Notes on Extensive Measurement

Among the elements of a set of physical objects or quantities – sets that are not mathematical entities like numbers -- comparisons (e.g. *is taller than, is more beautiful than, is better than*) are made by non-mathematical means. We use relations that are defined in physical, non-mathematical terms, but which nevertheless amount to orderings. Typically, these are named in natural language by ***comparative adjectives***, which are often associated with sets of what are called by linguists ***scalar adjectives***. Scalar adjectives are used to name regions of various ranks in an ordering. For example, the comparative adjective *happier than* describes a physical ordering in nature. Associated with it is the family of scalars adjectives *ecstatic, happy, content, so-so, sad, unhappy, miserable*.

Ordering relations $\leq$ vary and may posses a more or less rigorous structure -- from pre-orderings (reflexive, transitive) to the continuous order of real numbers. Here we shall investigate the standard features of natural structures that make them amenable to numerical measurements open that allow arithmetical computations on the measurement values by means of the standard arithmetical operations like addition and multiplication.

First let us do some ontology to figure out exactly what sort of thing is being ordered when we order quantities. Socrates and Plato are individual human beings, things in the world, who possess properties. Some of these properties admit of comparisons of "more" or "less," as in *Socrates is taller than Plato, Plato is more handsome than Socrates*. Some admit of numerical comparisons by so-called measure phrases, as in *Socrates is five inches taller than Plato.* Ontologically what is being ordered are the something like the extent to which some thing possesses a property, and what is being measured numerically is a quantity. Language possess mass nouns formed from adjectives for this purpose. What is it that Socrates has more of than Plato? Tallness. How much more tallness does he have? Five inches. What is ranked by the relevant ordering relation associated with a comparative adjective is "quantities" of the "mass" indicated by its associated mass noun.

For the purpose of clarifying the conditions necessary for the numerical measure of "masses" of this sort, social scientists make use of a special operation binary operation $\blacklozenge$ defined on quantities of a given "mass." If we let x and y represent two quantities of a given mass, like heat, weight, or length, we represent the quantity obtained by combining x and y by $x \blacklozenge y$. If $\blacklozenge$ is defined for weight, $x \blacklozenge y$ could be defined as the combined distance on a scale of the two distances x and y. If it is defined for volume, $x \blacklozenge y$ could be defined as the amount of water displaced first by x and then by y. The capture the fact that $x \blacklozenge y$ joins together two quantities, it could be called "combination", but its standard name in measurement theory is ***concatenation***.

Mathematically the challenge for measurement theory is to state the conditions under which an ordering relation $\leq$ on units of mass allows for those units to be quantified in such a way that numerical operations like addition and multiplication are meaningfully applicable to them. It is possible to state these conditions in terms of the properties of concatenation.[1] For this purpose we define an abstract structure composed of a set *B* (which are intuitively "quantities," "masses", or "extensions"), an ordering relation ≤ on these masses, and a concatenation operation ♦ that combines them. We also define the idea *nx* of concatenating the mass quantity *x* with itself *n*-times. Intuitively 2*x*, which is the same as *x*♦*x,* is "twice" the "mass" of *x.*

<*B*,≤,♦> is a ***(positive) closed extensive structure*** iff for any $x,y,u,v \in B$:

1. ≤ is a weak ordering on *B:* ≤ is transitive, and connected (*x*=*y*, *x*<*y* or *y*<*x*).
2. ♦ is associative: *x*♦*y* = *y*♦*x.*
3. ≤ is monotonic: *x*≤*y* iff (*x*♦*u*)≤(*y*♦*u*) iff (*u*♦*x*)≤(*u*♦*y*).
4. ♦ is positive: for any *x,y*∈*B* *x* ≤ *x*♦*y*. (Hence *x* ≤ *x*♦*x).*
5. ≤ is Archimedean: Let *nx* (read ***the concatenation of** x n **times***) be defined as follows (a) 1*x*=*x*, and (b) (*n*+1)*x*=*nx*♦*x.* If *x*<*y* , then for any *u,v*∈*B*, there exists a positive integer *n* such that (*nx*♦*u*) ≤ (*ny*♦*v*) .

Condition 1 is a minimal condition for considering ≤ to be an "ordering." Conditions 2-5 insure that concatenation provides the basis for mapping <*B*,≤,♦> onto a structure of numbers so the mapping, a "measure" assignment, showing that <*B*,≤,♦> reproduces numerical structure.

