Leibniz' *De arte combinatoria*

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I. Introduction

Logicians, philosophers and to judge from the Internet even the general public are vaguely aware that Leibniz held views about logic that anticipate modern ideas of proof system and algorithm. Though there are many places in Leibniz' works that might be cited as evidence for such claims, popular works cite virtually only two of Leibniz' shorter papers, *Characteristica universalis* and *De arte combinatoria.* Curiously, though there are hundreds, maybe thousands, of references to these papers, nothing serious has been written in recent decades about the papers themselves that could be called a professional exegesis or discussion of their logical content. The purpose of this short paper is to remedy that lack by offering a "reconstruction" of the system Leibniz sketches in *De arte combinatoria*, which of the two essays is the one more focused on the notions of proof and algorithm.

A point of caution about method should be made at the outset. Any modern "reconstruction" of views in the history of logic is by its nature a compromise. It is an attempt to preserve as much of the original content, including its terminology and formulas, as is possible while simultaneously meeting the standards of modern metatheory. For example, if it is possible to do justice to the original by observing standard formats, then they should be followed. For example, if it is fair to the text, it is desirable to define the syntax inductively, state definitions set theoretically, develop notions of proof within an axiom or natural deduction system, and define semantic ideas in a recursive manner parallel to syntax. It is largely the presence of these familiar frameworks that make reconstructions comparable in fruitful and interesting ways to modern logic. Fortunately Leibniz' theory lends itself to such a modern formulation – it is this fact after all that lies behind the claims that he anticipates modern ideas.

The reconstruction offered here is intended to capture the main logical ideas of *De arte combinatoria*, but it departs from the text in several ways. It simplifies some ideas, expands other to fill in what are from a modern perspective lacunae in the original, and it employs set theoretic definitions when doing so does not distort the original. It also supplements the relatively simple essay, which Leibniz wrote when only eighteen, with several ideas from his more mature metaphysics as developed in the *Monadology.* Included for example are the ideas of *infinite concepts, existence, God, positive* and *negative properties* and explicit analyses of *truth* and *necessity,* as these ideas are developed in this later work.

The concepts from his metaphysics are included because, as any student of Leibniz knows, they are closely related, even defined, in the larger metaphysical theory by reference to logical ideas. Necessary truth, possible world, essence, *a priori* knowledge, human epistemic imperfection, and compatibilistic freedom all depend on ideas from logic. But as any student of Leibniz also knows, the root logical ideas are not developed in the metaphysical works themselves. What is the sort of proof that God can do from necessary premises that humans cannot? Why are some proofs infinite? How could God evaluate an infinite proof? In what way do facts about individuals the actual world follow by necessity from a full specification of the essence of that world? The only accounts by Leibniz of the relevant logical concepts are found in the short exploratory essays like *De arte combinatoria*. These essays provide surprisingly clear – if technically limited and conceptually disputable – answers to these questions. These answers extrapolated from such essays cannot, of course, be taken as Leibniz' mature opinion because the essays are at best provisional. They are little more than exercise in which Leibniz tests how he might work out his early ideas of proof. Though some of his later papers are longer and more detailed, Leibniz never applied himself to writing what we would today consider a serious logical theory.

But his experiments are instructive anyway. They suggest, at the very least, the sort of logic Leibniz had in mind as underlying his other ideas. This reconstruction thus is offered as a kind of heuristic. It is an accessible modern statement of a miniature but rigorous logistic theory of the sort Leibniz had in mind as underlying his metaphysics. Its intention is to help readers understand more fully what Leibniz was getting at, both in his logic and his metaphysics. The system is also fun. It is elegant and clear. Much is entirely new in the history of syllogistic logic, and in parts anticipates work by Boole and Schröder.[1] Would that all eighteen-year-old logic students were as clever!

[1] See Volker Peckhaus, "19th Century Logic Between Logic and Mathematics" [1999]

A partial English translation of the text may be found in Parkinson (1966), and a partial edition of the original Latin text is currently on the Internet (see References).

II. Reconstruction

Syntax. The syntax begins by positing a set of basic terms that stand for primitive ideas:

First Terms: $t_1,...,t_n$. Among the first terms is *exists*.

Primitive terms may be joined together to make longer terms. In principle some of these longer terms may be infinitely long, though those of finite length are special. To define strings of first terms we make use of the concatenation operation: let $x^\frown y$ mean the result of writing (*concatenating*) x and y. (Later when there is no possibility of confusion, we shall suppress the concatenation symbol and refer to $a^\frown b^\frown c^\frown d$ as abcd.)

Finite Terms: If $t^1{}_1$ and $t^1{}_2$, are first terms, then $t^1{}_1{}^\frown t^1{}_2$ is a finite term.

If $t^n{}_i$ is a finite term and $t^1{}_j$ is a first term, then $t^n{}_i{}^\frown t^1{}_j$ is a finite term.

Nothing else is a finite term.

Infinite Terms: any countably infinite subset of First Terms.

Among the infinite terms is *God*.

Terms: the union Finite Terms and Infinite Terms.

Leibniz introduces a special vocabulary for discussing finite terms:

Terms of conXnation (defined inductively):

If t^1_i is a first term,

then t^1_i is a term of con1nation with exponent 1 and rank i.

If t^n_i is a term of conNnation and t^1_j is a term of con1nation,

then $t^n_i{\frown}t^1_j$ a term of conN+1nation,

with exponent n+1,

and a rank that is determined by three factors:

the ranks of t^n_i, t^i_j, and the ranks of those terms of

conN+1nation that have a lesser rank than $t^n_i{\frown}t^1_j$.

Nothing else is a term of conXnation.

Clearly the set of all terms of conXnation for some x is identical to the set *Finite Terms*. We let t^n_i refer to the term of conJnation of rank i.

Fraction notation: if t^{n+1}_k is some term $t^n_i{\frown}t^1_j$ of conN+1nation, then another name for t^{n+1}_k is $<i/n,t^1_j>$.

