Leibniz' *De arte combinatoria*

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I. Introduction

Logicians, philosophers and to judge from the Internet even the general public are vaguely aware that Leibniz held views about logic that anticipate modern ideas of proof system and algorithm. Though there are many places in Leibniz' works that might be cited as evidence for such claims, popular works cite virtually only two of Leibniz' shorter papers, *Characteristica universalis* and *De arte combinatoria.* Curiously, though there are hundreds, maybe thousands, of references to these papers, nothing serious has been written in recent decades about the papers themselves that could be called a professional exegesis or discussion of their logical content. The purpose of this short paper is to remedy that lack by offering a "reconstruction" of the system Leibniz sketches in *De arte combinatoria*, which of the two essays is the one more focused on the notions of proof and algorithm.

A point of caution about method should be made at the outset. Any modern "reconstruction" of views in the history of logic is by its nature a compromise. It is an attempt to preserve as much of the original content, including its terminology and formulas, as is possible while simultaneously meeting the standards of modern metatheory. For example, if it is possible to do justice to the original by observing standard formats, then they should be followed. For example, if it is fair to the text, it is desirable to define the syntax inductively, state definitions set theoretically, develop notions of proof within an axiom or natural deduction system, and define semantic ideas in a recursive manner parallel to syntax. It is largely the presence of these familiar frameworks that make reconstructions comparable in fruitful and interesting ways to modern logic. Fortunately Leibniz' theory lends itself to such a modern formulation – it is this fact after all that lies behind the claims that he anticipates modern ideas.

The reconstruction offered here is intended to capture the main logical ideas of *De arte combinatoria*, but it departs from the text in several ways. It simplifies some ideas, expands other to fill in what are from a modern perspective lacunae in the original, and it employs set theoretic definitions when doing so does not distort the original. It also supplements the relatively simple essay, which Leibniz wrote when only eighteen, with several ideas from his more mature metaphysics as developed in the *Monadology.* Included for example are the ideas of *infinite concepts, existence, God, positive* and *negative properties* and explicit analyses of *truth* and *necessity,* as these ideas are developed in this later work.

The concepts from his metaphysics are included because, as any student of Leibniz knows, they are closely related, even defined, in the larger metaphysical theory by reference to logical ideas. Necessary truth, possible world, essence, *a priori* knowledge, human epistemic imperfection, and compatibilistic freedom all depend on ideas from logic. But as any student of Leibniz also knows, the root logical ideas are not developed in the metaphysical works themselves. What is the sort of proof that God can do from necessary premises that humans cannot? Why are some proofs infinite? How could God evaluate an infinite proof? In what way do facts about individuals the actual world follow by necessity from a full specification of the essence of that world? The only accounts by Leibniz of the relevant logical concepts are found in the short exploratory essays like *De arte combinatoria*. These essays provide surprisingly clear – if technically limited and conceptually disputable – answers to these questions. These answers extrapolated from such essays cannot, of course, be taken as Leibniz' mature opinion because the essays are at best provisional. They are little more than exercise in which Leibniz tests how he might work out his early ideas of proof. Though some of his later papers are longer and more detailed, Leibniz never applied himself to writing what we would today consider a serious logical theory.

But his experiments are instructive anyway. They suggest, at the very least, the sort of logic Leibniz had in mind as underlying his other ideas. This reconstruction thus is offered as a kind of heuristic. It is an accessible modern statement of a miniature but rigorous logistic theory of the sort Leibniz had in mind as underlying his metaphysics. Its intention is to help readers understand more fully what Leibniz was getting at, both in his logic and his metaphysics. The system is also fun. It is elegant and clear. Much is entirely new in the history of syllogistic logic, and in parts anticipates work by Boole and Schröder.[1] Would that all eighteen-year-old logic students were as clever!

[1] See Volker Peckhaus, "19th Century Logic Between Logic and Mathematics" [1999]

A partial English translation of the text may be found in Parkinson (1966), and a partial edition of the original Latin text is currently on the Internet (see References).

