1. Russell's Problems. Russell offered what he took to be knock down arguments against the reduction of relational statements (e.g. aRb) to subject predicate statements (e.g. Fa, Fb, Ga etc.)[1]

Proposal 1. aRb **resolves to** $Fa \wedge Fb$ (for some F)

Theorem. $aRb \rightarrow bRa$

Proof: Suppose aRb. Then by def $Fa \wedge Fb$. Then $Fb \wedge Fa$. Then by def bRa. QED.

Bad Result: all relations are symmetric

Proposal 2. aRb **resolves to** $Fa \wedge Gb$ (for some F and G)

Theorem. $(aRb \wedge bRc) \rightarrow aRc$

Proof. Assume aRb and bRc. The by def $Fa \wedge Gb$, and $Fb \wedge Gc$. Then $Fa \wedge Gb$. Then by def aRb. QED.

Bad Result: all relations are transitive.

2. Leibniz' Account. Relational Propositions as Exponible (*exponibilia*) to Conjunctions of Subject Predicate Propositions.[2]

Case 1. R symmetric

a bears R to b **resolves to** a is P + preposition + b (in oblique case)
resolves to a is P + preposition + a thing which is b (nominative case)[3]
resolves to a is P $\wedge$ b is P

Example: *a is similar to b* **resolves to** *a is similar and b is similar*

Theorem. Every symmetric relation is transitive. (Proof obvious.)

[1] See Bertrand Russell, *Principles of Mathematics* [1903] (N.Y: Norton), Chapter II, §§ 23 & 24, and *A Critical Exposition of the Philosophy of Leibniz*.
[2] The exposition here follows Massimo Mugnai, *Leibniz' Theory of Relations*. 1992
[3] Leibniz holds that only nouns in the nominative case (not those in the so-classed "oblique cases" -- the genitive, dative, accusative, and ablative) stand for substances. In Latin, subjects (and predicates when joined by the verb *to be* as the copula) are in the nominative. Oblique cases are used to express, for example, the direct object (the accusitive), and various relations that in English we often express by prepositions, for example, the possessive and partive (the genitive), indirect object, opposition to, separation from (the dative), or means by which, agency, respect, place from (the ablative). Hence, Leibniz was careful to insert steps that transformed the terms describing the relata into nominative forms.

Case 2. R asymmetric:

Analysis of qua statements as Reduplicative or Reflexive:[4]

a is P qua Q **resolves to** a is P ∧ ["because"] a is Q ∧ ["as a rule"] (a is Q → a is P)
a is P ∧ a is Q ∧ (a is Q → a is P)

b is P qua S **resolves to** b is P ∧ ["because"] b is S ∧ ["as a rule"] (b is S → b is P)
b is P ∧ b is S ∧ (b is S → b is P)

A quatenus B **resolves to** A because B and as a rule B → A
A ∧ B ∧ (B → A)

a bears R to b **resolves to** a is P + preposition + b (in oblique case)
resolves to a is P + preposition + a thing which is b (nominative case)
resolves to a is P qua Q + quatenus + b is P qua S
resolves to (a is P qua Q) ∧ (b is P qua S) ∧ ((b is P qua S)→ (a is P qua Q)
resolves to (a is P ∧ a is Q ∧ (a is Q → a is P)) ∧
(b is P ∧ b is S ∧ (b is S → b is P)) ∧
((b is P ∧ b is S ∧ (b is S → b is P))→ (a is P ∧ a is Q ∧ (a is Q → a is P))
resolves to a is P ∧
a is Q ∧
b is P ∧
b is S ∧
(a is Q → a is P)) ∧
(b is S → b is P)) ∧
((b is P ∧ b is S ∧ (b is S → b is P))→ (a is P ∧ a is Q ∧ (a is Q → a is P))

If → is material equivalence, which is probably isn't,

((A∧B∧(B→A))→(C∧D∧(D→C)) is tautologically equivalent to (A∧B)→(C∧D)

and then:

a bears R to b **resolves to** a is P ∧
a is Q ∧
b is P ∧
b is S ∧
(a is Q → a is P)) ∧
(b is S → b is P)) ∧
((b is P ∧ b is S)→ (a is P ∧ a is Q)

Example: *a is wiser than b.*

a is wiser with respect to b **resolves to**

a is wiser than a thing which is b;

a is wise qua greater insofar as b is wise qua lessor;

a is greater and as a rule what is grater must be wise, insofar as b is lesser and as a rule what is lesser must be wise;

a is wise and b is wise, a is greater and b is lesser, and as a rule what is greater is wise and what is lesser is wise, and as a rule if b is wise qua lesser then a is wise qua greater.