We shall adopt some special notation for infinite terms. If $\{t^1_1,....,t^1_n,...\}$ is an infinite term (a set of first terms) we shall refer it briefly as $\{t\}_i$. A *proposition* is any expression t **is** t'such that t and t' are terms. It is permitted that these terms be infinite. A *finite proposition* is any: t^i ***is*** t^j, such that t^i is a term of conInation and t^j is a term of conJnation, for natural numbers i and j. An infinite propositions is any t^i ***is*** t^j such that both t^i and t^j are either finite or infinite terms and at least one of t^i and t^j is infinite. Notice that it follows from the definitions that though there are a finite number of first terms, there are an infinite number of finite terms and of finite propositions. A proposition that is not finite is said to be *infinite*. Such propositions will contain at least one infinite term.

Intensional Semantics

Conceptual Structure. For an intensional semantics we posit a set $\mathcal{C}$ of concepts for which there is a binary inclusion relation $\leq$ and a binary operation + of concept composition or addition. In modern metalogic the way to do this is to specify the relevant sort of "structure" understood as an abstract structure with certain specified structure features governing $\mathcal{C}$, $\leq$ and +. We also distinguish between positive and negative concepts and add concepts of existence and God. By *a Leibnizian intensional structure* is meant any structure $<\mathcal{C},\leq,+,\mathcal{G}>$ such that

1. $<\mathcal{C},\leq>$ is a partially ordered structure:

   $\leq$ is reflexive, transitive and anti-symmetric;

2. $<\mathcal{C},\wedge>$ is an infinite join semi-lattice determined by $<\mathcal{C},\leq>$:

   if $\mathcal{A}$ is an infinite subset of $\mathcal{C}$ (in which case we call $\mathcal{A}$ an *infinite concept*), then there is a least upper bound of $\mathcal{A}$ (briefly, a lub$\mathcal{A}$) in $\mathcal{C}$ (here the *least upper bound* of $\mathcal{A}$ in $\mathcal{C}$ is defined as the unique $z\in \mathcal{C}$ such that for any c in $\mathcal{A}$, $c\leq z$, and for any w, if for all c in $\mathcal{A}$, $c\leq w$, then $z\leq w$);

3. for any $c_1,...,c_n$ in $\mathcal{C}$, $c_1+...+c_n$ is defined as lub{ $c_1,...,c_n$},

   for any infinite subset $\mathcal{A}$ of $\mathcal{C}$, $+\mathcal{A}$ is defined as lub$\mathcal{A}$;

4. $\mathcal{G}$ (called the concept of *God*) is $+\mathcal{C}$

Theorem: If $<\mathcal{C},\leq,+,\mathcal{G}>$ is an intensional structure and let $c,d\in \mathcal{C}$, it follows that:

1. $c \leq d$ iff $c = c+d$,

2. $\mathcal{C}$ is closed under +, and + is idempotent, commutative, and associative,

3. if $\mathcal{A}$ is an infinite concept, then $c \in \mathcal{A}$ only if $c \leq +\mathcal{A}$,

4. $+\mathcal{C}$ is a supremum in $\mathcal{C}$ (i.e. for any $c \in \mathcal{C}$, $c \leq +\mathcal{C}$ and $+\mathcal{C} \in \mathcal{C}$);

Let a,b,c and d range over $\mathcal{C}$. It is also useful to have a notion of concept *subtraction.* Let c-d be defined as follows:

if c is a finite concept, c-d is that concept b such that d+b=c, if there is such a concept, and c-d is undefined otherwise;

if c is an infinite concept $\mathcal{A}$ then c-d is $\mathcal{A}$–{d}, i.e. it is the set theoretic relative complementation of $\mathcal{A}$ and {d} (i.e. c-d = $\{d|e \in c \text{ and } e \neq d\}$).

Theorem. For any c and c in $\mathcal{C}$, either $c \leq d$ or $c \leq +\mathcal{C}-d$

Intensional Interpretations.[2] By an intensional interpretation we mean any assignment of concepts to terms that mirrors their internal structure. That is, an *intensional interpretation* is any function Int with domain Terms and range $\mathcal{C}$ such that:

1. If t_i is a first term (i.e. term of con1nation), then $\text{Int}(t_i) \in \mathcal{C}$.

2. If t_k is some term $t_i{}^\frown t^1{}_j$ of conN+1nation, then $\text{Int}(t_k)=\text{Int}(t_i)+\text{Int}(t^1{}_j)$.

3. If $\{t\}_i$ is some infinite term, $\text{Int}(\{t\}_i)=+\{\text{Int}(t^1) | t^1 \in \{t\}_i\}$.

[2] The terminology and basic semantic framework used here is adapted from that of Rudolf Carnap, *Meaning and Necessity*, and Richard Montague, "Intensional Logic" [1970], reprinted in Thomason 1974.

4. Int(*God*)=$\mathcal{G}$.

(Algebraically, an intensional interpretation Int is what is called a *homomorphism* from the grammatical structure $<\text{Terms}, \cap>$ to the conceptual structure $<\mathcal{C},+>$.) Since Leibniz' languages are ideal, it is also plausible to require the stronger condition that the mapping Int be 1 to 1 (and hence an isomorphism), though since this extra condition plays no role here is will not be formally required.

Leibniz frequently identifies truth with conceptual inclusion. For some purposes it might be important to build the notion of an "atomic" concept into the definition of the intensional structure, but for our purposes here we shall refer to an *atomic concept* as any c in $\mathcal{C}$ that is the intension of some first term (i.e. such that for some first term t_i, Int(t_i)=c). Following modern usage, let us reserve the term analytic truth for this idea:

t_i ***is*** t_j is said to be *analytically true* for interpretation Int iff Int(t_j)≤Int(t_i).