II. Reconstruction

Syntax. The syntax begins by positing a set of basic terms that stand for primitive ideas:

First Terms: $t_1,...,t_n$. Among the first terms is *exists*.

Primitive terms may be joined together to make longer terms. In principle some of these longer terms may be infinitely long, though those of finite length are special. To define strings of first terms we make use of the concatenation operation: let $x^\frown y$ mean the result of writing (*concatenating*) x and y. (Later when there is no possibility of confusion, we shall suppress the concatenation symbol and refer to $a^\frown b^\frown c^\frown d$ as abcd.)

Finite Terms: If $t^1{}_1$and $t^1{}_2$, are first terms, then $t^1{}_1{}^\frown t^1{}_2$ is a finite term.

If $t^n{}_i$ is a finite term and $t^1{}_j$ is a first term, then $t^n{}_i{}^\frown t^1{}_j$ is a finite term.

Nothing else is a finite term.

Infinite Terms: any countably infinite subset of First Terms.

Among the infinite terms is *God*.

Terms: the union Finite Terms and Infinite Terms.

Leibniz introduces a special vocabulary for discussing finite terms:

Terms of conXnation (defined inductively):

If t^1_i is a first term,

then t^1_i is a term of con1nation with exponent 1 and rank i.

If t^n_i is a term of conNnation and t^1_j is a term of con1nation,

then $t^n_i \cap t^1_j$ a term of conN+1nation,

with exponent n+1,

and a rank that is determined by three factors:

the ranks of t^n_i, t^i_j, and the ranks of those terms of

conN+1nation that have a lesser rank than $t^n_i \cap t^1_j$.

Nothing else is a term of conXnation.

Clearly the set of all terms of conXnation for some x is identical to the set *Finite Terms*. We let t^n_i refer to the term of conJnation of rank i.

Fraction notation: if t^{n+1}_k is some term $t^n_i \cap t^1_j$ of conN+1nation, then another name for t^{n+1}_k is $<i/n, t^1_j>$.

We shall adopt some special notation for infinite terms. If $\{t^1_1,....,t^1_n,...\}$ is an infinite term (a set of first terms) we shall refer it briefly as $\{t\}_i$. A *proposition* is any expression t **is** t'such that t and t' are terms. It is permitted that these terms be infinite. A *finite proposition* is any: t^i ***is*** t^j, such that t^i is a term of conInation and t^j is a term of conJnation, for natural numbers i and j. An infinite propositions is any t^i ***is*** t^j such that both t^i and t^j are either finite or infinite terms and at least one of t^i and t^j is infinite. Notice that it follows from the definitions that though there are a finite number of first terms, there are an infinite number of finite terms and of finite propositions. A proposition that is not finite is said to be *infinite*. Such propositions will contain at least one infinite term.

Intensional Semantics

Conceptual Structure. For an intensional semantics we posit a set $\mathcal{C}$ of concepts for which there is a binary inclusion relation $\leq$ and a binary operation + of concept composition or addition. In modern metalogic the way to do this is to specify the relevant sort of "structure" understood as an abstract structure with certain specified structure features governing $\mathcal{C}$, $\leq$ and +. We also distinguish between positive and negative concepts and add concepts of existence and God. By *a Leibnizian intensional structure* is meant any structure $<\mathcal{C},\leq,+,\mathcal{G}>$ such that

1. $<\mathcal{C},\leq>$ is a partially ordered structure:

   $\leq$ is reflexive, transitive and anti-symmetric;

2. $<\mathcal{C},\wedge>$ is an infinite join semi-lattice determined by $<\mathcal{C},\leq>$:

   if $\mathcal{A}$ is an infinite subset of $\mathcal{C}$ (in which case we call $\mathcal{A}$ an *infinite concept*), then there is a least upper bound of $\mathcal{A}$ (briefly, a lub$\mathcal{A}$) in $\mathcal{C}$ (here the *least upper bound* of $\mathcal{A}$ in $\mathcal{C}$ is defined as the unique $z \in \mathcal{C}$ such that for any c in $\mathcal{A}$, $c\leq z$, and for any w, if for all c in $\mathcal{A}$, $c\leq w$, then $z\leq w$);

