Notes on Infinity

Infinity in Classical and Medieval Philosophy

Aristotle (Metaphysics 986^{a}22) reports that *limited* (peras, πέρας) and *unlimited* or *"infinite"* (apeiron, ἄπειρον) occur as the first pairing in the Pythagorean table of ten opposites. In the table the limited occurs in the positive or "good" column along with *one* and *good*, while *unlimited* occurs in the negative or "bad" column along with *many* and *bad*. Throughout classical philosophy the infinite or boundless is viewed as a privation of what is positive or good. Around the close of the classical period, however, the infinite became viewed as something positive, and eventually was understood to be one of god's properties. On a Neoplatonic understanding of god he is an entity that possesses all non-privative properties including being, goodness, and beauty. According to this understanding these properties occur in various degrees along what has been called the "great chain of being'. God is at the top of the ordering; ideas in his mind occupy a lower stage' spirits (angles and immortal souls) occupy a yet lower level; material substances occur at a lower level still. Evil is understood as the total privation of all positive properties and occupies the lowest level. In this framework the infinite is an ordering rather than a numerical concept. To say god is infinite in this sense means that he possesses all positive properties to the highest degree. By the 17th century the infinite acquired its modern mathematical meaning in which it refers to an uncountable quantity. This is a quantity so large that it cannot be completely counted. Today we would say this is a quantity so large that no matter how many of its elements are counted there is always one more.

The Natural Numbers and Countably Infinite Sets

According to both raw intuition and to the modern mathematics as well, the number of elements in a set is determined by counting its elements. Counting is understood as assigning numbers (the so-called "counting numbers") progressively to elements in the set, much as you count sheep when going to sleep. Mathematicians make this idea precise in steps. First they define a number *n* so that it is itself a set that contains exactly *n* elements. This point is important. The number *n* will be defined so that it contains exactly *n* things. A set *A* then can be said to have *n* elements if its elements can be matched 1 to 1 with the elements of *n.* In this case mathematicians say that the set and *n* are in a 1 to 1 correspondence with *n.* Logicians start counting with 0 rather than 1 and call the counting numbers the *natural numbers* or **Nn:** 0,1,2,3,… .

In addition to containing exactly *n* elements the definition of *n* will make sure that these elements are exactly the number that are less than *n.* How many numbers are there that are less than *n*? Consider the number 12. What are the numbers less than 12? They are 1,2,3,4,5,6,7,8,9,10, and 11. There are eleven of these. Let's add 0. Then, {0,1,2,3,4,5,6,7,8,9,10,11} is the set of all natural numbers less than 12, and there are 12 of them. Mathematicians will define the number 12 to be the set {0,1,2,3,4,5,6,7,8,9,10,11}. It follows that if *n* is less than *m* then *n* will be an element of *m.* There is more. Notice that 10 will be {0,1,2,3,4,5,6,7,8,9} , which contains 10 elements. Moreover {0,1,2,3,4,5,6,7,8,9} is a subset of {0,1,2,3,4,5,6,7,8,9,10,11}.

The definition of arbitrary numbers starts by defining 0 as the set with nothing in it. That is, 0 is the empty set $\varnothing$. The definition next defines ***S***(*n*), the successor of *n.* ***S***(*n*), or *n*+1, is defined so that it adds one more element to *n.* What element should this be? It does not matter so long as it is not already in *n.* The new element logicians have decided to use is {*n*}, the set that contains the number *n.* Beside not being already in *n* another reason for adding the element {*n*} to *n* is that it is then easy to define ***S***(*n*). ***S***(*n*),or *n*+1 is just $n\cup\{n\}$, the set you get by combining (taking the union of) *n* and the set containing *n.* That is, $n\cup\{n\}$ is just adding to *n* the new element that is *n* itself.

First we need some notation:

{*a,…,b*} is the set containing *a,…,b;*
$a\in B$ means that the element *a* is in the set *B;*
$A\subseteq B$ means *A* is a subset of *B;*
$A\subset B$ means *A* is a ***proper subset*** of *B,* i.e. that *A* is a subset of *B* but $A\neq B$;
$A\cup B$ means ***the union*** of the sets *A* and *B,* i.e. the set you get when you combine the elements of both sets;
P(*A*) is the set of all subsets of *A* and is called ***the power set of*** *A.*

We can now define:

Nn	the set of natural numbers,
≤	the less than relation,
S	the so-called successor operation,
+	the addition operation,
•	the multiplication operation,
0	zero or the additive identity,
1	one or the multiplicative identity.