Extensional Semantics (Possible Worlds)

Possible Worlds. In modern logic, possible worlds would be understood as extensional "models" that conform to the restrictions of a given intensional interpretation. Given the interpretation, a possible world will consist of an assignment of sets (extensions) to concepts (and hence to terms) in a manner that mirrors their internal structure. Let us define a *possible world* relative to an intensional interpretation Int to be any W that assigns "extensions" to concepts as follows: W is a function with domain $\mathcal{C}$ such that

1. If c is an atomic concept, then W(c) is some set D of possible

objects ("the objects that fall under c in the world W");

2. If c is some concept a+b, then $W(c)=W(a)\cap W(b)$.

3. if c is some infinite concept $\mathcal{A}$, then $W(c)=\bigcap\{W(d)|d\leq\mathcal{A}\}$

Finally, the extensional interpretation of the syntax in a possible world W for Int assigns to a term the set determined by its concept and a truth-value to a proposition accordingly to whether the extension of the predicate embraces than of the subject. By the *extensional interpretation* Ext_W for the possible world W relative to intensional interpretation Int assigns extensions to terms and truth-values to propositions as follows:

1. If t_i is a term, $Ext_W(t_i)=W(Int(t_i))$;

2. If t_i ***is*** t_j is a proposition, $Ext_W(t_i \textbf{ is } t_j)=T$ if $Ext_W(t_i) \subseteq Ext_W(t_j)$,

$Ext(t_i \textbf{ is } t_j)=F$ if $not(Ext_W(t_i) \subseteq Ext_W(t_j))$.

Logical Truth. Let a proposition P be called a *logical truth* relative to Int (briefly, $\models$P) iff, for all possible worlds W of Int, $Ext_W(P)=T$.

Theorem. 1. t_i ***is*** t_j is an analytic truth relative to Int iff it is a logical truth relative to Int.

2. If $Int(t_j)\leq Int(t_i)$, then for W relative to Int, $Ext_W(t_i)\subseteq Ext_W(t_j)$.

Remark. Leibniz allows for possible worlds to vary in "perfection, " and for the use of negations to describe privations of such perfection. These ideas are essentially Neoplatonic. Logically they presuppose a ranking on "worlds" and a Neoplatonic privative negation. Such theories may be developed coherently by imposing additional features to the syntax and semantic structure, but are not developed here because they play no role in the points to be made.[3]

Proof Theory, Necessity and Contingency. Although Leibniz frequently says that all truth is conceptual inclusion, i.e. that truth is analytic truth, he also makes a distinction between necessary and contingent truths. Ordinarily in modern logic, necessary truth is identified with what we have called logical truth, and contingent truth with truth in a possible world. If all truths were analytic and necessary truth was the same logical truth, then truth and necessity collapse, and there could be no contingent truths. Leibniz avoids this problem by adopting what is now a non-standard notion of necessary truth. Leibniz defends what we would call today a proof theoretic concept of necessity by identifying necessity with provability. To do so Leibniz forges a distinction between truth defined semantically (e.g. analytic and logical truth) and a purely syntactically definable notion of a proposition's having a proof. He is arguably the first philosopher to do so clearly, and to complete the project we present here a version of his proof theory.

Proof Theory. Leibniz understands proofs to be syntactic derivations of propositions. They take what he calls "identity" propositions as axioms. Inferences progress by adding first terms to the subject of earlier propositions in the proof, or by subtracting first terms from the predicates of earlier lines. We begin by defining the set of axiom as the set of identity propositions axioms:

Basic Propositions (Axioms): any finite proposition of the form t_i ***is*** t_i.

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[3] See John Martin, "Proclus and the Neoplatonic Syllogistic" [2001].

(Also called *identity propositions*.)

Inferences proceed by adding and subtracting first terms to subjects and predicates respectively.

Inference Rule:

from $t_i{}^\frown t_j$ ***is*** t_k infer $t_i{}^\frown t^{1\frown} t_j$ ***is*** t_j;

from infer t_k *is* $ti^\frown t^{1\frown} t_j$ infer t_k ***is*** $t_i{}^\frown t_j$;

The process is complicated somewhat because Leibniz envisages language as containing abbreviations in which shorter expressions are used in place of long terms for which they are synonymous. As defined in the syntax, genuine terms (in the set Terms) are all finite concatenations of first terms. These expressions we shall say are in *primitive notation*. Let us now allow that such terms may be abbreviated by a single expression. Let a *defined term* be any expression E that is defined as abbreviating a term t_i (in Terms) by means of a definition of the form: $E =_{def} t_i$. (For example we might have the definition: $A =_{def}$ abcd.) We draw together all definitions into a set that we call the *Lexicon*. Note that the Lexicon could be infinitely large. It is a standard rule in logic (and mathematics) that it is permissible to replace a term in any line of a proof by either its abbreviation (its *definiendum*) if it is a primitive term, or by its analysis into primitive notation (its *definiens*) if it is a defined term. Let P[t] be a proposition containing a term t and P[E] be like P[t] except for containing E at one or more places where P[t] contains t.

Rule of Definition: if $E =_{def} t_i$, from P[t] infer P[E], and from P[E] infer P[t].

A proof may now be defined as any derivation from the axioms by the rules:

Proof: any finite series of propositions such that each is a basic proposition or follows by the inference rules (including the Rule of Definition) from previous members of the series.

Let us say a proposition P is (*finitely*) p*rovable* (alternative terminology is P is a *theorem*, is *necessary* or in symbols $\vdash$P) iff P is the last line of some proof.

Examples: Here are four proofs (read down each column). Let A $=_{def}$ abcd:

a is a	abcd is abcd	ab is ab	A is A
ab is a	abcd is abc	abc is ab	A is abcd
abc is a	abcd is ab	abcd is ab	A is abc
abcd is a	abcd is a	A is ab	A is ab

(Following Aristotle's usage in the *Prior Analytics*, Leibniz himself talks of “reductions” instead of "proofs". A reduction is just an upside down proof in which the first line is what is to be proved and you work down the page to the basic identity axiom.) Note that it follows from the definition of proof that all proofs have a finite number of lines. It is very important for Leibniz that necessity is *finitely* provable. *Contingent* propositions, he says, are ones that are true in his sense (i.e. analytically true) but for which there is no finite proof. The concept of God or of a possible world for Leibniz are infinite concepts and the term *God* abbreviates an infinite terms standing for an infinite concept.

