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A SYNTACTIC CHARACTERIZATION OF KLEENE'S STRONG CONNECTIVES WITH TWO DESIGNATED VALUES

by JOHN N. MARTIN in Cincinnati, Ohio (U.S.A.)

Recently, a number of theorists have attempted to admit category mistakes as non-bivalent and explain the projection of truth-values to molecular sentences by means of S. C. KLEENE's strong connectives. See, for example, L. ÅQVIST [1], K. DONNELLAN [2], and R. L. MARTIN [5]. From a philosophical point of view, a desideratum of any such projection is the preservation of as much of classical 2-valued logic as possible. For a defense of this view see R. THOMASON [8]. Below is reported the degree of success to which this goal may be attained on one definition of valid argument for KLEENE's connectives. Specifically, semantic entailment with 1 and 1/2 designated is characterized in terms of classical semantic entailment and syntactical features of the sentences in question. Since the semantic entailment relation for classical logic can itself be syntactically characterized, the result provides, in effect, an axiomatization of KLEENE's system.

Let the set of formulas be inductively defined over a denumerable set of atomic formulas such that $-A$ and $(A \cdot B)$ are formulas if A and B are. Let $A \vee B$ be short for $-(-A \cdot -B)$.

Let the matrix for *classical logic* be $\mathfrak{C} = \langle\{0, 1\}, \{1\}, \&, \sim\rangle$ such that & and $\sim$ are operations on $\mathfrak{C}$ conforming to the classical truth-tables for conjunction and negation. The set of classical valuations $V_{\mathfrak{C}}$ consists of just those functions v on formulas such that for all atomic formulas A, $v(A) \in \{0, 1\}$, and $v(A \cdot B) = v(A) \,\&\, v(B)$, and $v(-A) = \sim v(A)$. *A semantically entails B* in $\mathfrak{C}$, briefly $A \Vdash_{\mathfrak{C}} B$, iff

$$(\forall v \in V_{\mathfrak{C}})\ (v(A) = 1 \rightarrow v(B) = 1).$$

Likewise, A is *valid* in $\mathfrak{C}$, briefly $\Vdash_{\mathfrak{C}} A$, iff $(\forall v \in V_{\mathfrak{C}})\ (v(A) = 1)$.

Let the matrix for the *strong connectives* be $\mathfrak{K} = \langle\{0, 1/2, 1\}\,\{1/2, 1\}, \wedge, \neg\rangle$ such that $\wedge$ and $\neg$ are operations on $\{0, 1/2, 1\}$ conforming to the following tables:

$\neg$			$\wedge$	0	1/2	1
0	1			0	0	0
1/2	1/2			0	1/2	1/2
1	0			0	1/2	1.

See KLEENE [3] and compare J. ŁUKASIEWICZ [4]. The set $V_{\mathfrak{K}}$ of $\mathfrak{K}$-valuations consists of just those functions on formulas such that for all atomic formulas A, $v(A) \in \{0, 1/2, 1\}$, and $v(A \cdot B) = v(A) \wedge v(B)$ and $v(-A) = \neg v(A)$. We choose $\{1/2, 1\}$ as the set of designated values, abstracting from classical logic the fact that valid arguments never lead from non-false premises, to false conclusion. *A semantically entails B* in $\mathfrak{K}$ iff $(\forall v \in V_{\mathfrak{K}})\ (v(A) \in \{1/2, 1\} \rightarrow v(B) \in \{1/2, 1\})$. Likewise, A is *valid* in $\mathfrak{K}$ iff $(\forall v \in V_{\mathfrak{K}})$ $(v(A) \in \{1/2, 1\})$. Again, we abbreviate $A \Vdash_{\mathfrak{K}} B$ and $\Vdash_{\mathfrak{K}} A$.

Theorem 1. *The semantic entailment relation* $\Vdash_{\mathfrak{K}}$ *is a proper subset of that* $\Vdash_{\mathfrak{C}}$.

Proof. First we show it is a subset. Assume $A \Vdash_{\mathfrak{K}} B$ but not $(A \Vdash_{\mathfrak{C}} B)$. Then, there exists a $v \in V_{\mathfrak{C}}$ such that $v(A) = 1$ and $v(B) = 0$. Let v' be a function in $V_{\mathfrak{K}}$ such that v' is like v for all atomic sentences in A and B. Since $\& \subseteq \wedge$ and $\sim \subseteq \neg$, $v'(A) = 1$ and $v'(B) = 0$. Hence not $(A \Vdash_{\mathfrak{K}} B)$ which is absurd. Hence $\Vdash_{\mathfrak{K}} \subseteq \Vdash_{\mathfrak{C}}$. That it is proper inclusion is shown by the argument from $A \cdot -A$ to B which holds in $\mathfrak{C}$ but fails in $\mathfrak{K}$ in the case where $v(A) = 1/2$ and $v(B) = 0$. For then $v(A \cdot -A) = 1/2 \in \{1/2, 1\}$, but $v(B) = 0$.

Though the two relations fail to coincide in general they are identical in a special case. First, some preliminary remarks. DeMorgan's laws, double negation, associativity, commutativity, and distribution through conjunction and disjunction all hold in both $\mathfrak{C}$ and $\mathfrak{K}$, as is easily shown by truth-tables. In fact, A and its normal forms continue to have the same truth-values under every $\mathfrak{K}$-valuation. We shall use this fact below. Let a *disjunctive normal form of A*, briefly dnf (A), be any formula resulting from applications to A of DeMorgan's laws, double negation, associativity, commutativity, and distribution through conjunction and disjunction, and that has the form of a disjunction of terms D_i such that each D_i is a conjunction made up of atomic sentences and negations of atomic sentences.