The definition of **Nn** then has three steps. First it says that 0 is in **Nn**. Second it says that if any number *n* is in **Nn**, then so its successor ***S***(*n*). Lastly it says that nothing else in is **Nn**. The definition of **Nn** below is said to be recursive.

Definition. The structure <**Nn** ≤,***S***,+,•,0,1> is defined as follows:
1. ***S*** is a unary operation (the ***successor*** operation) on sets such that ***S***(*x*) $=x\cup\{x\}$
2. $0=\varnothing$ and 1= ***S***(0)
3. **Nn** (the set of ***natural numbers***) is the least set *B* such that
   a. $\varnothing\in B$
   b. for any *x*, if $x\in B$, then ***S***(*x*)$\in B$,
   c. nothing else is in *B*
4. ≤ is a binary relation on **Nn** (the ***less than*** relation) defined as follows: $x\leq y$ iff $x\in y$. (*B*y convention $x<y$ abbreviates $x\leq y$ and not $x=y$.)

2. + is a binary operation of **Nn** (***addition***) defined (recursively):
   a. for all x in **Nn** , $x + 0 = x$,
   b. for all x and y **Nn** , $x + \boldsymbol{S}(y) = \boldsymbol{S}(x + y)$
3. • is a binary operation (***multiplication***) of **Nn** defined (recursively):
   a. for all x in **Nn** , $x \bullet 0 = 0$,
   b. for all x and y **Nn** , $x \bullet \boldsymbol{S}(y) = (x \bullet y) + x$

Definitions 2= ***S***(1), 3= ***S***(2), 4= ***S***(3), etc.

Theorems

Each natural number is a set. The set definition is designed to insure that their set theoretic properties coincide with numerical properties that we are more familiar with.

0 is the empty set ∅. Hence 0 is the set with no members.

1 is the set containing 0, i.e. 1 is the set containing ∅: 1={0}={∅}. Hence 1 is a set with just one member. Note also that since ∅ is a subset of every set, ∅ is a subset of {∅}.

2 is the set containing 0 and 1, i.e. 2={0,1}={∅,{∅}}. Hence 2 is a set containing just two members, and is in fact the set containing all the natural numbers less than itself. Note also that both ∅, which is 0, and {∅}, which is 1, are subsets of {∅,{∅}}, which is 2. Hence the relation ⊆ of *subset* captures the *less than* relation ≤ for numbers less than 2.

3 is the set containing 0,1, and 2, i.e. 3={0,1,2}={∅,{∅},{∅,{∅}}}. Hence 3 is a set that contains just three elements, namely all the natural numbers less than 3. Note also that ∅, which is 0, and {∅}, which is 1, and {∅,{∅}}, which is 2, are subsets of {∅,{∅},{∅,{∅}}}, which is 3. Hence the relation ⊆ of *subset* captures the *less than* relation ≤ for numbers less than 3.

Nn={0,1,2,3,…}.

In general, the definitions insure that a natural number n is a set that contains exactly n elements, and that these are exactly all the natural numbers less than n. Moreover, the numbers are defined as sets in such a way that any number less than n is a subset of n.

Theorem. For any natural numbers n and m the following are equivalent:

$n \leq m$,

$n \in m$,

$n \subseteq m$.

Definitions

A is ***equipollent to*** *B* or *A* ***has the same cardinality as*** *B* (in symbols $A \approx B$) iff there is a 1-1 mapping (correspondence) from the elements of *A* to those of *B*.

A**<**B iff there is $C \subset B$ such that $A \approx C$.

A is ***countable*** (or ***denumerable***) iff for some $n \in \mathbf{Nn}$, $n \approx A$

A is ***infinite*** iff for some $B \subset A$, $A \approx B$.

A is ***countably infinite*** iff for some $n \in \mathbf{Nn}$, $n \approx A$.

It is a relatively easy matter to prove that **Nn** is infinite in the sense of Cantor. Though we shall not prove so here **Nn** is also the smallest infinite set – any set that is infinite is either larger than or equipollent to it.

Theorems.