Remark. Notice since infinite terms are literally infinite lists of basic terms, they are infinite in length and hence are precluded from appearance in a proof. Thus thought the following inference rules that employ infinite terms are valid, they are not proof theoretical acceptable:

from $\{t\}_i$ ***is*** t_k infer $\{t\}_i \cup \{t^1\}$ is t_k;

from t_k is $\{t\}_i$ infer t_k ***is*** $\{t\}_i–\{t^1\}$.

Theorem. The notion of proof is sound and complete for finite propositions, i.e. provability and logical (and hence analytic) truth coincide:

Finite Soundness:

if P is finite and $\vdash$P (equivalently, P is necessary),

then $\models$P (equivalently, P is analytic).

Finite Completeness:

if P is finite and $\models$P (equivalently, P is analytic),

then $\vdash$P (equivalently, P is necessary).

Theorem. If P is infinite, then not($\vdash$P)

Proof. Let P contain an infinite term $\{t\}_i$, and assume for a *reductio* that $\vdash$P. Then there is some proof of P. Moreover, if $\{t\}_i$ is the subject of P, there is for every first term t^1 in $\{t\}_i$ at line introducing that term to the subject. But then since there are an infinite number of such first terms in $\{t\}_i$ there an infinite number of lines in the proof. But a proof is only finitely long. Hence by *reductio.* There is no proof of P. The reasoning is similar if $\{t\}_i$ occurs as the predicate of P. Q.E.D.

Theorem. Soundness holds for both finite and infinite propositions, but completeness fails for infinite propositions:

Soundness:

For any P, if $\vdash$P, the $\models$P (equivalently, P is analytic)

Failure of Completeness: There is some infinite propositions P such that $\models$P(equivalently, P is analytic) but not($\vdash$P).

Exercises:

1. If all truth is conceptual inclusion (analytic truth), is there any notion in Leibniz for “truth in a possible world” (modern day contingent truth)? (Perhaps adding the indexical modal operator *actually* would reintroduce the distinction.)

2. Is the proposition *God exists* true (i.e. analytic)? Is it provable? Is it necessary? (Prove your answer to each.) Is the constellation of answers odd? Explain.

References

Partial Latin text of *De arte combinatoria* (web page)

Leibniz, G.W. *Dissertatio de Arte Combinatoria, cum Appendice* St.-Michaels-Gymnasium Metten, Startseite,Mathematik, Naturwissenschaften, Leibniz, [1666] [cited http://home.t-online.de/home/099191080-0002/de_arte_.htm.

Partial English translation of *De arte combinatoria* and other essays in logic:

Parkinson, G.H.P. *Leibniz, Logical Papers*. Oxford: Clarendon Press, 1966.

Secondary Sources on Leibniz' logic:

Carnap, Rudolf. *Meaning and Necessity*. Chicago: University of Chicago Press, 1956 [1952].

Couturat, Louis. *La Logique de Leibniz*. Hildesheim: Georg Olms Verlag, 1985 [1901].

Lewis, C. I. *A Survey of Symbolic Logic*. N.Y.: Dover, 1960 [1918].

Martin, John N. "Proclus and the Neoplatonic Syllogistic", *Journal of Philosophical Logic,* 30 (2001), pp. 187-240.

Mates, Benson. *The Philosophy of Leibniz: Metaphysics and Language*. N.Y.: Oxford University Press, 1986.

Peckhaus, Volker. "19th Century Logic Between Logic and Mathematics," *Bulletin of Symbolic Logic,* 5 (1999), pp. 433-450.

Russell, Bertrand. *A Critical Exposition of the Philosophy of Leibniz*. London: George, Allen and Unwin, 1967 [1900].

Thomason, Richmond. *Formal Philosophy*. New Haven: Yale University Press, 1974.

GOTTFRIED WILHELM LEIBNIZ

Philosophical Papers and Letters

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VOLUME I

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1

DISSERTATION ON THE ART OF COMBINATIONS
1666
(Selections)

The Dissertatio de arte combinatoria, which Leibniz published in 1666, was an expansion of the dissertation and theses submitted for disputation the same year to qualify for a position in the philosophical faculty at Leipzig. The work contains the germ of the plan for a universal characteristic and logical calculus, which was to occupy his thinking for the rest of his life. That project is here conceived as a problem in the arithmetical combination of simple into complex concepts, Leibniz deriving basic theorems on permutation and combination and applying them to the classification of cases in logic, law, theology, and other fields of thought. His later judgment on the work was that in spite of its immaturity and its defects, especially in mathematics, its basic purpose was sound.

Three introductory sections which supply the metaphysical and logical foundations of the work are given here. They are (I) a demonstration of the existence of God with which he prefaced the work; (II) the "corollaries" prepared for the disputation; and (III) the definitions introducing the work itself. The solution of the first two problems and several applications are also included.

* * *

[G., IV, 32-33]

I

DEMONSTRATION OF THE EXISTENCE OF GOD

Hypotheses [Praecognita]:[1]

1. Definition 1. God is an incorporeal substance of infinite

power [virtus].

2. Definition 2. I call substance whatever moves or is moved.

3. Definition 3. Infinite power is an original capacity [potentia] to move the infinite. For power is the same as original capacity; hence we say that secondary causes operate by virtue [virtus] of the primary.

4. Postulate. Any number of things whatever may be taken simultaneously and yet be treated as one whole. If anyone makes bold to deny this, I will prove it. The concept of parts is this: given a plurality of beings all of which are understood to have something in common; then, since it is inconvenient or impossible to enumerate all of them every time, one name is thought of which takes the place of all the parts in our reasoning, to make the expression shorter. This is called the whole. But in any number of given things whatever, even infinite, we can understand what is true of all, since we can enumerate them all individually, at least in an infinite time. It is therefore permissible to use one name in our reasoning in place of all, and this will itself be a whole.[2]

5. Axiom 1. If anything is moved, there is a mover.

6. Axiom 2. Every moving body is being moved.

7. Axiom 3. If all its parts are moved, the whole is moved.

8. Axiom 4. Every body whatsoever has an infinite number of parts; or, as is commonly said, the continuum is infinitely divisible.

