Examples: FOL Truth-Conditional and Validity Metatheorems

MT. $\mathfrak{M} \models P(c)$ iff, for all g, $\mathfrak{M}(c) \in P^{\mathfrak{M}}$

Proof:

$\mathfrak{M} \models P(c)$	iff	for all g, $\mathfrak{M} \models P(c)[g]$
	iff	for all g, $\llbracket c \rrbracket_g^{\mathfrak{M}} \in P^{\mathfrak{M}}$
	iff	for all g, $\mathfrak{M}(c) \in P^{\mathfrak{M}}$

MT. $\mathfrak{M} \models \neg P(c)$ iff, for all g, $\mathfrak{M}(c) \notin P^{\mathfrak{M}}$

Proof:

$\mathfrak{M} \models \neg P(c)$	iff	for all g, $\mathfrak{M} \models \neg P(c)[g]$
	iff	for all g, $\llbracket c \rrbracket_g^{\mathfrak{M}} \notin P^{\mathfrak{M}}$
	iff	for all g, $\mathfrak{M}(c) \notin P^{\mathfrak{M}}$

MT. $\mathfrak{M} \models P(x)$ iff, for all g, $d \in P^{\mathfrak{M}}$

Proof:

$\mathfrak{M} \models P(x)$	iff	for all g, $\mathfrak{M} \models P(x)[g]$
	iff	for all g, $\llbracket x \rrbracket_{g\,(x/d)}^{\mathfrak{M}} \in P^{\mathfrak{M}}$
	iff	for all g, $d \in P^{\mathfrak{M}}$

MT. $\mathfrak{M} \models \neg P(x)$ iff, for all g, $d \notin P^{\mathfrak{M}}$

Proof:

$\mathfrak{M} \models \neg P(x)$	iff	for all g, $\mathfrak{M} \models \neg P(x)[g]$
	iff	for all g, $\llbracket x \rrbracket_{g\,(x/d)}^{\mathfrak{M}} \notin P^{\mathfrak{M}}$
	iff	for all g, $d \notin P^{\mathfrak{M}}$

MT. $\mathfrak{M} \models \forall x P(x)$ iff, for all g, $\mathfrak{M}(c) \in P^{\mathfrak{M}}$

Proof:

$\mathfrak{M} \models \forall x P(x)$	iff	for all g, $\mathfrak{M} \models \forall x P(x)[g]$
	Iff	for all g, for all d in $D^{\mathfrak{M}}$, $\mathfrak{M} \models P(x)[g(x/d)]$
	iff	for all g, for all d in $D^{\mathfrak{M}}$, $\llbracket x \rrbracket_{g\,(x/d)}^{\mathfrak{M}} \in P^{\mathfrak{M}}$
	iff	for all g, for all d in $D^{\mathfrak{M}}$, $d \in P^{\mathfrak{M}}$

MT. $\mathfrak{M} \models \forall x \neg P(x)$ iff, for all g, for all d in $D^{\mathfrak{M}}$, $d \notin P^{\mathfrak{M}}$

Proof:

$\mathfrak{M} \models \forall x \neg P(x)$	iff	for all g, $\mathfrak{M} \models \forall x \neg P(x)[g]$
	Iff	for all g, for all d in $D^{\mathfrak{M}}$, $\mathfrak{M} \models \neg P(x)[g(x/d)]$
	iff	for all g, for all d in $D^{\mathfrak{M}}$, $\llbracket x \rrbracket_{g\,(x/d)}^{\mathfrak{M}} \notin P^{\mathfrak{M}}$
	iff	for all g, for all d in $D^{\mathfrak{M}}$, $d \notin P^{\mathfrak{M}}$

MT. $\mathfrak{M} \models \neg \forall x P(x)$ iff, for all g, for some d in $D^{\mathfrak{M}}$, $d \notin P^{\mathfrak{M}}$

Proof:

