An Example of an Axiom System in the language of First-Order Logic

The System $\mathcal{F}$

1. The inference rules of $\mathcal{F}$. The set **PRF** of inference rules contains just one rule:

If $\vdash_{\mathcal{F}} P$ and $\vdash_{\mathcal{F}} P\rightarrow Q$, then $\vdash_{\mathcal{F}} Q$ (*modus ponens*)

2. The Axioms of $\mathcal{F}$. The set $\mathbf{Ax}_{\mathcal{F}}$ of axioms consist of all sentences of the following forms:

For the Propositional Calculus (Sentential Logic)

1. $\vdash_{\mathcal{F}} P\rightarrow(Q\rightarrow P)$
2. $\vdash_{\mathcal{F}}(P\rightarrow(Q\rightarrow R))\rightarrow((P\rightarrow Q)\rightarrow(P\rightarrow R))$
3. $\vdash_{\mathcal{F}}(\sim P\rightarrow\sim Q)\rightarrow(Q\rightarrow P)$

For Quantification Theory (First-Order, Predicate Logic) with Identity

4. $\vdash_{\mathcal{F}} \forall x(P\rightarrow Q)\rightarrow(\forall xP\rightarrow\forall xQ)$
5. $\vdash_{\mathcal{F}} P\rightarrow\forall xP$ where x is not free in P
6. $\vdash_{\mathcal{F}}\forall xP[x]\rightarrow P[y]$ where $P[y]$ is like $P[x]$ except for containing free occurrences of y where $P[x]$ contains free occurrences of x
7. $\vdash_{\mathcal{F}} \forall x(x=x)$
8. $\vdash_{\mathcal{F}} \forall x(x=x \wedge P[x]) \rightarrow P[y]$ where $P[y]$ is like $P[x]$ except for containing free occurrences of y where $P[x]$ contains free occurrences of x

For (Naive) Set Theory (Russell's version of Frege, 1903)

9. $\vdash_{\mathcal{F}}\forall y(y\in\{x|P[x]\}\leftrightarrow P[y]))$, for any $P[x]$
10. $\vdash_{\mathcal{F}} \forall x\forall y[x=y\leftrightarrow\forall z(z\in x\leftrightarrow z\in y)]$

3. The set $\mathcal{Th}_{\mathcal{F}}$ of theorems of $\mathcal{F}$ is the closure of $\mathbf{Ax}_{\mathcal{F}}$ under the rule in **PRF** .