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[4] *Qua (as), quatenus (insofar as), seducum quod (according to which), in quantum (inasmuch as)* are explained by Leibniz ,"in the general expression *quatenus* means: with regard to the sentence which follows; for instance; *man is immortal quatenus man has a mind*. That is, man is immortal in respect to a relation to this: man has a mind. Hence I often use this in resolving *(in resolvendo)*." See Mugnai, p 66.

Theorem: if R is asymmetric, then R is transitive:

Proof. Let R be asymmetric and assume aRb and bRc.
Since a bears R to b, it follows that:

1 a is P $\wedge$
2 a is Q $\wedge$
3 b is P $\wedge$
4 b is S $\wedge$
5 (a is Q $\rightarrow$ a is P)) $\wedge$
6 (b is S $\rightarrow$ b is P)) $\wedge$
7 ((b is P $\wedge$ b is S $\wedge$ (b is S $\rightarrow$ b is P))$\rightarrow$ (a is P $\wedge$ a is Q $\wedge$ (a is Q $\rightarrow$ a is P))

Since b bears R to c, it follows that:

8 b is P $\wedge$
9 b is Q $\wedge$
10 c is P $\wedge$
11 c is S $\wedge$
12 (b is Q $\rightarrow$ b is P)) $\wedge$
13 (c is S $\rightarrow$ c is P)) $\wedge$
14 ((c is P $\wedge$ c is S $\wedge$ (c is S $\rightarrow$ c is P))$\rightarrow$ (b is P $\wedge$ b is Q $\wedge$ (b is Q $\rightarrow$ b is P))

The defining conditions of a bears R to c are therefore satisfied:

15 a is P $\wedge$ from 1
16 a is Q $\wedge$ from 2
17 c is P $\wedge$ from 10
18 c is S $\wedge$ from 11
19 (a is Q $\rightarrow$ a is P)) $\wedge$ from 5
20 (c is S $\rightarrow$ c is P)) $\wedge$ from 13
21 ((c is P $\wedge$ c is S $\wedge$ (c is S $\rightarrow$ c is P))$\rightarrow$ (a is P $\wedge$ a is Q $\wedge$ (a is Q $\rightarrow$ a is P))
from 7 and 14. QED.

Theorem. All relations are transitive. (By combining the two precious theorems.)

Open Questions: Leibniz hoped to be able to use analyses like these to provide relational resolutions of the following argument forms that Leibniz (and others) accepted as valid but which do not fit the valid forms of the syllogistic:

Argumentum a recto ad obliquum

Christ is God
Therefore, the mother of Christ is the mother of God

Painting is an art
Therefore, he who learns painting learns an art

Every nut is a fruit
Therefore, every nut eater is a fruit eater.

Inversio relationis

David is the father of Solomon
Therefore, Solomon is the son of David

Let us adopt the following set theoretic definitions:

$F(R)=\{z|\exists x(<x,z>\in R) \vee \exists y(<z,y>\in R)\}$ F(R) is the *field* of R

$R"\{y\}=\{x|<x,y>\in R\}$ R"{y} is the *image* of {y} under R

$\{x\}"R=\{y|<x,y>\in R\}$ {x}"R" is the *counter-image* of {x} under R

The following are trivial consequences of the defintions:

$R"\{y\}\subseteq F(R)$

$\{x\}"R"\subseteq F(R)$

$<x,y>\in R$ iff $x\in R"\{y\}$ iff $y\in\{x\}"R$

$[<x,y>\in R \vee x\in R"\{y\} \vee y\in\{x\}"R] \rightarrow (x\in F(R) \wedge y\in F(R))$

Let us adopt the following translations into set theory with repect to a give relational Leibniz's analysist of the assertion"a bears R to b":