$\mathfrak{M} \models \neg \forall x P(x)$	iff	for all g, $\mathfrak{M} \models \neg \forall x P(x)[g]$
	iff	for all g, not ($\mathfrak{M} \models \forall x P(x)[g]$)
	Iff	for all g, not(for all d in $D^{\mathfrak{M}}$, $\mathfrak{M} \models P(x)[g(x/d)]$)
	iff	for all g, for some d in $D^{\mathfrak{M}}$, not ($\mathfrak{M} \models P(x)[g(x/d)]$)
	iff	for all g, for some d in $D^{\mathfrak{M}}$, $\llbracket x \rrbracket_{g\,(x/d)}^{\mathfrak{M}} \notin P^{\mathfrak{M}}$
	iff	for all g, for some d in $D^{\mathfrak{M}}$, $d \notin P^{\mathfrak{M}}$

MT. $\mathfrak{M} \models \forall x(P(x) \rightarrow Q(x))$ iff, for all g, for all d in $D^{\mathfrak{M}}$, $d \in P^{\mathfrak{M}}$ or $d \notin Q^{\mathfrak{M}}$

Proof:

$\mathfrak{M} \models \forall x(P(x) \rightarrow Q(x))$ iff for all g, $\mathfrak{M} \models \forall x(P(x) \rightarrow Q(x))\ [g]$

iff for all g, for all d in $D^{\mathfrak{M}}$, $\mathfrak{M} \models P(x) \rightarrow Q(x)[g(x/d)]$

iff for all g, for all d in $D^{\mathfrak{M}}$, either $\mathfrak{M} \models P(x)[g(x/d)]$ or not $\mathfrak{M} \models Q(x)[g(x/d)]$

iff for all g, for all d in $D^{\mathfrak{M}}$, $[\![x]\!]^{\mathfrak{M}}_{g\,(x/d)} \notin P^{\mathfrak{M}}$ or $[\![x]\!]^{\mathfrak{M}}_{g\,(x/d)} \in Q^{\mathfrak{M}}$

iff for all g, for all d in $D^{\mathfrak{M}}$, $d \notin P^{\mathfrak{M}}$ or $d \in Q^{\mathfrak{M}}$

MT. $\mathfrak{M} \models \forall x(P(x) \vee Q(x))$ iff, for all g, for all d in $D^{\mathfrak{M}}$, $d \in P^{\mathfrak{M}}$ or $d \in Q^{\mathfrak{M}}$

Proof:

$\mathfrak{M} \models \forall x(P(x) \vee Q(x))$ iff for all g, $\mathfrak{M} \models \forall x(P(x) \vee Q(x))\ [g]$

iff for all g, for all d in $D^{\mathfrak{M}}$, $\mathfrak{M} \models P(x) \vee Q(x)[g(x/d)]$

iff for all g, for all d in $D^{\mathfrak{M}}$, either $\mathfrak{M} \models P(x)[g(x/d)]$ or $\mathfrak{M} \models Q(x)[g(x/d)]$

iff for all g, for all d in $D^{\mathfrak{M}}$, $[\![x]\!]^{\mathfrak{M}}_{g\,(x/d)} \in P^{\mathfrak{M}}$ or $[\![x]\!]^{\mathfrak{M}}_{g\,(x/d)} \in Q^{\mathfrak{M}}$

iff for all g, for all d in $D^{\mathfrak{M}}$, $d \in P^{\mathfrak{M}}$ or $d \in Q^{\mathfrak{M}}$

MT. $\mathfrak{M} \models \exists x P(x)$ iff, for all g, for some d in $D^{\mathfrak{M}}$, $d \in P^{\mathfrak{M}}$

Proof:

$\mathfrak{M} \models \exists x P(x)$ iff for all g, $\mathfrak{M} \models \exists x P(x)[g]$

