Syntax of Sentential Logic

A ***syntax* SL *for sentential logic*** is a structure $<\mathbf{AF_{SL}},\mathbf{R_{SL}},\mathbf{F_{SL}}>$ such that:

$\mathbf{AF_{SL}}$ (the set of ***atomic formulas*** of **SL**) is some subset of the sentence letters: $P_1,\ldots, P_n,\ldots$.

$\mathbf{R_{SL}}$ (the set of ***grammatical rules*** of **SL**) is set of function $\{R_{\sim},R_{\wedge},R_{\vee},R_{\rightarrow},R_{\leftrightarrow}\}$ defined:

- $R_{\sim}$ constructs $\sim x$ from any string x; *i.e.* $R_{\sim}(x)=\sim x$
- $R_{\wedge}$ constructs $(x\wedge y)$ from strings x and y; *i.e.* $R_{\wedge}(x,y)=(x\wedge y)$
- $R_{\vee}$ constructs $(x\vee y)$ from strings x and y; *i.e.* $R_{\vee}(x,y)=(x\vee y)$
- $R_{\rightarrow}$ constructs $(x\rightarrow y)$ from strings x and y; *i.e.* $R_{\rightarrow}(x,y)=(x\rightarrow y)$
- $R_{\leftrightarrow}$ constructs $(x\leftrightarrow y)$ from strings x and y; *i.e.* $R_{\leftrightarrow}(x,y)=(x\leftrightarrow y)$

$\mathbf{F_{SL}}$ (the set of ***well-formed formulas*** or ***wffs*** of **SL**) is defined inductively as follows:

1. **Basis Clause.** All formulas in $\mathbf{AF_{SL}}$ are in $\mathbf{F_{SL}}$.
2. **Inductive Clause.** If P and Q are in $\mathbf{F_{SL}}$, then the results of applying the rules $R_{\sim},R_{\wedge},R_{\vee},R_{\rightarrow}$, and $R_{\leftrightarrow}$ to them, namely $\sim P$, $(P\wedge Q)$, $(P\vee Q)$, $(P\rightarrow Q)$, $(P\leftrightarrow Q)$, are all in $\mathbf{F_{SL}}$;
3. Nothing is in $\mathbf{F_{SL}}$ except by clauses 1 and 2.

The Semantics (Model Theory) for Sentential Logic.

Let the set $\{f_{\sim},f_{\wedge},f_{\vee},f_{\rightarrow},f_{\leftrightarrow}\}$ of ***truth-functions*** be defined: $f_{\sim}=\{<T,F>,<F,T>\}$

$f_{\wedge}=\{<T,T,T><T,F,F>,<F,T,F>,<F,F,F>\}$ $f_{\vee}=\{<T,T,T><T,F,T>,<F,T,T>,<F,F,F>\}$

$f_{\rightarrow}=\{<T,T,T><T,F,F>,<F,T,T>,<F,F,T>\}$ $f_{\leftrightarrow}=\{<T,T,T><T,F,F>,<F,T,F>,<F,F,T>\}$

A formula P is ***true in a model*** $\mathfrak{A}$ (written briefly, $\mathfrak{A}\models P$) and a ***valuation*** function $\Im$ from $\mathbf{F_{SL}}$ to {T,F} are defined ("recursively") as follows:

Basis Clause. For any atomic formula P,

either $\mathfrak{A}\models P$ or not($\mathfrak{A}\models P$), and

either $\Im(P)$=T or $\Im(P)$=F.

Inductive Clauses. The cases for molecular formulas are broken down:

$\mathfrak{A}\models\sim P$ iff not $\mathfrak{A}\models P$	$\Im(\sim P)=f_{\sim}(\Im(P))$, *i.e.*	$\Im(\sim P)$=T iff $\Im(P)\neq$T
$\mathfrak{A}\models P\wedge Q$ iff ($\mathfrak{A}\models P$ and $\mathfrak{A}\models Q$)	$\Im(P\wedge Q)=f_{\wedge}(\Im(P),\Im(Q))$, *i.e.*	$\Im(P\wedge Q)$=T iff, $\Im(P)$=T and $\Im(Q)$=T
$\mathfrak{A}\models P\vee Q$ iff ($\mathfrak{A}\models P$ or $\mathfrak{A}\models Q$)	$\Im(P\vee Q)=f_{\vee}(\Im(P),\Im(Q))$, *i.e.*	$\Im(P\vee Q)$=T iff, $\Im_{\mathbf{s}}(P)$=T or $\Im(Q)$=T
$\mathfrak{A}\models P\rightarrow Q$ iff (not $\mathfrak{A}\models P$ or $\mathfrak{A}\models Q$)	$\Im(P\rightarrow Q)=f_{\rightarrow}(\Im(P),\Im(Q))$, *i.e.*	$\Im(P\rightarrow Q)$=T iff, $\Im(P)\neq$T or $\Im(Q)$=T
$\mathfrak{A}\models P\leftrightarrow Q$ iff ($\mathfrak{A}\models P$ iff $\mathfrak{A}\models Q$)	$\Im(P\leftrightarrow Q)=f_{\leftrightarrow}\Im(P),\Im(Q))$, *i.e.*	$\Im(P\leftrightarrow Q)$=T iff, $\Im(P)$=T iff $\Im(Q)$=T

P is an SL ***logical truth*** (abbreviated $\models_{\mathbf{SL}}P$) means:

for all models $\mathfrak{A}$, $\mathfrak{A}\models P$, or in alternative notation,

for all $\Im$, $\Im(P)$=T.

The argument from premises $P_1,\ldots P_n$ to conclusion Q is SL ***valid*** (briefly, $P_1,\ldots P_n\models_{\mathbf{SL}}Q$) means:

for all models $\mathfrak{A}$, if for all $i=1,\ldots,n$ $\mathfrak{A}\models P_i$, then $\mathfrak{A}\models Q$, or in alternative notation,

for all $\Im$, if for all $i=1,\ldots,n$, $\Im(P_i)$=T, then $\Im(Q)$=T ,

A set X of formulas $P_1,\ldots P_n$ is SL ***satisfiable*** (*i.e.* "semantically consistent") means:

there is some $\mathfrak{A}$ such that for all $i=1,\ldots,n,$, $\mathfrak{A}\models P_i$, or in alternative notation

there is some $\Im$ such that for all $i=1,\ldots,n,\Im(P_i)$=T .

Metatheorem: Truth-Functionality. For any interpretation $\Im$, $\Im$ is a homomorphism from the structure $<\mathbf{F_{SL}},R_{\sim},R_{\wedge},R_{\vee},R_{\rightarrow},R_{\leftrightarrow}>$ to the structure $<\{T,F\}, f_{\sim}, f_{\wedge}, f_{\vee}, f_{\rightarrow}, f_{\leftrightarrow}>$, *i.e.*

$\Im$ maps $\mathbf{F_{SL}}$ into {T,F} and

For any R_i, $\Im(R_i(P_1,\ldots P_n))= f_i(\Im(R_i(P_1),\ldots, R_i(P_n))$

Metatheorem: Substitutivity of Material Equivalents. Let $Q(P)$ be a formula containing P,and let $Q(P')$ be like $Q(P)$ except for containing P' at some place that $Q(P)$ contains P.

If $\Im(P)=\Im(P')$, then $\Im(Q(P))=\Im(Q(P'))$, or equivalently,

if $\Im(P\leftrightarrow P')$=T, then $\Im(Q(P)\leftrightarrow Q(P'))$=T.