1. **Nn** is infinite.
2. For any *A*, if *A* is infinite, then either $A \approx \mathbf{Nn}$ or $\mathbf{Nn} < A$.

Definition. A set *A* is ***countably infinite*** or ***denumerable*** iff it can be put into 1-1 correspondence with **Nn** , i.e. $A \approx \mathbf{Nn}$

Transfinite and Uncoutably Infinite Sets

Georg Cantor (1845 –1918) proved that some infinite sets are larger than other.

Theorem (Cantor). For any set *A*, $A < \mathbf{P}(A)$

Proof. We show first that it is not the case that $A \approx \mathbf{P}(A)$. We do so by a reduction to the absurd. To begin the proof, we assume the opposite, that $A \approx \mathbf{P}(A)$. Then, there is a 1-1 mapping ***f*** from *A* onto **P**(*A*). Now consider the set:

$$B = \{x \mid x \in A \ \& \ {\sim}x \in \boldsymbol{f}(x)\}.$$

Clearly *B* is a subset of *A*. Hence, since ***f*** maps *A* onto **P**(*A*), there must be some *y* in *A*, such that $\boldsymbol{f}(y)=B$. Consider now two alternatives.

I. Suppose first that $y \in \boldsymbol{f}(y)$. Then, since $\boldsymbol{f}(y)=B$, we may substitute identities and obtain $y \in B$. But then by the definition of *B*, ${\sim}y \in \boldsymbol{f}(y)$. Hence, $y \in \boldsymbol{f}(y) \rightarrow {\sim}y \in \boldsymbol{f}(y)$.

II. Suppose the opposite, alternative, namely that ${\sim}y \in \boldsymbol{f}(y)$. Now, since $y \in A$ by hypothesis, *y* meets the conditions for membership in *B*, briefly $y \in B$. Then, since $\boldsymbol{f}(y)=B$, by Substitutivity of identity, $y \in \boldsymbol{f}(y)$. Hence, ${\sim}y \in \boldsymbol{f}(y)$ iff $y \in \boldsymbol{f}(y)$.

By I and II, it follows that $y \in \boldsymbol{f}(y)$ iff ${\sim}y \in \boldsymbol{f}(y)$. But this is a contradiction. Hence the original hypothesis is false, and we have established what we set out to prove, namely it is not the case that $A \approx \mathbf{P}(A)$. There remain two possibilities: either $\mathbf{P}(A) < A$ or $A < \mathbf{P}(A)$. However, we can apply the argument above to any $B \subseteq A$, showing that it is not the same size as **P**(*A*). Hence we may generalize that for all $B \subseteq A$, ${\sim}[B \approx \mathbf{P}(A)]$. But logically, this fact entails that there no proper subset B of A such that $B \approx \mathbf{P}(A)$. We have therefore eliminated the possibility that $\mathbf{P}(A) < A$. It follows that the only remaining alternative must be true, namely that $A < \mathbf{P}(A)$. **QED**

Notation for number sets:

- **Nn** the set of ***natural numbers***: 0,1,2,3,… ;
- $\mathbb{Z}$ the set of i***nteger***s: …,-3,-2,-1,0,1,2,3,… ;
- $\mathbb{Q}$ the ***rational***, i.e. the integers and all whole number fractions of integers, both positive and negative;
- $\mathbb{R}$ the set of ***reals***, i.e. the set of rationals and irrationals like $\sqrt{2}$ and π.

It is not difficult to prove several counter-intuitive results. First even though the natural numbers is a proper subset of the integers and the rationals, and the integers are a proper subset of the rational, there are exactly as many natural number as there are integers and rationals. However, there are more reals than natural numbers, integers, and rationals.

Theorems

1. The natural numbers, integers, and rationals are all countably infinite and equipollent: $\mathbf{Nn} \approx \mathbb{Z} \approx \mathbb{Q}$
2. The cardinality of the set of reals $\mathbb{R}$ is greater than that of the natural numbers **Nn**, the integers $\mathbb{Z}$, and the rational $\mathbb{Q}$, i.e. $\mathbf{Nn} < \mathbb{R}$, $\mathbb{Z} < \mathbb{R}$, and $\mathbb{Q} < \mathbb{R}$.

A conjecture in set theory is that the set $\mathbb{R}$ of the reals, called ***the continuum***, is the smallest transfinite set. This hypothesis is so far neither are proven or refuted:

The Continuum Hypothesis: There is no A such that $\mathbf{Nn} < A < \mathbb{R}$.