Iff for all g, for some d in $D^{\mathfrak{M}}$, $\mathfrak{M} \models P(x)[g(x/d)]$

iff for all g, for some d in $D^{\mathfrak{M}}$, $[\![x]\!]^{\mathfrak{M}}_{g\,(x/d)} \in P^{\mathfrak{M}}$

iff for all g, for some d in $D^{\mathfrak{M}}$, $d \in P^{\mathfrak{M}}$

MT. $\mathfrak{M} \models \exists x \neg P(x)$ iff, for all g, for some d in $D^{\mathfrak{M}}$, $d \notin P$

Proof:

$\mathfrak{M} \models \exists x \neg P(x)$ iff for all g, $\mathfrak{M} \models \exists x \neg P(x)[g]$

Iff for all g, for some d in $D^{\mathfrak{M}}$, $\mathfrak{M} \models \neg P(x)[g(x/d)]$

iff for all g, for some d in $D^{\mathfrak{M}}$, $[\![x]\!]^{\mathfrak{M}}_{g\,(x/d)} \notin P^{\mathfrak{M}}$

iff for all g, for some d in $D^{\mathfrak{M}}$, $d \notin P^{\mathfrak{M}}$

MT. $\mathfrak{M} \models \neg\exists x P(x)$ iff, for all g, for all d in $D^{\mathfrak{M}}$, $d \notin P^{\mathfrak{M}}$

Proof:

$\mathfrak{M} \models \neg\exists x P(x)$ iff for all g, $\mathfrak{M} \models \neg\exists x P(x)[g]$

iff for all g, not ($\mathfrak{M} \models \exists x P(x)[g]$)

Iff for all g, not(for some d in $D^{\mathfrak{M}}$, $\mathfrak{M} \models P(x)[g(x/d)]$)

iff for all g, for all d in $D^{\mathfrak{M}}$, not ($\mathfrak{M} \models P(x)[g(x/d)]$)

iff for all g, for all d in $D^{\mathfrak{M}}$, $[\![x]\!]^{\mathfrak{M}}_{g\,(x/d)} \notin P^{\mathfrak{M}}$

iff for all g, for all d in $D^{\mathfrak{M}}$, $d \notin P^{\mathfrak{M}}$

Examples: FOL Truth-Conditional and Validity Metatheorems

MT. $\mathfrak{M} \models \exists x(P(x) \wedge Q(x))$ iff, for all *g*, for some *d* in $D^{\mathfrak{M}}$, $d \in P^{\mathfrak{M}}$ and $d \in Q^{\mathfrak{M}}$

Proof:

$\mathfrak{M} \models \exists x(P(x) \wedge Q(x))$ iff for all *g*, $\mathfrak{M} \models \exists x P(x) \wedge Q(x))\ [g]$

iff for all *g*, for some *d* in $D^{\mathfrak{M}}$, $\mathfrak{M} \models P(x) \wedge Q(x)[g(x/d)]$

iff for all *g*, for some *d* in $D^{\mathfrak{M}}$, either $\mathfrak{M} \models P(x)[g(x/d)]$ and $\mathfrak{M} \models Q(x)[g(x/d)]$

iff for all *g*, for some *d* in $D^{\mathfrak{M}}$, $[\![x]\!]^{\mathfrak{M}}_{g\,(x/d)} \in P^{\mathfrak{M}}$ and $[\![x]\!]^{\mathfrak{M}}_{g\,(x/d)} \in Q^{\mathfrak{M}}$

iff for all *g*, for some *d* in $D^{\mathfrak{M}}$, $d \in P^{\mathfrak{M}}$ and $d \in Q^{\mathfrak{M}}$

