Reduction of Arithmetic to Logic and Set Theory

In the 19th century Giuseppe Plano (1858-1932) axiomatized arithmetic and Georg Cantor (1845-1918) worked out in a non-axiomatic way the fundamental ideas of set theory. The stage was set for a remarkable synthesis. The German logician Gottleb Frege (1848-1925) was the first to see that the two theories could be combined by means of symbolic logic into a single axiom system in a way that “reduced” arithmetic to logic and set theory. In the last decade of the 19th century, Frege published an important work in which he deduced as theorems Peano’s postulates for arithmetic from a handful of more basic axioms from logic and set theory. On the basis of his technical accomplishment, he advanced a hypothesis about the nature of mathematics generally. Mathematics, he suggested, was a part of logic. This thesis, known as ***logicism***, is rich in implications for mathematics, logic, and philosophy.

For the mathematician logicism explains what mathematics is all about, and what its methods should be. Math turns out to consist of the working out of reason’s implications. Its method is the production of axiom systems that, in principle at least, could be formulated in symbolic logic. Non-Euclidean geometry then proves to have been a misleading storm in a teacup. Whatever the peculiarities of geometry, arithmetic, the heart of mathematics, remains groundable in *a priori* truths of reason.

For philosophy logicism breathes new life into a species of rationalism. There still seems to be an important branch of science, namely mathematics, which consists of working out the implications of the self-evident principles of pure thought.

For logic logicism is the supreme validation. Logic becomes the science of pure *a priori* reason. Logic provides the symbolic language, reasoning patterns, and axiomatic method applicable to all the sciences, and for non-empirical mathematics it provides in addition its basic truths.

To show how brief and elegant Frege’s sort of theory can be, I will now provide a statement of a basic axiom set sufficient for his purposes. The system will be called **F** (for Fregean Arithmetic). We begin by specifying the set of sentences $\mathbf{L_F}$ of the system. Only two primitive symbols are necessary beyond those of logic, and these two concern sets: the set membership symbol $\in$ and the set abstract $\{x|P[x]\}$, which in symbols says “the class of all x such that $P[x]$. Here $P[x]$ is what is called “an open sentence”. The letter *P* represents some sentence – it can be any sentence – and the [*x*] indicates that the sentence *P* contains the variable *x*.

Primitive Symbols of Fregean Arithmetic (the System F):

Primitive Symbol:	English Translation:	Symbol Name & Idea:	Example:	Translation of the Example in English:
$\in$	*is a member of*	set membership	$x \in A$	*x is a member of A*
$\{x \mid P[x]\}$	*set of x such that P[x]*	set abstract	$\{x \mid \exists y(x=2y)\}$	*the even numbers*

The set $\mathbf{L_F}$ of sentences in the formal language will be all sentences of symbolic logic that we can make up from the primitive terms $\in$ and $\{x|P[x]\}$ by means of the usual expressions of symbolic logic. These include variables *x,y,z,* etc. and logical symbols:

Sentential Connectives:

- $\sim$ the negation symbol for “not” or “it is not the case that”,
- $\wedge$ the conjunction symbol for “and”
- $\vee$ the disjunction symbol for “or”
- $\rightarrow$ the conditional symbol for “if … then”
- $\leftrightarrow$ the biconditional symbol for “… if and only if …”

Quantifiers:

- $\forall$ the universal quantifier symbol for “for all …”
- $\exists$ the existential quantifier symbol “for some …” or “there exists an…such that”

A Relational Symbol:

- $=$ the identity symbol for “… is identical to …”

To define the axiom system, it is necessary to specify three things:

(1) a set of axioms,
(3) rules of inference for deducing theorems from the axioms, and
(3) definitions for abbreviating longer expressions into shorter.

In place of Peano’s axioms using primitive ideas from arithmetic, Frege uses axioms from logic and set theory. These may be divided into three sorts.