9. Observation. There is a moving body.

Proof [Ἔκθεσις]:

1. Body A is in motion, by hypothesis No. 9.
2. Therefore there is something which moves it, by No. 5,
3. and this is either incorporeal
4. because it is of infinite power, by No. 3;
5. since A, which it moves, has infinite parts, by No. 8;
6. and is a substance, by No. 2.
7. It is therefore God, by No. 1. Q.E.D.
8. Or it is a body,

9. which we may call B.
10. This is also moved, by No. 6,
11. and what we have demonstrated about body A again applies, so that
12. either we must sometime arrive at an incorporeal power, as we showed in the case of A, in steps 1-7 of the proof, and therefore at God;
13. or in the infinite whole there exist bodies which move each other continuously.
14. All these taken together as one whole can be called C, by No. 4.
15. And since all the parts of C are moved, by step 13,
16. C itself is moved, by No. 6,
17. and by some other being, by No. 5,
18. namely, by an incorporeal being, since we have already included all bodies, back to infinity, in C, by step 14. But we need something other than C, by 17 and 19,
19. which must have infinite power, by step No. 3, since C, which is moved by it, is infinite, by steps 13 and 14;
20. and which is a substance, by No. 2,
21. and therefore God, by No. 1.

Therefore, God exists. Q.E.D.[3]

II

COROLLARIES FOR DISPUTATION[4]

[G., IV, 41-43]

An Arithmetical Disputation on Complexions, which Mr. Gottfried Wilhelm Leibniz of Leipzig will hold in the famous university of Leipzig, by permission of its distinguished philosophical faculty, on March 7, 1666.

I. Logic.

1. There are two primary propositions. The first is the principle of all theorems or necessary propositions: what is (so)

either is or is not (so), or conversely. The other is the basis of all observations or contingent propositions: something exists.

2. Perfect demonstrations are possible in all disciplines.

3. If we regard the disciplines in themselves, they are all theoretical; if their application, they are all practical. Those, however, from which the application follows more immediately are rightly called practical par excellence.

4. Although every method can be employed in every discipline, as we follow the traces either of our own investigation or of the producing nature in our treatment, it yet happens in the practical disciplines that the order of nature and that of knowledge coincides, because here the nature of the thing itself originates in our thought and production. For the end in view both moves us to produce the means and leads us to know them, which is not true in the matters which we can merely know but cannot also produce. Moreover, although every method is allowed, not every one is expedient.

5. The end of logic is not the syllogism but simple contemplation. The proposition is, in fact, the means to this end, and the syllogism is the means to the proposition.

II. Metaphysics.

1. One infinite is greater than another. (Cardan, Pract. Arith., chap. 66, nn. 165 and 260. Seth Ward is said to dissent in his Arithmetic of Infinites.)[5]

2. God is substance; creature is accident.

3. A discipline concerning created beings in general is needed, but this is nowadays usually included in metaphysics.

4. It is very improbable that the term cause expresses an unequivocal concept to cover efficient, material, formal, and final causes. For what is the word influx, more than a mere word?[6]

III. Physics.

1. Since we may observe that other cosmic bodies move about their own axes, it is not absurd that the same should be true

of the earth; but neither is the contrary.

2. Since the most general difference between bodies is that of density and rarity,[7] the four primary qualities may obviously be explained as follows: the humid is the rare, the dry is the dense, the warm is the rarefying, and the cold is the condensing. Everything rare is easily confined within external boundaries, but with difficulty within its own boundaries; everything dense, the contrary. In the rare, everything that rarefies facilitates the quickening of the homogeneous with respect to itself and the separation of the heterogenous; in the dense the way to this is blocked. A reason is thus supplied for the Aristotelian definitions. Nor does fire, which seems to be rare but must actually be dry, provide an exception to this, for I reply that one thing is to be said about fire per se and another of fire which inheres in other bodies, for in this case it follows the nature of these bodies. Thus it is clear that a flame, which is nothing but burning air, must be fluid just as is air itself. On the other hand, the fire which consists of burning iron is like iron itself.

3. It is a fiction that the force of the magnet is checked by steel.

IV. Practical.

1. Justice (particular) is a virtue serving the mean in the affections of one man toward another, the affections of enjoying and of harming, or those of good will and hate. The rule of the mean is to gratify another (or myself) as long as this does not harm a third person (or another). This must be noted in order to defend Aristotle against the cavil of Grotius, who speaks as follows in the Prolegomena of his de Jure belli et pacis (Sec. 4):

> That this principle (that virtue consists in the mean) cannot correctly be assumed as universal is clear even in the case of justice. For since he (Aristotle) was unable to find the opposites of excess and defect in the affections and the actions which follow from them, he sought them both in the things themselves with which justice is concerned. But this is obviously to leap from one genus of things to another, a fault which he rightly criticizes in others.[8]

Grotius, namely, maintains that it is inconsistent to introduce into the species of a classification something which is derived by another principle of classification; he calls this, not too philosophically, "leaping over into another genus." Certainly the mean in affections is one thing, the mean in things another, and virtues are habits, not of things but of minds. Therefore I show that justice is also found in a moderation of the affections.

2. Thrasymachus well says, in Plato's Republic, Book i, that justice is what is useful to the more powerful. For in a proper and simple sense, God is more powerful than others. In an absolute sense one man is not more powerful than another, since it is possible for a strong man to be killed by a weak one. Besides, usefulness to God is not a matter of profit but of honor. Therefore the glory of God is obviously the measure of all law. Anyone who consults the theologians, moralists, and writers on cases of conscience will find that most of them base their arguments on this. Once this principle is established as certain, therefore, the doctrine of justice can be worked out scientifically. Until now this has not been done.[9]

III

Cum Deo!

[G., IV, 35-75]

1. Metaphysics, to begin at the top, deals with being and with the affections of being as well. Just as the affections of a natural body are not themselves bodies, however, so the affections of a being are not themselves beings.

2. An affection (or mode) of a being, moreover, is either something absolute, which is called quality, or something relative, and this latter is either the affection of a thing relative to its parts if it has any, that is, quantity, or that of one thing relative to another, relation. But if we speak more accurately and assume a part to be different from the whole, the quantity of a thing is also a relation to its part.

