In addition, for all w, $\mathtt{v}_w(\bot)=0$.

$\mathtt{v}_w(\sim P)=1$ iff $\forall w'\leq w$, $\mathtt{v}_{w'}(\sim P)=0$,

$\mathtt{v}_w(P\wedge Q)=1$ iff $\mathtt{v}_w(P)=1$ and $\mathtt{v}_w(P)=1$,

$\mathtt{v}_w(P\vee Q)=1$ iff $\mathtt{v}(P)=1$ or $\mathtt{v}_w(P)=1$,

$\mathtt{v}_w(P\to Q)=1$ iff $\forall w'\leq w$, if $Int_{w'}(P)=1$, then $Int_{w'}(Q)=1$.

b. $\mathbf{Int}_{<K,\leq>}(P)=\{w\in K| \mathtt{v}_w(P)=1\}$,

c. $\mathbf{Tr_w}=\{ P | \forall w'\leq w, \mathtt{v}_{w'}(P)=1\}$

d. $X\models_w P$ iff for any $\mathtt{v}\in\mathtt{Val}_{<K,\leq>}$, $\mathtt{v}_w(Q)=1$ for all $Q\in X$, only if $\mathtt{v}_w(P)=1$

e. $X\models_{<K,\leq>}P$ iff for w∈K, $X\models_w P$

Theorem. $X\models_w P$ iff $\forall w'\leq w$, $X\models_{w'} P$

Definition. An atomic rule $\dfrac{\varnothing\vdash P_1>, \quad .... \quad \varnothing\vdash P_k}{\varnothing\vdash Q}$ ***hold at* w** iff $P_1,\ldots,P_n\models_w Q$

Theorem. For any Kripke world structure <K,≤>, any w∈K, and any atomic rule R that holds in w,

$$X\vdash_{I+R}P \text{ iff } X\models_w P$$

The proof uses Henkin methods and is relatively straightforward. Soundness is uncomplicated. The maximalization lemma in the completeness proof replaces the classical maximally consistent set with an intuitionistic version that is consistent, closed under $\vdash_{I+R}$ and which verifies a disjunction only if it verifies one of its disjuncts. The satisfiability lemma proceeds by constructing a Kripke world system from worlds that are themselves sets of sentences (the atomic sentences entailed by sets closed under R).[3]

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[2] See S. Kripke, "Semantics Analysis of Intuitionistic Logic, I" in J.N. Crossley and M. Dummett, eds, *Formal Systems and Recursive Functions, II* (Amsterdam: North Holland, 1967).
[3] See Neil Tennant, *Natural Logic*, (Edinburgh: Edinburgh University Press, 1978).

3. Beth Trees

E.W. Beth has developed the strongest possible world semantics for intuitionistic logic in the sense that it provides an argument soundness and completeness result for full first-order logic, for which Kripke's methods fail. Beth's trees too are naturally understood as structures of information points. The truth-value of a sentence once determined at a point continues the same at all lower points. Conceptually, the difference lies in the treatment of the connectives. Instead of determining inevitability (which structurally is much like "necessity with respect to the future"in tense logic with a branching future) by the inspection of all lower worlds in the structure, the semantics makes use of what is conceptually a more economic idea. Given that information is never lost, all that is needed to determine that a sentence *P* must will be true is that we can show that it turns up true at some point on any path proceeding to the "future." A collection of such points is said to "bar" the tree in the sense that a subset D of K bars the tree from w reference point w if there is no way to proceed from w without meeting one point in D. (The formal definition was stated earlier with the introduction of tree concepts.) If an atomic sentence is true at w, it will then be true at any lower w′. Conversely, if there is some set D that bars the tree from the perspective of w and every w′ in D makes *P* true, then *P* is "inevitable" and is accordingly true at w as well. (True to the finitistic intutions behind intuitionism built into the relevant concept of inevitablity is finiteness -- improvements in information that are not finitely "near" would be of little use.) Similarly, it will be sufficient to know the truth of a disjunction $P \vee Q$ at w regardless of what we know of the truth of *P* or of *Q* individually at w, if we know the is some set D that bars the tree from the perspective of w and each world in D verifies one or the other (or both) of *P* or *Q*. Disjunction then joins the ranks of negation and the conditional in becoming non-extensional.

Definition.
Let ***Beth world structure*** be any (possibly infinite) tree $<K,\leq>$.

Let $\mathcal{Syn}_{SL}$ be a sentential language and $<K.\leq>$ be a Beth world structure.

a. The set **B-**$\mathtt{Val}_{<K,\leq>}$ of ***Beth valuations*** relative $<K,\leq>$ is the set of all $\mathtt{v}$ on $\mathcal{Sen}_{SL}$ such that

For any atomic P ,

$\mathtt{v}_w(P)\in(0,1\}$,

$\mathtt{v}_w(P)=1$ only if $\forall w'\leq w$, $\mathtt{v}_{w'}(P)=1$,

if there is some $D\subseteq K$ s.t. D bars w and ($\forall w'\in D$, $\mathtt{v}_{w'}(P)=1$), then $\mathtt{v}_w(P)=1$,

In addition, for all w, $\mathtt{v}_w(\perp)=0$.

$\mathtt{v}_w(\sim P)=1$ iff $\forall w'\leq w$, $\mathtt{v}_{w'}(\sim P)=0$,

$\mathtt{v}_w(P\wedge Q)=1$ iff $\mathtt{v}_w(P)=1$ and $\mathtt{v}_w(P)=1$,

$\mathtt{v}_w(P\vee Q)=1$ iff there is some $D\subseteq K$ s.t. D bars w and ($\forall w'\in D$, $\mathtt{v}_{w'}(P)=1$ or $\mathtt{v}_{w'}(Q)=1$),

$\mathtt{v}_w(P\rightarrow Q)=1$ iff $\forall w'\leq w$, if $\mathrm{Int}_{w'}(P)=1$, then $\mathrm{Int}_{w'}(Q)=1$.

b. $X\models^{B}_{w}P$ iff for any $\mathtt{v}\in\mathtt{Val}_{<K,\leq>}$, $\mathtt{v}_w(Q)=1$ for all $Q\in X$, only if $\mathtt{v}_w(P)=1$

c. $X\models_{B,<K,\leq>}P$ iff for $w\in K$, $X\models_w P$

Theorem. Relative to a Beth world structure $<K,\leq>$, $X\models^{B}_{w}P$ iff $\forall w'\leq w$, $X\models_{w'}P$

Definition. If $<K.\leq>$ is a Beth world structure and $w\in K$, we say P is ***finitiely inevitable from*** w in $<K,\leq>$ iff there is some $D\subseteq K$ s.t. D bars w and $\forall w'\in D$, $\mathtt{v}_{w'}(P)=1$.

Theorem. Relative to a Beth world structure $<K,\leq>$, $X\models^{B}_{w}P$ iff P is finitely inevitable from w.

Theorem. Argument Soundness and Completeness. For any Beth world structure $<K,\leq>$,

$X\vdash_I P$ iff for any for any Beth world structure $<K,\leq>$, $X\models_{B,<K,\leq>}P$

It should be remarked that though this theorem is provable, Beth's proof does not use finitistic means and is therefore not intuitionistically valid.[4]

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[4] See E.W. Beth, *The Foundations of Mathematics* (Amsterdam: North Holland, 1968). For a discussion of the limitations of Beth's methods and intuitionistic attempts to replicate it see Michael Dummett, *Elements of Intuitionism* (Oxford: Clarendon Press, 1977).