MT. $\forall x(P(x) \models P(c)$

Proof : Let $\mathfrak{M}$ be arbitrary. Assume for a conditional proof that $\mathfrak{M} \models \forall x P(x)$. Then by previous truth-conditional metatheorems: for all *g,* for all *d* in $D^{\mathfrak{M}}$, $d \in P^{\mathfrak{M}}$. Since the quantification over *g* is vacuous, it may be dropped: for all *d* in $D^{\mathfrak{M}}$, $d \in P^{\mathfrak{M}}$. Since the quantification over $D^{\mathfrak{M}}$ is universal, it may be instantiated by the instance $\mathfrak{M}(c)$: $\mathfrak{M}(c) \in P^{\mathfrak{M}}$, which may be vacuously quantified: for all *g*, $\mathfrak{M}(c) \in P^{\mathfrak{M}}$. But this means by definition that $\mathfrak{M} \models P(c)$. Hence by conditional proof in the metalanguage, if $\mathfrak{M} \models \forall x P(x)$, then $\mathfrak{M} \models P(c)$. Since $\mathfrak{M}$ is arbitrary, this fact may be universally generalized: for any $\mathfrak{M}$, if $\mathfrak{M} \models \forall x P(x))$, then $\mathfrak{M} \models P(c)$. Hence by definition of $\models$, $\forall x P(x) \models P(c)$. QED.

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MT. $\forall x(P(x) \rightarrow Q(x)), P(c) \models Q(c)$

Proof : Let $\mathfrak{M}$ be arbitrary. Assume for a conditional proof that $\mathfrak{M} \models \forall x(P(x) \rightarrow Q(x))$ and $\mathfrak{M} \models P(c)$. Then by previous truth-conditional metatheorems: for all *g,* for all *d* in $D^{\mathfrak{M}}$, $d \notin P^{\mathfrak{M}}$ or $d \in Q^{\mathfrak{M}}$, and for all *g,* $\mathfrak{M}(c) \in P^{\mathfrak{M}}$. Since the quantifications over g are vacuous, they may be dropped: for all *d* in $D^{\mathfrak{M}}$, $d \notin P^{\mathfrak{M}}$ or $d \in Q^{\mathfrak{M}}$, and $\mathfrak{M}(c) \in P^{\mathfrak{M}}$. Since the quantification over $D^{\mathfrak{M}}$ is universal, it may be instantiated by the instance $\mathfrak{M}(c)$: $\mathfrak{M}(c) \notin P^{\mathfrak{M}}$ or $\mathfrak{M}(c) \in Q^{\mathfrak{M}}$. This combines with the earlier fact that $\mathfrak{M}(c) \in P^{\mathfrak{M}}$ to entail by truth-functional logic in the metalanguage that $\mathfrak{M}(c) \in Q^{\mathfrak{M}}$, which may in turn be vacuously quantified: for all *g*, $\mathfrak{M}(c) \in Q^{\mathfrak{M}}$. But this means by definition that $\mathfrak{M} \models Q(c)$. Hence by conditional proof in the metalanguage, if $\mathfrak{M} \models \forall x(P(x) \rightarrow Q(x))$ and $\mathfrak{M} \models P(c)$, then $\mathfrak{M} \models Q(c)$. Since $\mathfrak{M}$ is arbitrary, this fact may be universally generalized: for any $\mathfrak{M}$, if $\mathfrak{M} \models \forall x(P(x) \rightarrow Q(x))$ and $\mathfrak{M} \models P(c)$, then $\mathfrak{M} \models Q(c)$. Hence by definition of $\models$, $\forall x(P(x) \rightarrow Q(x)), P(c) \models Q(c)$. QED.