Transfinite Numbers

The existence of infinite sets of increasing "size" has lead in mathematics to the discovery of numbers suitable for "counting" or "measuring" the "size" of such sets. There are two concepts of "size" that in the case of finite sets coincide but that in the case of infinite sets diverge, and these two concepts have lead to the development of two distinct concept of "number" appropriate to measuring size in the two different sense.

Cardinal Numbers. The cardinality of a set is roughly how large a set is if you count its members. Using this sense of cardinality the set of contains two elements has a "smaller cardinality" than one that contains four elements. It is in this sense that the counter-intuitive result holds that some infinite sets are the same size even though one is a proper subset of the other, e.g. the set of integers is the same "size" as the set of rationals despite the fact that the integers form a proper subset of the rationals. There is a set of numbers called the ***cardinal numbers*** that are appropriate for measuring sets according to this concept of size.

For finite sets the ordinary definition of the natural numbers – the counting numbers – is fine. Therefore the cardinal numbers start with the series 0,1,2,3,… . For infinite sets however new numbers are necessary. The first cardinal number is the one appropriate for measuring the set of natural numbers itself, which, recall, is the set we use to define when a set is infinite. (Recall that a set is countably infinite if it is in 1-1 correspondence to the set of natural numbers.) Therefore, the first *infinite* cardinal number, with is called $\aleph_0$ (read "aleph zero" or "aleph null" – $\aleph$ is the first letter of the Hebrew alphabet) simply identified with the set of natural numbers itself: $\aleph_0=\{0,1,2,3,\ldots\}$. Before explaining the next larger cardinal number, which measures the size of the next largest infinite seit **P**(Nn), which in alternative notation is **P**($\aleph_0$)), we must turn to the second sense of "larger than."

Ordinal Numbers and Transfinite Cardinal Numbers. The cardinal numbers appropriate for measuring size according to the number of elements in a set are defined by
The second sense of "size," with its own set of numbers, is defined in terms of the subset relation or – what is the same thing – whether one set has one more element than another. Numbers appropriate for measuring size in this sense are called ordinal numbers. In the case of finite sets there is no difference between the size of a set if it is measured by its cardinality – by how many elements it contains – or by its ordinality – by whether it is a subset of another set. The number 2 is defined as $\{\varnothing,\{\varnothing\}\}$ and the number three is defined as $\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}$. Given this definition 2 is less than 3 in cardinality because 2 has fewer elements that 3, and 2 is less than 3 in ordinality because 2 is a subset of 3. However, it is possible to add an extra element to an infinite set without altering its cardinality. Recall that one way to add a new element to a set is by means of the successor operation *S.* This operation adds to a set *A* the set itself: $S(A)=A\cup\{A\}$. In ordinal number theory it is customary to use the symbol ω (omega) name the set of natural numbers (0 plus the positive integers). It follows that ω is countably infinite. Let us add one more element to ω and call it $\omega+1$, i.e. $\omega+1= S(\omega)=A\cup\{\omega\}$. It is easy to put ω and $\omega+1$ in 1-1 correspondence, and therefore the two infinite sets have the same cardinality. However, $\omega \subset \omega+1$. Therefore, ω is less than $\omega+1$ as measured by ordinality.

As in the case of cardinal numbers the standard natural numbers are perfectly adequate for measuring the ordinality size of finite sets. Therefore the set of ordinal numbers, like the set of cardinal numbers, starts with the natural numbers: 0,1,2,3,… . The first infinite set appropriate for measuring ordinality, again like the case of cardinal numbers, is the set of natural numbers itself. The first ordinal number ω or in alternative notation ω_0, is identified with {0,1,2,3,…), which recall is also the same as $\aleph_0$. That is, $\omega=\omega_0=\{0,1,2,3,\ldots\}=\aleph_0=$Nn.