3. Therefore, it is obvious that neither quality nor quantity nor relation is a being; it is their treatment in a signate actuality that belongs to metaphysics.

4. Furthermore, every relation is either one of union or one of harmony [convenientia]. In union the things between which there is this relation are called parts, and taken together with their union, a whole. This happens whenever we take many things simultaneously as one. By one we mean whatever we think of in one intellectual act, or at once. For example, we often grasp a number, however large, all at once in a kind of blind thought, namely, when we read figures on paper which not even the age of Methuselah would suffice to count explicitly.

5. The concept of unity is abstracted from the concept of one being, and the whole itself, abstracted from unities, or the totality, is called number.[10] Quantity is therefore the number of parts. Hence quantity and number obviously coincide in the thing itself, but quantity is sometimes interpreted extrinsically, as it were, in a relation or ratio to another quantity, to aid us, namely, when the number of parts is unknown.

6. This is the origin of the ingenious specious analysis[11] which Descartes was the first to work out, and which Francis Schotten and Erasmus Bartholin later organized into principles, the latter in what he calls the Elements of Universal Mathematics. Analysis is thus the science of ratios and proportions, or of unknown quantity, while arithmetic is the science of known quantity, or numbers. But the Scholastics falsely believed that number arises only from the division of the continuum and cannot be applied to incorporeal beings. For number is a kind of incorporeal figure, as it were, which arises from the union of any beings whatever; for example, God, an angel, a man, and motion taken together are four.

7. Since number is therefore something of greatest universality, it rightly belongs to metaphysics, if you take metaphysics to be the science of those properties which are common to all

classes of beings. For to speak accurately, mathematics (adopting this term now) is not one discipline but small parts taken out of different disciplines and dealing with the quantity of the objects belonging to each of them. These parts have rightly grown together because of their cognate nature. For as arithmetic and analysis deal with the quantity of beings, so geometry deals with the quantity of bodies, or of the space which is coextensive with bodies. Far be it from us, certainly, to destroy the social distribution of disciplines among the professions, which has followed convenience in teaching rather than the order of nature.

8. Furthermore, the whole itself (and thus number or totality) can be broken up into parts, smaller wholes as it were. This is the basis of *complexions*, provided you understand that there are common parts in the different smaller wholes themselves. For example, let the whole be *ABC*; then *AB*, *BC*, and *AC* will be smaller wholes, its parts. And the disposition of the smallest parts, or of the parts assumed to be smallest (that is, the unities) in relation to each other and to the whole can itself also be varied. Such a disposition is called *situs*.[12]

9. So there arise two kinds of *variation*: *complexion* and *situs*. And viewed in themselves, both complexion and situs belong to metaphysics, or to the science of whole and parts. If we look at their variability, however, that is, at the quantity of variation, we must turn to numbers and to arithmetic. I am inclined to think that the science of complexions pertains more to pure arithmetic, and that of situs to an arithmetic of figure. For so we understand unities to produce a line. I want to note here in passing, however, that unities can be arranged either in a straight line or in a circle or some other closed line or lines which outline a figure. In the former case they are in absolute situs or that of parts to the whole, or *order*; in the latter they are in relative situs or that of parts to parts, or *vicinity*. In definitions 4 and 5, below, we shall tell how these differ. Here these preliminary remarks will suffice to bring to light the discipline upon which our subject matter is based.[13]

Definitions

1. *Variation* here means change of relation. For change may be one of substance, or of quantity, or of quality; still another kind changes nothing in the thing but only its relation, its situs, its conjunction with some other thing.

2. *Variability* is the quantity of all variations. For the limits of powers taken in abstraction denote their quantity; so it is frequently said in mechanics that the power of one machine is double that of another.

3. *Situs* is the location of parts.

4. Situs is either absolute or relative; the former is that of the parts with respect to the whole, the latter that of parts to parts. In the former the number of places is considered, and the distance from the beginning and the end; in the latter neither the beginning nor the end is considered, but only the distance of one part from another part is viewed. Hence the former is expressed by a line or by lines which do not inclose a figure or close upon themselves, and best by a straight line; the latter is expressed by a line or lines inclosing a figure, and best by a circle. In the former much consideration is given to priority and posteriority; in the latter, none. We will therefore do well to call the former *order*,

5. And the latter *vicinity*. The former is disposition; the latter, composition. Thus by reason of order the following situses are different: *abcd*, *bcda*, *cdab*, *dabc*. But in vicinity there can be no variation but only situs, namely, this: $\begin{array}{ccc} & b & \\ a & & c \\ & d & \end{array}$. Thus when the very witty Taubman was dean of the philosophical faculty at Wittenberg, he is said to have placed the names of Master's candidates on the public program in a circular arrangement, so that eager readers should not learn who held the position of "swine."[14]

6. We will usually mean the variability of order when we take variations par excellence; for example, 4 things can be arranged in 24 ways.[15]

7. The variability of a complex we call *complexions*; for ex-

ample, 4 things can be put together in 15 different ways.[16]

8. The number of varying things we shall call simply number; for example, 4 in the case proposed.

9. A complexion is the union of a smaller whole within the greater, as we have said in the introduction.

10. In order to determine a certain complexion, however, the greater whole is to be divided into equal parts assumed as minima (that is, parts now not to be considered as further divisible). Of these parts it is composed, and by the variation of them the complexion or lesser whole may be varied. Because the lesser whole itself is greater or less according as more parts are included at any time, we call the number of parts or unities to be connected together at one time the exponent, after the example of a geometric progression. For example, let the whole be ABCD. If the lesser whole is to consist of two parts, for example, AB, AC, AD, BC, BD, CD, the exponent will be 2; if of three parts, for example, ABC, ABD, ACD, BCD, the exponent will be 3.

11. We shall write the complexions with a given exponent as follows: if the exponent is 2, com2nation (combination); if 3, con3nation (conternation); if 4, con4nation; etc.

12. Complexions taken simply are all the complexions computed for all exponents; for example, 15 of the number 4. These consist of 4 units, 6 com2nations, 4 con3nations, 1 con4nation.