Intuitively, Condition 5 is the most difficult to understand. Intuitively, it says that no matter how big a head start (*u*) you give a lesser extension (*x*), you can always find enough units, namely *n,* of the larger extension *y* (possibly enlarged by *v*) so that *n* units of *y* together with *v* will be bigger than *n* units of *x* with its the head start *u.*

There is a less general by more intuitive way to state the idea. Condition 5 insures the following "Archimedean" result: if *x*<*y* in a physical sense in which we can compare physical sizes, then we can extend *x* by some finite number *n* of iterations so that the result *nx* is bigger than *y.* That is, Condition 4 entails the following theorem:

Theorem. Let <*B*,≤,♦> be a positive extensive structure, and $x,y \in B$. If *x*<*y* , then there exists a positive integer *n* such that *y*≤*nx*

Proof. Assume

(1) *x*<*y*.

By Condition 4, (1) entails for *u*=*v*=*x,*

(2) $\exists n\ (x_1 \blacklozenge \ldots \blacklozenge\ x_{n+1} \leq y_1 \blacklozenge \ldots \blacklozenge\ y_n \blacklozenge\ x)$,

By Condition 3, ≤ is monotonic; hence (2) entails:

(3) $\exists n\ (x_1 \blacklozenge \ldots \blacklozenge\ x_n \leq y_1 \blacklozenge \ldots \blacklozenge\ y_n)$.

Let *m* be the least such *n,* so that:

(4) $x_1 \blacklozenge \ldots \blacklozenge x_m \leq y_1 \blacklozenge \ldots \blacklozenge\ y_m$, and

(5) not $(x_1 \blacklozenge \ldots \blacklozenge\ x_{\tilde{m}1} \leq y_1 \blacklozenge \ldots \blacklozenge\ y_{\tilde{m}1})$

By Condition 1, ≤ is complete; hence (5) entails:

(6) $y_1 \blacklozenge \ldots \blacklozenge\ y_{\tilde{m}1} \leq\ x_1 \blacklozenge \ldots \blacklozenge\ x_{\tilde{m}1.}$

By Condition 5, the structure is positive; hence,

(7) $y \leq\ y_1 \blacklozenge \ldots \blacklozenge\ y_{\tilde{m}1}$.

By Condition 1, ≤ is transitive; hence by (6) and (7):

(8) $y \leq\ x_1 \blacklozenge \ldots \blacklozenge\ x_{\tilde{m}1}$

By definition, (8) may be rephrased:

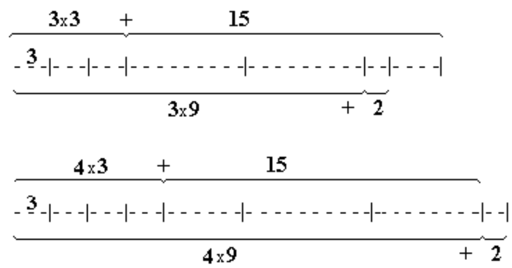
(9) $y \leq\ (\tilde{m}!\ x.$ QED.

[1]Reference: David H. Krantz, R. Duncan Luce, Patrick Suppes, and Amos Taversku, *Foundations of the Theory of Measurement.* vol I. (N.Y.: Academic Press).

Example. To see that the Archimedean property as stated in Condition 5 fits measurement cases, consider the example of $3 \leq 9$. Even though $3 \leq 9$ we can increase the size of 3, say three times, and then add 6 we get 15, a number that is larger than a similar augmentation of the larger number 9 by the same factor and then adding a lesser number, say 2:

$$3x3 + 6 > 3x9 + 2$$

But eventually the fact that 9 is larger than 3 emerges if we up the increase, say in this case from 3 times to 4. We then arrive at the number predicted by the Archimedean property, namely one that makes the increase of the larger number, in this case 9, with 2 added to it , greater than the increase of the smaller number, in this case 3, with a number greater than 2, in this case 15, added to it: $3x3 + 6 \leq 4x9 + 2$



Any number *n* greater than 4 will continue to fit the Archimedean inequality. The fact that this sort of property holds depends on the numerical measurability of concatenation increments and (somewhat surprisingly given the arcaneness of its formulation) is characteristic of structures that admit of numerical measurement.

That the notion of extensive structure captures the necessary and sufficient conditions for the possibility of arithmetical measurement is shown by the following theorem:

Theorem. Let *B* be a non-empty set, $\leq$ a binary relation on *B* and ♦ a binary operation closed on *B*. Then $<B, \leq, ♦>$ is a closed extensive structure iff there exists a function *m* mapping *B* into the set of real numbers such that for all $x, y \in B$,

1. $x \leq y$ iff $m(x) \leq m(y)$,
2. $m(x ♦ y) = m(x) + m(y)$.

Further, a function *m′* satisfies 1 and 2 iff there exists an *n* such that $0 < n$ and for any $x \in B$, $m'(x) = nm(x)$. (That is, intuitively, any other measurement assignment *m′* will be a “scale” value of *m.*) Moreover, the structure is positive iff any $x \in B$, $0 < m(x)$.