MT. $\forall x(P(x) \rightarrow Q(x)), \neg Q(c) \models \neg P(c)$

Proof : Let $\mathfrak{M}$ be arbitrary. Assume for a conditional proof that $\mathfrak{M} \models \forall x(P(x) \rightarrow Q(x))$ and $\mathfrak{M} \models \neg Q(c)$. Then by previous truth-conditional metatheorems: for all *g,* for all *d* in $D^{\mathfrak{M}}$, $d \notin P^{\mathfrak{M}}$ or $d \in Q^{\mathfrak{M}}$, and for all *g,* $\mathfrak{M}(c) \notin Q^{\mathfrak{M}}$. Since the quantifications over g are vacuous, they may be dropped: for all *d* in $D^{\mathfrak{M}}$, $d \notin P^{\mathfrak{M}}$ or $d \in Q^{\mathfrak{M}}$, and $\mathfrak{M}(c) \notin Q^{\mathfrak{M}}$. Since the quantification over $D^{\mathfrak{M}}$ is universal, it may be instantiated by the instance $\mathfrak{M}(c)$: $\mathfrak{M}(c) \notin P^{\mathfrak{M}}$ or $\mathfrak{M}(c) \in Q^{\mathfrak{M}}$. This combines with the earlier fact that $\mathfrak{M}(c) \notin Q^{\mathfrak{M}}$ to entail by truth-functional logic in the metalanguage that $\mathfrak{M}(c) \notin P^{\mathfrak{M}}$, which may in turn be vacuously quantified: for all *g*, $\mathfrak{M}(c) \notin P$. But this means by definition that $\mathfrak{M} \models \neg P(c)$. Hence by conditional proof in the metalanguage, if $\mathfrak{M} \models \forall x(P(x) \rightarrow Q(x))$ and $\mathfrak{M} \models \neg Q(c)$, then $\mathfrak{M} \models \neg P(c)$. Since $\mathfrak{M}$ is arbitrary, this fact may be universally generalized: for any $\mathfrak{M}$, if $\mathfrak{M} \models \forall x(P(x) \rightarrow Q(x))$ and $\mathfrak{M} \models \neg Q(c)$, then $\mathfrak{M} \models \neg P(c)$. Hence by definition of $\models$, $\forall x(P(x) \rightarrow Q(x)), \neg Q(c) \models \neg P(c)$. QED

MT. $P(c) \models \exists x(P(x)$

Proof : Let $\mathfrak{M}$ be arbitrary. Assume for a conditional proof that $\mathfrak{M} \models P(c)$. By previous truth-conditional metatheorems: $\mathfrak{M}(c) \in P^{\mathfrak{M}}$. Let us give a name of convience d' to $\mathfrak{M}(c)$ in $D^{\mathfrak{M}}$. Hence $d' \in P^{\mathfrak{M}}$. This d' may be existentially generalized: for some *d* in $D^{\mathfrak{M}}$, $d' \in P^{\mathfrak{M}}$. A vacuous quantificastion over *g* may now be added: for all *g,* for some *d* in $D^{\mathfrak{M}}$, $d \in P^{\mathfrak{M}}$. Then, by an earlier truth-conditional metatheorem, $\mathfrak{M} \models \exists x P(x)$.

Hence by conditional proof in the metalanguage, if $\mathfrak{M}\models P(c)$, then $\mathfrak{M}\models\exists xP(x)$. Since $\mathfrak{M}$ is arbitrary, this fact may be universally generalized: for any $\mathfrak{M}$, if $\mathfrak{M}\models P(c)$, then $\mathfrak{M}\models\exists xP(x)$. Hence by definition of $\models$, $P(c)\models\exists xP(x)$. QED.

MT. $\forall$x(P(x)$\models\exists x(x)$

Proof : Let $\mathfrak{M}$ be arbitrary. Assume for a conditional proof that $\mathfrak{M}\models\forall xP(x)$. Then by previous truth-conditional metatheorems: for all *g,* for all *d* in $D^{\mathfrak{M}}$, $d\in P^{\mathfrak{M}}$. Since the quantification over *g* is vacuous, it may be dropped: for all *d* in $D^{\mathfrak{M}}$, $d\in P^{\mathfrak{M}}$. Since the quantification over $D^{\mathfrak{M}}$ is universal, it may be instantiated by the instance d' : $d'\in P^{\mathfrak{M}}$. This d' may be existentially generalized: for some *d* in $D^{\mathfrak{M}}$, $d'\in P^{\mathfrak{M}}$. A vacuous quantificastion over *g* may now be added: for all *g,* for some *d* in $D^{\mathfrak{M}}$, $d\in P^{\mathfrak{M}}$. Then, by an earlier truth-conditional metatheorem, $\mathfrak{M}\models\exists xP(x)$. Hence by conditional proof in the metalanguage, if $\mathfrak{M}\models\forall xP(x)$, then $\mathfrak{M}\models\exists xP(x)$. Since $\mathfrak{M}$ is arbitrary, this fact may be universally generalized: for any $\mathfrak{M}$, if $\mathfrak{M}\models\forall xP(x)$, then $\mathfrak{M}\models\exists xPxc)$. Hence by definition of $\models$, $\forall xP(x)\models\exists xP(x)$. QED.