13. A useful (useless) variation is one which can (cannot) occur because of the nature of the subject matter; for example, the four [physical] elements can be com2ned six times, but two com2nations are useless, namely, those in which the contraries fire and water and the contraries air and earth are com2ned. . . .

Problems

Three things should be considered: problems, theorems, and applications. We have added the application to individual problems wherever it seemed worth while, and the theorems also. To some of the problems, however, we have added a demonstration. Of these,

we owe the latter part of the first problem, and the second and fourth, to others; the rest we ourselves have discovered. We do not know who was the first to discover them. Schwenter (Delic., Book i, Sec. 1, prop. 32) says they exist in Jerome Cardan, John Buteonis, and Nicolas Tartalea. But we have not found them in Cardan's Arithmetica practica, published in Milan in 1539. Christopher Clavius set forth especially clearly what has been found recently, in his Commentarium in Sphaeram Joannis de Sacro Bosco, published in Rome in 1585, pages 33 ff.[17]

Problem I

To Discover the Complexions for a Given Number and Exponent

1. There are two ways of solving this problem, one for all complexions, the other for com2nations only. The former is more general, but the latter requires fewer data, namely, only the number and the exponent, while the former also presupposes the discovery of antecedent complexions.[18]

2. We have developed the more general method; the special one is popularly known. The more general solution is this: Add the complexions of the number preceding the given number, by the given exponent and by the exponent preceding it; the sum will be the desired complexions. For example, let the given number be 4 and the exponent 3; add the 3 com2nations and the 1 con3nation of the preceding number 3; (3 + 1 = 4). The sum 4 will be the answer.

3. But since the complexions of the preceding number are required for this solution, Table ℵ must be constructed. In it the top line contains the numbers from 0 to 12 inclusive from left to right (we believe this is far enough, since it is easily extended); the vertical line at the left contains the exponents from 0 to 12, reading from top to bottom; and the bottom line, from left to right, contains the total complexions [complexiones simpliciter]. The lines between contain the complexions for the number given at the

head of the corresponding column and for the exponent given at the left.[19]

4. The reason for this solution, and the basis of the table, will be clear if we demonstrate that the complexions for a given number and exponent arise from the sum of the complexions of the preceding number, for both the given and the preceding exponents. Taking the given number as 5 and the given exponent as 3, the antecedent number will be 4; it will have 4 con3nations and 6 com2nations, by Table ℵ. Now the number 5 has all the con3nations of the preceding number (since the part is contained in the whole), namely, 4, and it has besides as many con3nations as the preceding number has com2nations, since the unit by which the number 5 exceeds 4, added to each of the individual com2nations of 4, will make the same number of con3nations. Thus 6 + 4 = 10. Therefore the complexions for a given number, etc. Q.E.D.

TABLE ℵ

Exponents	Complexions												
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7n	8u	9m	10b	11e	12r
2	0	0	1	3	6	10	15	21	28	36	45	55	66
3	0	0	0	1	4	10	20	35	56	84	120	165	220
4	0	0	0	0	1	5	15	35	70	126	210	330	495
5	0	0	0	0	0	1	6	21	56	126	252	462	792
6	0	0	0	0	0	0	1	7	28	84	210	462	924
7	0	0	0	0	0	0	0	1	8	36	120	330	792
8	0	0	0	0	0	0	0	0	1	9	45	165	495
9	0	0	0	0	0	0	0	0	0	1	10	55	220
10	0	0	0	0	0	0	0	0	0	0	1	11	66
11	0	0	0	0	0	0	0	0	0	0	0	1	12
12	0	0	0	0	0	0	0	0	0	0	0	0	1
*	0	1.	3.	7.	15.	31.	63.	127.	255.	511.	1023.	2047.	4095.
†	1.	2.	4.	8.	16.	32.	64.	128.	256.	512.	1024.	2048.	4096.

*The complexions taken simply (or the sum of the complexions of all given exponents), added to 1, equal the total of a geometric progression with base 2†.[20] . . .

Problem II

To Discover the Complexions Taken Simply for a Given Number

Seek the given number among the exponents of a geometric progression with base 2; then the total of complexions sought will be the number or term of the progression whose exponent is the given number, minus 1. It is difficult to understand the reason or demonstration for this, or to explain it if it is understood. The fact, however, is apparent from Table ℵ. For, when added together, and the sum added to unity, the particular complexions of a given number always constitute, when one is added, the term of that geometric progression with base 2, whose exponent is the given number. But if anyone is interested in seeking the reason for this, it will have to be found in the process of resolving used in the *Practica italica*, *vom Zerfallen*. This must be such that a given term of the geometrical progression is separated into more parts by one than there are units (i.e., numbers) in its exponent. The first of these must always be equal to the last, the second to the next to the last, the third to the third from the last, etc., until, if it is broken up into an equal number of parts, the exponent or number of things being odd, the two parts in the middle will be equal (for example, 128 or 2^7 may be broken up into eight parts according to Table ℵ: 1, 7, 21, 35, 35, 21, 7, 1); or, if the exponent is even and it must be broken into an odd number, the number left in the middle will have none corresponding to it (for example, 256 or 2^8 may be broken up into nine parts according to Table ℵ: 1, 8, 28, 56, 70, 56, 28, 8, 1). Someone may therefore think that this brings to light a new method which is absolute for solving problem 1; namely, by breaking up the complexions taken simply, or the terms of a geometric progression with base 2, by a method discovered with the aid of algebra. In fact, however, there are not sufficient data, and the same number can be broken up in several ways yet according to the same form.

Application of Problems I and II

Since everything which exists or which can be thought must be compounded of parts, either real or at least conceptual, whatever differs in kind must necessarily either differ in that it has other parts, hence the use of complexions; or by another situs, hence the use of dispositions. The former are judged by the diversity of matter; the latter, by the diversity of form. With the aid of complexions, indeed, we may discover not only the species of things but also their attributes. Thus almost the whole of the inventive part of logic is grounded in complexions—both that which concerns simple terms and that which concerns complex terms; in a word, both the doctrine of divisions and the doctrine of propositions; not to mention how much we hope to illumine the analytic part of logic, or the logic of judgment, by a diligent examination of the modes of the syllogism in Example VI.