Note something that it will be important in a minute: ω is special in that though it is a "number," it is not obtained by the successor operation. Rather it is defined by the special property that it is the union (combination of elements of) all the numbers in {0,1,2,3,…}, each of which is the set of its predicessors. That is

$$\omega_0=\mathrm{U}\{0,1,2,3,\ldots\}=\mathrm{U}\{0,0+1,0+2,\ldots,0+n,\ldots\}$$

Now, unlike cardinals we can increase the "size" of an ordinal in the sense of making a set with a new member by simply adding a new element. We can do this to ω_0. $\boldsymbol{S}(\omega_0)=\omega_0+1=\omega_0\cup\{\omega_0\}$ is "larger than" ω_0 in this sense because $\omega_0\subset(\omega_0\cup\{\omega_0\})$. We can make a series of such ordinals of increasing ordinal "size":

$\omega_0,\omega_0+1,\omega_0+2,\ldots,\omega_0+n,\ldots$

Each ordinal in the series is a proper subset of its successor, and therefor the successor is larger in ordinality than its predecessor because it contains its predecessor as a proper subset. However, perhaps suprisingly, all of these sets are the same size as measured by cardinality. Each set in the series can be put into 1-1 correspondence with any other member of the series.

Moreover if we take the union of the series, we obtain a larger set and thereofore a new ordinal number:

$\omega_1 = \mathrm{U}\{\omega_0,\omega_0+1,\omega_0+2,\ldots,\omega_0+n,\ldots\}$

This set is larger than any previous ordinal because unlike its predecessors it conatins all its predecessors.

Notice that like $\omega_0=\mathrm{U}\{0,0+1,0+2,\ldots,0+n,\ldots\}$, ω_1 not formed by the successor operation, but rather is formed by taking the union of the set of prior ordinals. It is not the successor of any prior ordinal. It turns out, moreover, ω_1 is also larger than its predecessors in cardinality. It is in fact equipollent to – can be put into 1-1 correspondence – to **P**(Nn), which in alternative notiation is with $\mathbf{P}(\aleph_0)$, which recall is larger in cardinality than Nn. There is moreover no set of cardinality between that of Nn and **P**(Nn), or in alternative notation between $\aleph_0$, and $\mathbf{P}(\aleph_0)$. For this reason ω_1 Is indentified $\aleph_1$,the next are cardinal number larger in cardinality than $\aleph_0$.

Obviously, by the successor operation we can now start adding elements to ω_1 and make yet another series of sets of increasing ordinality,

$\omega_1,\omega_1+1,\omega_1+2,\ldots,\omega_1+n,\ldots$

All these too are of the same cardianality because they too can be put into 1-1 correspondenceand with one another. Moreover, we can take their union to make yet anther ordinal:

$\omega_2 = \mathrm{U}\{\omega_1,\omega_1+1,\omega_1+2,\ldots,\omega_1+n,\ldots\}$.

which turns out to have the next higher cardinality compared to all its predecessors. It turns out to be equipollent to **P**(**P**(Nn)) and is identified with $\aleph_0$, the next higher cardinal mumber.

Clearly, there is a pattern hear.

The set of all can be defined:

Definition of Ordinal Number:

0 is an ordinal;

if *n* is an ordinal, then $\boldsymbol{S}(n)$ is an ordinal;

if *A* is a set of ordinals, then U*A* is an ordinal;
nothing else is an ordinal.

An ordinals formed by union but not by the successor operation are called limit ordinal. It is customary to refer to limit ordinals by the Greek letter lambda λ:

Definition. If *n* is an ordinal but is not the successor of any ordinal then *n* is a ***limit ordinal.***

It turns out that the series of increasingly larger sets as measured by cardinality turns out to be equipollent to the the infinite series of limit ordinals. For this reason the infinite series of transfinite cardinal numbers is identified with the infinite series of limit ordinals:

Definition of the Transfine Cardinals:

$\aleph_n = \omega_n$.

Conclusions for the Philosophy of Religion

- The major lesson of this review of modern mathematics is that the concept of the infinite is now extremely well understood.

- The concept of infinity as it is traditionally applied to god in theology and philosophy turns out to have nothing to do with infinity in well understood mathematical sense. Strictly speaking it is nonsense to say that god is infinite in the mathematical sense because god is not a set containing elements.

- The theological notion of infinity is quite different from the mathematical sense. This sense presupposes an ordering relation: “having more being or perfection than”. An infinite being in the theological sense, then, would be the maximal or highest element in this ordering.

- There is nothing mathematically or logically contradictory or absurd to the notion of a series of cause or instances of time that recede infinitely into the past. Such a regression would simply be equipollent to the negative integers or perhaps to the negative reals.