MT. (Barbara) $\forall x(M(x)\rightarrow P(x))$, $\forall x(S(x)\rightarrow M(x))\models\forall x(S(x)\rightarrow P(x))$

Proof : Let $\mathfrak{M}$ be arbitrary. Assume for a conditional proof that $\mathfrak{M}\models\forall x(M(x)\rightarrow P(x))$ and $\mathfrak{M}\models\forall x(S(x)\rightarrow M(x))$. Then by previous truth-conditional metatheorems: for all *g,* for all *d* in $D^{\mathfrak{M}}$, $d\notin M^{\mathfrak{M}}$ or $d\in P^{\mathfrak{M}}$, and for all *g,* for all *d* in $D^{\mathfrak{M}}$, $d\notin S^{\mathfrak{M}}$ or $d\in M^{\mathfrak{M}}$. Since the quantifications over g are vacuous, they may be dropped: for all *d* in $D^{\mathfrak{M}}$, $d\in M^{\mathfrak{M}}$ or $d\notin P^{\mathfrak{M}}$, and for all *d* in $D^{\mathfrak{M}}$, $d\in S^{\mathfrak{M}}$ or $d\notin M^{\mathfrak{M}}$. Since the quantification over $D^{\mathfrak{M}}$ is universal, it may be instantiated to the arbitrary instance d': $d'\notin M^{\mathfrak{M}}$ or $d'\in P^{\mathfrak{M}}$, and $d'\notin S^{\mathfrak{M}}$ or $d'\in M^{\mathfrak{M}}$. By truth-functional logic in the metalanguage, it follows that $d'\notin S^{\mathfrak{M}}$ and $d'\in P^{\mathfrak{M}}$. Since d' is arbitrary, this may be universally generalized: for any d', $d'\notin S^{\mathfrak{M}}$ and $d'\in P^{\mathfrak{M}}$. Consider now an arbitrary $g'(x/d')$ such that Clearly $g'(x/d')(x)=d'$, and hence by substitutivity of =, $g'(x/d')(x)\notin S^{\mathfrak{M}}$ and $g'(x/d')(x)\in P^{\mathfrak{M}}$. Then by the turh-conditions for $\rightarrow$, $\mathfrak{M}\models S(x)\rightarrow M(x)[\,g'(x/d')\,]$ Since d' is arbitrary this may be universally generalized, for any *d* in $D^{\mathfrak{M}}$, $\mathfrak{M}\models S(x)\rightarrow M(x)[\,g'(x/d')\,]$. But then by the truth-conditions for $\forall$, $\mathfrak{M}\models\forall x(S(x)\rightarrow M(x))[\,g'\,]$. Now, since g' is also arbitray, it too may be universally generalized: for any *g,* $\mathfrak{M}\models\forall x(S(x)\rightarrow M(x))[g]$. But then by definition, $\mathfrak{M}\models\forall x(S(x)\rightarrow M(x))$. Hence by conditional proof in the metalanguage, if $\mathfrak{M}\models\forall x(M(x)\rightarrow P(x))$ and $\mathfrak{M}\models\forall x(S(x)\rightarrow M(x))$ then $\mathfrak{M}\models\forall x(S(x)\rightarrow M(x))$. Since $\mathfrak{M}$ is arbitrary, this fact too may be universally generalized: for any $\mathfrak{M}$, if $\mathfrak{M}\models\forall x(M(x)\rightarrow P(x))$ and $\mathfrak{M}\models\forall x(S(x)\rightarrow M(x))$ then $\mathfrak{M}\models\forall x(S(x)\rightarrow M(x))$. Hence by definition of $\models$, $\forall x(M(x)\rightarrow P(x))$, $\forall x(S(x)\rightarrow M(x))\models\forall x(S(x)\rightarrow P(x))$. QED.