The Three Kinds of Axioms for the Axiom System F

- Axioms for sentence logic (which was then called the *propositional calculus*)
- Axioms for predicate logic and identity, (then called the *predicate calculus* or *quantification theory* and now called *first-order logic*)
- Axioms for set theory

Since Frege’s original work in the 1890’s the required axiom set has been reduced and simplified.[1] In place of Frege’s original five axioms of the propositional calculus, here we

[1] Gottlob Frege, *Grundgesetze der Arithmetik*, vol. I (1893), vol. II (1903) (Jena: Verlag Hermann Pohle). (A partial translation is avialable in Montgonery Furth, *The Basic Laws of Arithmetic* (Berkeley: Univ. of

shall use a three axiom simplification first proposed by the Polish logician Jan Lukasiewicz in 1930. Frege's original axioms for the quantifiers were reduced and stated in a rigorously logistic system by David Hilbert and Wilhelm Ackermann in 1922. The three axiom version used here is due to W.V.O.Quine (1940). For naive set theory we shall use Bertrand Russell's two axioms of 1903. Strictly speaking the axioms are called ***axiom schemata*** because each schema validate a set of axioms, namely the set of all sentences that have the same form as the schema. (More precisely, an axiom is an instance of a schema obtained as the result of uniformly replacing in the schema all occurrences of non-logical letters by descriptive expressions of the appropriate grammatical type.) One reason I have chosen this particular set of axioms is that it needs only the one rule of inference, *modus ponens*.

California Press, 1964). Jan Lukasiewicz and A. Tarski, "Untersuchugen über den Aussagenlalkül," *C. R. Soc. Sci. Varsovie* 23 (1930). David Hilbert and Wilhem Ackermann, *Mathematical Logic* [1928] (N.Y.: Chelsea, 1950). W.V.O.Quine, *Mathematical Logic* [First ed.,1940] (N.Y.:Harper, revised ed.1951; Bertrand Russell, *op. cit*.

The System F for Arithmetic. (Modeled on Frege's, *Grundgesetze der Arithmetik*, 1893,1903):

1. The inference rules of F. $\mathbf{R_F}$ contains just one rule:

If $\vdash_F P$ and $\vdash_F P\to Q$, then $\vdash_F Q$ (*modus ponens*)

2. The Axioms of F. The set $\mathbf{Ax_F}$ of axioms consist of all sentences of the following forms:

Axioms of the Propositional Calculus (Sentence Logic) (Lukasiewicz , 1930)

1. $\vdash_F P\to(Q\to P)$
2. $\vdash_F (P\to(Q\to R))\to((P\to Q)\to(P\to R))$
3. $\vdash_F (\sim P\to\sim Q)\to(Q\to P)$

Axioms of First-Order with Identity (Quine, 1940)

4. $\vdash_F \forall x(P\to Q)\to(\forall xP\to\forall xQ)$ — If every *P is* Q, then if everything is, it is Q.
5. $\vdash_F P\to\forall xP$ where *x* is not free in *P.* — If P then everything is P
6. $\vdash_F \forall xP[x]\to P[y]$ where *P*[*y*] is like *P*[*x*] except for containing free occurrences of *y* where *P*[*x*] contains free occurrences of *x.* — If everything is P, then a particular case of it for y is true.
7. $\vdash_F \forall x(x=x)$ — Everything is identical to itself.
8. $\vdash_F \forall x\forall y(x=y \wedge P[x]) \to P[y])$ where *P*[*y*] is like *P*[*x*] except for containing free occurrences of *y* where *P*[*x*] contains free occurrences of *x.* — If *x* and *y* are identical, you can substitute *y* for *x* in P.

Axioms of (Naive) Set Theory (Russell's version of Frege, 1903)

9. $\vdash_F \exists A\forall x(x\in A \leftrightarrow P[x])$ — For any sentence P there is a set that contains all and only those things that P is true of. You can make any sentence the defining conditions for a set.
10. $\vdash_F A=B\leftrightarrow\forall y(y\in A\leftrightarrow y\in B)$ — Two sets are identical if and only if they have the same members

Definition (Set Abstract)

$Q[\{x|P[x]\}] \leftrightarrow_{df} \exists A(\forall x(x\in A \leftrightarrow P[x])\wedge\forall B(\forall x(x\in A \leftrightarrow P[x])\to B=A)\wedge P[A]$

To say the set of P's has the property Q means that there is a set of P's, there is only one, and it has Q.