3. for any $c_1,...,c_n$ in $\mathcal{C}$, $c_1+...+c_n$ is defined as lub$\{c_1,...,c_n\}$,

   for any infinite subset $\mathcal{A}$ of $\mathcal{C}$, $+\mathcal{A}$ is defined as lub$\mathcal{A}$;

4. $\mathcal{G}$ (called the concept of *God*) is $+\mathcal{C}$

Theorem: If $<\mathcal{C},\leq,+,\mathcal{G}>$ is an intensional structure and let $c,d\in \mathcal{C}$, it follows that:

1. $c \leq d$ iff $c = c+d$,

2. $\mathcal{C}$ is closed under +, and + is idempotent, commutative, and associative,

3. if $\mathcal{A}$ is an infinite concept, then $c \in \mathcal{A}$ only if $c \leq +\mathcal{A}$,

4. $+\mathcal{C}$ is a supremum in $\mathcal{C}$ (i.e. for any $c \in \mathcal{C}$, $c \leq +\mathcal{C}$ and $+\mathcal{C} \in \mathcal{C}$);

Let a,b,c and d range over $\mathcal{C}$. It is also useful to have a notion of concept *subtraction.* Let c-d be defined as follows:

> if c is a finite concept, c-d is that concept b such that d+b=c, if there is such a concept, and c-d is undefined otherwise;
>
> if c is an infinite concept $\mathcal{A}$ then c-d is $\mathcal{A}$–{d}, i.e. it is the set theoretic relative complementation of $\mathcal{A}$ and {d} (i.e. c-d = $\{d | e \in c \text{ and } e \neq d\}$).

Theorem. For any c and c in $\mathcal{C}$, either $c \leq d$ or $c \leq +\mathcal{C}-d$

Intensional Interpretations.[2] By an intensional interpretation we mean any assignment of concepts to terms that mirrors their internal structure. That is, an *intensional interpretation* is any function Int with domain Terms and range $\mathcal{C}$ such that:

1. If t_i is a first term (i.e. term of con1nation), then $Int(t_i) \in \mathcal{C}$.

2. If t_k is some term $t_i{}^\frown t^1{}_j$ of conN+1nation, then $Int(t_k)=Int(t_i)+Int(t^1{}_j)$.

3. If $\{t\}_i$ is some infinite term, $Int(\{t\}_i)=+\{Int(t^1) | t^1 \in \{t\}_i\}$.

[2] The terminology and basic semantic framework used here is adapted from that of Rudolf Carnap, *Meaning and Necessity*, and Richard Montague, “Intensional Logic” [1970], reprinted in Thomason 1974.

4. Int(*God*)=$\mathcal{G}$.

(Algebraically, an intensional interpretation Int is what is called a *homomorphism* from the grammatical structure $\langle \text{Terms}, {}^\frown \rangle$ to the conceptual structure $\langle \mathcal{C},+ \rangle$.) Since Leibniz' languages are ideal, it is also plausible to require the stronger condition that the mapping Int be 1 to 1 (and hence an isomorphism), though since this extra condition plays no role here is will not be formally required.

Leibniz frequently identifies truth with conceptual inclusion. For some purposes it might be important to build the notion of an "atomic" concept into the definition of the intensional structure, but for our purposes here we shall refer to an *atomic concept* as any c in $\mathcal{C}$ that is the intension of some first term (i.e. such that for some first term t_i, Int(t_i)=c). Following modern usage, let us reserve the term analytic truth for this idea:

t_i ***is*** t_j is said to be *analytically true* for interpretation Int iff Int(t_j)≤Int(t_i).