"x is P" as $x\in F(R)$

"x is Q" as $x\in R"\{y\}$

"x is S" as $y\in\{x\}"R$

a is P qua Q **resolves to** a is P $\wedge$ a is Q $\wedge$ (a is Q $\rightarrow$ a is P)
translates as $a\in F(R)\wedge a\in R"\{b\}\wedge R"\{b\}\subseteq F(R)$

b is P qua S **resolves to** b is P $\wedge$ b is S $\wedge$ (b is S $\rightarrow$ b is P)
translates as $b\in F(R)\wedge b\in\{a\}R"\wedge\{a\}R"\subseteq F(R)$

A quatenus B **resolves to** A because B and as a rule B $\rightarrow$ A

a bears R to b **resolves to** a is P qua Q + quatenus + b is P qua S
resolves to (a is P $\wedge$ a is Q $\wedge$ (a is Q $\rightarrow$ a is P)) $\wedge$
(b is P $\wedge$ b is S $\wedge$ (b is S $\rightarrow$ b is P)) $\wedge$
((b is P $\wedge$ b is S $\wedge$ (b is S $\rightarrow$ b is P))$\rightarrow$ (a is P $\wedge$ a is Q $\wedge$ (a is Q $\rightarrow$ a is P))

translates as $a\in F(R)\wedge a\in R"\{b\}\wedge R"\{b\}\subseteq F(R) \wedge$
$b\in F(R)\wedge b\in\{a\}R"\wedge\{a\}R"\subseteq F(R) \wedge$
$[b\in F(R)\wedge b\in\{a\}R"\wedge\{a\}R"\subseteq F(R]\rightarrow[a\in F(R)\wedge a\in R"\{b\} \wedge R"\{b\}\subseteq F(R)]$

<u>Remark.</u> The success of the analysis turns on trivial properties of relations. Every (two-place) relation R determines a monadic non-relational subject-predicate property, namely being a member of the relation's field F(R), i.e. of

standing in tha relation to something or other. That is, part of what it means to say that x stands in a relation R is that x is a member of the field of R. Now, the fact that x is in F(R) can be further specified so as to limit which entity it is that x stands to under R . This may be done by appleal to a second-order properties of F(R) – or as medivals would say, by appleal to one of its properties in second intension, a modes of a mode. For each entity in the range of R (defined as $\{y|\exists x(<x,y>\in R)\}$), there is a second order set. Ler r range of relations. We can define this set as $K_y=\{r|\exists x(x \text{ bears } r \text{ to } y)\}$. (Equivalently, for any R, there is a second order property of R defined by the open sentence $\exists x(x \text{ bears } R \text{ to } y)$. Thus, the fact that x bears R to y, can be expressed as the conjunction of two facts: that a first order property holds of x, namely $x\in F(R)$, and that a second order property holds of R, namely $R\in K_y$. Thus, in an Aristotelian ontology the relational fact that x bears R to y is explicable interms of substances, their modes, and modes of modes.

A simplification, however, is possible because when x bears R to y, the second order fact that $R\in K_y$ biuniquely determines a first order fact about x, namely that x is a member of a first order set, the image of {y} under R. Thus, the fact that x bears R to y may be formulated as the conjuction of two first-order facts: $x\in F(R)\wedge x\in R''\{y\}$. Thus, this monadic or subject-predicate analysis of relations may be understood in the context of an Aristotelian ontology as committed at a miminum to substances and their first order modes.

Exactly similar facts hold for any entity y that is in the range of R. To say x bears R to Y is equivalent to saying that As befor in an Aristotelian ontologically the theory could be understood as presupposing substance, and at most their first and second order modes.

Moreover, in this analysis the general rules layed down within Leibniz as part of his analysis of “qua” and “quatanus” hold as consequences of the set theoretic definitions. The “qua” relation holds because when $x\in F(R)$ qua $x\in R''\{y\}$ it automatically holds that $R''\{y\}\subseteq F(R)$ and that when $y\in F(R)$ qua $y\in\{x\}''R$, that $\{x\}''R\subseteq F(R)$. Likewise Leibniz’ “quatanus” requirment that when x bears R to y the moadic facts of y entail those of x hold because again in set theory $y\in F(R)\wedge y\in\{x\}''R$ entails $x\in F(R)\wedge x\in R''\{y\}$.

Theorem. “a bears R to b” as translated above is equivalent to $<a,b>\in R$

Remark. Since this result is completely general for any relation R, it does not entail that R has an properties that hold for only a proper subset of relations, e.g. refelxitivy, symmetry, or transitivity.