Theorems (Abstraction for Set Abstracts)

$\vdash_F \exists A(A=\{x|P[x]\}$ — The set of things that are P exists.

$\vdash_F \forall y(y\in\{x|P[x]\}\leftrightarrow P[y])$ — Something is in the set of P's exactly when it is P.

Theorem (Extensionality for Set Abstracts)

$\vdash_F \{x|P[x]\}=\{x|Q[x]\}\leftrightarrow \forall y(y\in\{x|P[x]\}\leftrightarrow y\in\{x|Q[x]\})$ — The set of P's is identical to the set of Q's if and only if they have the same members.

If we now add several of the elementary definitions of arithmetical ideas, we can state some of the theorems provable within the system.

3. "Bridging" Definitions within F. Concepts of Arithmetic defined in Set Theory and Logic.

	English Translation:	**Definition:**	
S(n)	*the successor of* n	$n \cup \{n\}$	The successor operation adds another element to *n,* i.e. it increases its size by one.
0	*zero*	$\varnothing$, the empty set, i.e. = $\{x \mid x \neq x\}$.	The empty set has 0 elements.
1	*one*	***S***(0)	This set has one more element than 0.
2	*two*	***S***(1)	This set has one more element than 1.
3	*three*	***S***(2), etc.	
Nn	*the natural numbers*	the least set A such that $0 \in A$ & $(x \in A \rightarrow S(x) \in A)$	
$n+m$	*the sum of* n *and* m	the element e of **Nn** such that for some non-overlapping sets A and B, A maps 1-1 to *n*, B maps 1-1to *m*, and maps 1-1 to $(A \cup B)$	
$n \leq m$	n *is less than* m	$n \subseteq m$ (for *n* and *m* in **Nn**)	

Given these axioms and definitions it is possible to prove as theorems Peano's postulates for arithmetic and from them in turn the truths of the simple arithmetic of the natural numbers.

Theorems in F.

Peano's Postulates are theorems of **F.**

$\vdash_F 0 \in Nn$ 0 is a natural number.

$\vdash_F \forall x[x \in Nn \rightarrow \boldsymbol{S}(x) \in Nn)]$ The successor of a natural number is a natural number.

$\vdash_F \forall x[x \in Nn \rightarrow \sim \boldsymbol{S}(x)=0)]$ 0 is not the successor of any natural number.

$\vdash_F \forall x \forall y([\boldsymbol{S}(x)=\boldsymbol{S}(y)] \rightarrow x=y)$ Two successor of *x* and *y* are identical only if *x=y.*

$\vdash_F \{0 \in \boldsymbol{A} \wedge \forall x \forall y[x \in Nn \wedge y \in Nn \wedge x \in \boldsymbol{A} \wedge \boldsymbol{S}(x)=y] \rightarrow y \in \boldsymbol{A})\} \rightarrow \forall x(x \in Nn \rightarrow x \in \boldsymbol{A})$

The Principle of Mathematical Induction: if 0 is in a set, and if a number is in the set only if its successor is, then every natural number is in that set.

Theorems of **F** that were also Peano's Theorems – they are "reduced to" the system **F.**

$\vdash_F 1 \leq 3$

$\vdash_F 2+2=4$

Since most mathematicians would identify the heart of mathematics with arithmetic, the success of this derivation was thought to show that mathematics is "part of" logic, and that the methods of mathematics should be those of the axiomatic logician. These hopes were dashed, and logicism refuted, when Kurt Gödel (1906-1978) later proved his very famous incompleteness theorem (1931) that no axiom system can contain all the truths of Peano arithmetic, and that, therefore, the system **F** is incomplete.