Extensional Semantics (Possible Worlds)

Possible Worlds. In modern logic, possible worlds would be understood as extensional "models" that conform to the restrictions of a given intensional interpretation. Given the interpretation, a possible world will consist of an assignment of sets (extensions) to concepts (and hence to terms) in a manner that mirrors their internal structure. Let us define a *possible world* relative to an intensional interpretation Int to be any W that assigns "extensions" to concepts as follows: W is a function with domain $\mathcal{C}$ such that

1. If c is an atomic concept, then W(c) is some set D of possible

objects ("the objects that fall under c in the world W");

2. If c is some concept a+b, then $W(c)=W(a)\cap W(b)$.

3. if c is some infinite concept $\mathcal{A}$, then $W(c)=\bigcap\{W(d)|d\leq\mathcal{A}\}$

Finally, the extensional interpretation of the syntax in a possible world W for Int assigns to a term the set determined by its concept and a truth-value to a proposition accordingly to whether the extension of the predicate embraces than of the subject. By the *extensional interpretation* Ext_W for the possible world W relative to intensional interpretation Int assigns extensions to terms and truth-values to propositions as follows:

1. If t_i is a term, $Ext_W(t_i)=W(Int(t_i))$;

2. If t_i ***is*** t_j is a proposition, $Ext_W(t_i$ ***is*** $t_j)=T$ if $Ext_W(t_i) \subseteq Ext_W(t_j)$,

   $Ext(t_i$ ***is*** $t_j)=F$ if $not(Ext_W(t_i) \subseteq Ext_W(t_j))$.

Logical Truth. Let a proposition P be called a *logical truth* relative to Int (briefly, $\models P$) iff, for all possible worlds W of Int, $Ext_W(P)=T$.

Theorem. 1. t_i ***is*** t_j is an analytic truth relative to Int iff it is a logical truth relative to Int.

2. If $Int(t_j)\leq Int(t_i)$, then for W relative to Int, $Ext_W(t_i)\subseteq Ext_W(t_j)$.

Remark. Leibniz allows for possible worlds to vary in "perfection, " and for the use of negations to describe privations of such perfection. These ideas are essentially Neoplatonic. Logically they presuppose a ranking on "worlds" and a Neoplatonic privative negation. Such theories may be developed coherently by imposing additional features to the syntax and semantic structure, but are not developed here because they play no role in the points to be made.[3]

Proof Theory, Necessity and Contingency. Although Leibniz frequently says that all truth is conceptual inclusion, i.e. that truth is analytic truth, he also makes a distinction between necessary and contingent truths. Ordinarily in modern logic, necessary truth is identified with what we have called logical truth, and contingent truth with truth in a possible world. If all truths were analytic and necessary truth was the same logical truth, then truth and necessity collapse, and there could be no contingent truths. Leibniz avoids this problem by adopting what is now a non-standard notion of necessary truth. Leibniz defends what we would call today a proof theoretic concept of necessity by identifying necessity with provability. To do so Leibniz forges a distinction between truth defined semantically (e.g. analytic and logical truth) and a purely syntactically definable notion of a proposition's having a proof. He is arguably the first philosopher to do so clearly, and to complete the project we present here a version of his proof theory.

Proof Theory. Leibniz understands proofs to be syntactic derivations of propositions. They take what he calls "identity" propositions as axioms. Inferences progress by adding first terms to the subject of earlier propositions in the proof, or by subtracting first terms from the predicates of earlier lines. We begin by defining the set of axiom as the set of identity propositions axioms:

Basic Propositions (Axioms): any finite proposition of the form t_i ***is*** t_i.

[3] See John Martin, "Proclus and the Neoplatonic Syllogistic" [2001].

(Also called *identity propositions*.)

Inferences proceed by adding and subtracting first terms to subjects and predicates respectively.

Inference Rule:

from $t_i{}^\frown t_j$ ***is*** t_k infer $t_i{}^\frown t^{1\frown} t_j$ ***is*** t_j;

from infer t_k *is* $ti^\frown t^{1\frown} t_j$ infer t_k ***is*** $t_i{}^\frown t_j$;

The process is complicated somewhat because Leibniz envisages language as containing abbreviations in which shorter expressions are used in place of long terms for which they are synonymous. As defined in the syntax, genuine terms (in the set Terms) are all finite concatenations of first terms. These expressions we shall say are in *primitive notation*. Let us now allow that such terms may be abbreviated by a single expression. Let a *defined term* be any expression E that is defined as abbreviating a term t_i (in Terms) by means of a definition of the form: $E =_{def} t_i$. (For example we might have the definition: $A =_{def}$ abcd.) We draw together all definitions into a set that we call the *Lexicon*. Note that the Lexicon could be infinitely large. It is a standard rule in logic (and mathematics) that it is permissible to replace a term in any line of a proof by either its abbreviation (its *definiendum*) if it is a primitive term, or by its analysis into primitive notation (its *definiens*) if it is a defined term. Let P[t] be a proposition containing a term t and P[E] be like P[t] except for containing E at one or more places where P[t] contains t.