MT. (Celarent) $\neg\exists x(M(x)\wedge P(x))$, $\forall x(S(x)\rightarrow M(x))\models\neg\exists x(S(x)\wedge P(x))$

Proof : Let $\mathfrak{M}$ be arbitrary. Assume for a conditional proof that $\mathfrak{M}\models\neg\exists x(M(x)\wedge P(x))$ and $\mathfrak{M}\models\forall x(S(x)\rightarrow M(x))$. Then by previous truth-conditional metatheorems: not (for some *g,* for all *d* in $D^{\mathfrak{M}}$, $d\in M^{\mathfrak{M}}$ and $d\in P^{\mathfrak{M}}$), and for all *g,* for all *d* in $D^{\mathfrak{M}}$, $d\notin S^{\mathfrak{M}}$ or $d\in M^{\mathfrak{M}}$. By quantifier negation and DeMorgans Law in the metalanguage, for all *g,* for some *d* in $D^{\mathfrak{M}}$, $d\notin M^{\mathfrak{M}}$ and $d\notin P^{\mathfrak{M}}$. Since the quantifications over g are vacuous, they may be dropped: for all *d* in $D^{\mathfrak{M}}$, $d\notin S^{\mathfrak{M}}$ or $d\in M$, and for some *d* in $D^{\mathfrak{M}}$, $d\notin M^{\mathfrak{M}}$ and $d\notin P^{\mathfrak{M}}$. Let us give a name of convenience to this "some *d*" and call it *d′*, and universally instantiate the "for all *d*" to *d′* as well: $d' \notin S^{\mathfrak{M}}$ or $d' \in M^{\mathfrak{M}}$, and $d' \notin M^{\mathfrak{M}}$ and $d' \notin P^{\mathfrak{M}}$. Since $d' \notin M^{\mathfrak{M}}$, and either $d' \notin S^{\mathfrak{M}}$ or $d' \in M^{\mathfrak{M}}$, it follows by truth-conditional logic in the metalanguage that $d' \notin S$. Hence, $d' \notin S^{\mathfrak{M}}$ and $d' \notin P^{\mathfrak{M}}$. This case of *d′* may be existentially generalized: for some *d* in $D^{\mathfrak{M}}$, $d \notin S^{\mathfrak{M}}$ and $d' \notin P^{\mathfrak{M}}$, to which a vacuous universal quantification over *g* may be added: for all *g,* for some *d* in $D^{\mathfrak{M}}$, $d \notin S^{\mathfrak{M}}$ and $d' \notin P^{\mathfrak{M}}$. Then by DeMorgan's Law and quantifier negation in the metalanguage, not (for some *g,* for all *d* in $D^{\mathfrak{M}}$, $d' \in S^{\mathfrak{M}}$ and $d' \in P^{\mathfrak{M}}$). Hence, by previous truth-conditional metatheorems, $\mathfrak{M} \models \neg\exists x(S(x) \wedge P(x))$. Hence by conditional proof in the metalanguage, if $\mathfrak{M} \models \neg\exists x(M(x) \wedge P(x))$ and $\mathfrak{M} \models \forall x(S(x) \rightarrow M(x))$ then $\mathfrak{M} \models \neg\exists x(S(x) \wedge M(x))$. Since $\mathfrak{M}$ is arbitrary, this fact too may be universally generalized: for any $\mathfrak{M}$, if $\mathfrak{M} \models \neg\exists x(M(x) \wedge P(x))$ and $\mathfrak{M} \models \forall x(S(x) \rightarrow M(x))$ then $\mathfrak{M} \models \neg\exists x(S(x) \wedge M(x))$. Hence by definition of $\models$, $\neg\exists x(M(x) \wedge P(x)), \forall x(S(x) \rightarrow M(x)) \models \neg\exists x(S(x) \wedge P(x))$. QED.