The use of complexions in divisions is threefold: (1) given the principle of one division, to discover its species; (2) given many divisions of the same genus, to discover the species mixed from different divisions (this we will treat in Problem III, however); (3) given the species, to discover the subaltern genera. Examples are scattered throughout all of philosophy, and we will show that they are not lacking in jurisprudence. And in medicine every variety of compounded medicaments and pharmaceuticals is made by mixing various ingredients, though the greatest care is necessary in choosing useful mixtures. First, therefore, we will give examples of species to be discovered by this principle.[21]

I. Among jurisconsults the following division is proposed (Digests, Gaius, XVII, 1, 2). A mandate is contracted in five ways: in favor of the mandator, of the mandator and mandatory, of a third person, of the mandator and a third person, of the mandatory and a third person. We shall seek out the adequacy of the division in this way: its basis is the question, for whom, or the person in whose favor the contract is made; there are three of these, the mandator, the mandatory, and a third person. But there are seven complexions of three things:

Three 1nions: since contract may be in favor of only (1) the mandator; (2) the mandatory; or (3) a third person.

The same number of com2nations: (4) in favor of the mandator and mandatory; (5) of the mandator and a third person; or (6) of the mandatory and a third person.

One con3nation: (7) in favor of the mandator, the mandatory, and a third person all together.

Here the jurisconsults reject as useless that 1nion in which the contract is in favor of the mandatory alone, because this would be advice rather than a mandate. There remain thus six classes. Why they kept only five, omitting the con3nation, I do not know.

II. Aristotle (On Generation and Corruption, Book ii), with Ocellus Lucanus the Pythagorean, deduces the number of elements, or of the mutable species of a simple body, from the number of primary qualities, of which he assumes there are four, but according to these laws: (1) that every element is to be a compound of two qualities and neither more nor less, and it is thus obvious that 1nions, con3nations, and the con4nation are to be discarded and only com2nations retained, of which there are six; and (2) that contrary qualities can never enter into one com2nation and that therefore two of the com2nations are useless because there are two contraries among these primary qualities. Hence there remain four com2nations, the same as the number of elements. . . . Moreover, as Aristotle derived the elements from these qualities, so Galen derived from them the four temperaments, the various mixtures of which later medics have studied; all of whom Claudius Campensis opposed in the past century, in his Animadversiones naturales in Aristotelem et Galenum (Leyden, 1576). . . .

IV. In wind organs we call a register, in German ein Zug, a little shaft by whose opening the sound may be varied, not with respect to the perceived melody or pitch itself, but in its basis in the pipe, so that sometimes a shaking, sometimes a whisper, is achieved. More than thirty of such qualities have been discovered by the industry of our contemporaries. Assume that there are in some organs only twelve such simple effects; then there will be in all about 4,095, as many as there are complexions taken simply of twelve things. So a great organist can vary his playing as he opens them together, sometimes many, sometimes a few, sometimes these, sometimes those.

V. Thomas Hobbes, *Elementa de corpore*, Part I, chapter 5, classifies the things for which there are terms built into a proposition, or, in his own terminology, the named things [*nominata*] for which there are names [*nomina*], into *bodies* (that is, substances, since for him every substance is a body), *accidents*, *phantasms*, and *names*. Thus a name is a name either of bodies, for example, man; or of accidents, for example, all abstractions, rationality, motion; or of phantasms, in which he includes space, time, all sensible qualities, etc.; or of names, in which he includes second intentions. Since these are com2ned with each other in six ways, there arise the same number of kinds of propositions, and adding to these the cases in which homogeneous terms may be com2ned (a body ascribed to body, accident to accident, phantasm to phantasm, secondary concept to secondary concept), namely, four, the total is ten. Out of these Hobbes thinks that only homogeneous terms can be usefully com2ned. If this is so, as the common philosophy certainly also acknowledges, and abstract and concrete, accident and substance, primary and secondary concepts, are wrongly predicated of each other, this will be useful for the art of discovering propositions or for observing the selection of useful com2nations out of the uncountable mixture of things. . . .

VIII. The eighth application is in the formation of cases by the jurisconsults. For one cannot always wait for the lawmaker when a case arises, and it is more prudent to set up the best possible laws without defects, from the first, than to intrust their restriction and correction to fortune; not to mention the fact that, in any state whatsoever, a judicial matter is the better treated, the less is left to the decision of the judge (Plato *Laws*, Book ix; Aristotle *Rhetoric*, Book i; Jacob Menochius, *De arbitrariis judicum questionibus et causis*, Book i, proem. 1).

Moreover, the art of forming cases is founded in our doctrine of complexions. For as jurisprudence is similar to geometry in other things, it is also similar in that both have elements and both have cases. The elements are simples; in geometry figures, a triangle, circle, etc.; in jurisprudence an action, a promise, a sale, etc. Cases are complexions of these, which are infinitely variable in either field. Euclid composed the *Elements of Geometry*, the elements of law are contained in the *Corpus Juris*, but in both works more complicated cases are added. The simple terms in the law, however, out of the combinations of which the rest arise, and, as it were, the *loci communes* and highest genera, have been collected by Bernhard Lavintheta, a Franciscan monk, in his commentary on the *Ars magna* of Lully (which see). To us it seems thus: the terms from whose complexions there arises the diversity of cases in the law are persons, things, acts, and rights. . . .

The basis of terms is the same in theology, which is, as it were, a kind of special jurisprudence, but fundamental for the same reason as the others. For theology is a sort of public law which applies in the Kingdom of God among men. Here the unfaithful are like rebels; the church is like good subjects; ecclesiastical persons, and indeed also the political magistrate, are like the subordinate magistrates; excommunication is like banishment; the teaching of Sacred Scripture and the Word of God is like that of the laws and their interpretation; that of the canon like the question of which of the laws are authentic; that of fundamental errors like that of capital crimes; that of the Final Judgment and the Last Day like that of the judiciary process and the rendered judgment; that of the remission of sins like that of the pardoning power; that of eternal punishment like that of capital punishment, etc. . . .