Rule of Definition: if $E =_{def} t_i$, from P[t] infer P[E], and from P[E] infer P[t].

A proof may now be defined as any derivation from the axioms by the rules:

Proof: any finite series of propositions such that each is a basic proposition or follows by the inference rules (including the Rule of Definition) from previous members of the series.

Let us say a proposition P is (*finitely*) p*rovable* (alternative terminology is P is a *theorem*, is *necessary* or in symbols $\vdash$P) iff P is the last line of some proof.

Examples: Here are four proofs (read down each column). Let A $=_{def}$ abcd:

a is a	abcd is abcd	ab is ab	A is A
ab is a	abcd is abc	abc is ab	A is abcd
abc is a	abcd is ab	abcd is ab	A is abc
abcd is a	abcd is a	A is ab	A is ab

(Following Aristotle's usage in the *Prior Analytics*, Leibniz himself talks of “reductions” instead of "proofs". A reduction is just an upside down proof in which the first line is what is to be proved and you work down the page to the basic identity axiom.) Note that it follows from the definition of proof that all proofs have a finite number of lines. It is very important for Leibniz that necessity is *finitely* provable. *Contingent* propositions, he says, are ones that are true in his sense (i.e. analytically true) but for which there is no finite proof. The concept of God or of a possible world for Leibniz are infinite concepts and the term *God* abbreviates an infinite terms standing for an infinite concept.

Remark. Notice since infinite terms are literally infinite lists of basic terms, they are infinite in length and hence are precluded from appearance in a proof. Thus thought the following inference rules that employ infinite terms are valid, they are not proof theoretical acceptable:

from $\{t\}_i$ ***is*** t_k infer $\{t\}_i \cup \{t^1\}$ is t_k;

from t_k is $\{t\}_i$ infer t_k ***is*** $\{t\}_i$–$\{t^1\}$.

Theorem. The notion of proof is sound and complete for finite propositions, i.e. provability and logical (and hence analytic) truth coincide:

Finite Soundness:

if P is finite and ├P (equivalently, P is necessary),

then ╞P (equivalently, P is analytic).

Finite Completeness:

if P is finite and ╞P (equivalently, P is analytic),

then ├P (equivalently, P is necessary).

Theorem. If P is infinite, then not(├P)

Proof. Let P contain an infinite term $\{t\}_i$, and assume for a *reductio* that ├P. Then there is some proof of P. Moreover, if $\{t\}_i$ is the subject of P, there is for every first term t^1 in $\{t\}_i$ at line introducing that term to the subject. But then since there are an infinite number of such first terms in $\{t\}_i$ there an infinite number of lines in the proof. But a proof is only finitely long. Hence by *reductio.* There is no proof of P. The reasoning is similar if $\{t\}_i$ occurs as the predicate of P. Q.E.D.

Theorem. Soundness holds for both finite and infinite propositions, but completeness fails for infinite propositions:

Soundness:

For any P, if ├P, the ╞P (equivalently, P is analytic)

Failure of Completeness: There is some infinite propositions P such that ╞P(equivalently, P is analytic) but not(├P).

Exercises:

1. If all truth is conceptual inclusion (analytic truth), is there any notion in Leibniz for “truth in a possible world” (modern day contingent truth)? (Perhaps adding the indexical modal operator *actually* would reintroduce the distinction.)

2. Is the proposition *God exists* true (i.e. analytic)? Is it provable? Is it necessary? (Prove your answer to each.) Is the constellation of answers odd? Explain.

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