Lemma. $(\forall v \in V_{\mathfrak{K}}) (v(A) = v(\mathrm{dnf}(A)))$.

Another interesting parallel between $\mathfrak{K}$ and $\mathfrak{C}$ is that A and $(A \cdot B . \vee . A \cdot -B)$ are both designated if one is. More precisely, the relation between these two sentences is that they both have the same truth-value whatever the value of B, except in one case: when A is 1 and B is 1/2, $(A \cdot B . \vee . A \cdot -B)$ is 1/2. Hence they mutually entail each other in $\mathfrak{K}$, and in certain cases atomic sentences absent from one of the disjuncts in the disjunctive normal form of A may be introduced without changing truth-values. For example, when p and q are 1 and r is 1, we can be sure that $p\bar{q} \vee \bar{p}\bar{q}$ has the same truth value as $p\bar{q}r \vee p\bar{q}\bar{r} \vee \bar{p}\bar{q}r \vee \bar{p}\bar{q}\bar{r}$. By a *strengthening of A by B* let us understand a dnf$(A \cdot B . \vee . A \cdot -B)$.

Lemma. *If* A^* *is a strengthening of* A *by all the members* B *of a set* X, *then all* $\mathfrak{K}$-*valuations conform to the following table:*

A	A^*
1	1 or 1/2
1/2	1/2
0	0

Syntactically the disjunctive normal form of A in $\mathfrak{K}$ will appear just like it does in $\mathfrak{C}$ except for one difference. Contradictory disjuncts cannot be dropped from the normal form in $\mathfrak{K}$, for they are $\mathfrak{K}$-satisfiable when the contradictory members are neither true nor false. Hence, just as any classical valuation assigning 1 to a disjunct of dnf(A) will assign 1 to A, so any $\mathfrak{K}$-valuation that designates a disjunct of dnf(A) will also designate A. The way to read dnf$(A) = D_1 \vee \cdots \vee D_n$ in $\mathfrak{K}$ is as follows. Each D_i determines a number of $\mathfrak{K}$-valuations v that satisfy A. Let v be one such. If D is non-contradictory, then v assigns to any atomic part p of D either 1/2 or 1 if p is unnegated, or either 1/2 or 0 if p is negated. If D is contradictory, v assigns 1/2 to each of the contradictory atomic parts and otherwise treats D as if it were non-contradictory. We may now show that the valid formulas of $\mathfrak{C}$ and $\mathfrak{K}$ are the same. Part II of the proof is due to B. C. VAN FRAASSEN.

Theorem 2. $\Vdash_{\mathfrak{K}} A$ *iff* $\Vdash_{\mathfrak{C}} A$.

Proof. I. Let $\Vdash_{\mathfrak{K}} A$ and not $\Vdash_{\mathfrak{C}} A$. Then, there is a $v \in V_{\mathfrak{C}}$ such that $v(A) = 0$. Let v' be a $\mathfrak{K}$-valuation that agrees with v on all atomic formulas in A. Then $v'(A) = 0$ and not $\Vdash_{\mathfrak{K}} A$ which is absurd. Hence $\Vdash_{\mathfrak{K}} A$ only if $\Vdash_{\mathfrak{C}} A$.

II. That $\Vdash_{\mathfrak{C}} A$ only if $\Vdash_{\mathfrak{K}} A$. Let A be $\mathfrak{C}$-valid and suppose that for some $\mathfrak{K}$-valuation v that $v(A) = 0$. Then, $v(\text{dnf}(A)) = 0$. Then, for each disjunct D_i $(i = 1, \ldots, n)$ of dnf(A), there is a conjunct $c^i_{j_i}$ that is also assigned 0 by v. Then, $v(c^1_{j_1} \vee \cdots \vee c^n_{j_n}) = 0$. Also, v agrees with some $\mathfrak{C}$-valuation v' on all these conjuncts. Now, $A \Vdash_{\mathfrak{C}} (c^1_{j_1} \vee \cdots \vee c^n_{j_n})$. Hence, $v'(A) = 0$, and not $\Vdash_{\mathfrak{C}} A$, which is absurd.

Thus, validity for $\mathfrak{K}$ is sufficiently like $\mathfrak{C}$ to capture all the truths of classical logic but sufficiently unlike $\mathfrak{C}$ to falsify some of the classically valid arguments. We turn now to a precise syntactic characterization of the relation $\Vdash_{\mathfrak{K}}$ in terms of the relation $\Vdash_{\mathfrak{C}}$. Given A and B, let A^* and B^* be the disjunctive normal forms of A and B strengthened so that each of their disjuncts contains all the atomic sentences appearing in either A or B. Further, the negated version of p is $\bar{p}$ and that of $\bar{p}$ is p.

Definition. $\langle A, B \rangle$ is $\mathfrak{K}$-*pernicious* iff there is a disjunct D of A^* such that

(1) D is contradictory;

(2) there is a largest, non-empty, consistent set C formed from the non-contradictory conjuncts of D;

(3) for any disjunct D' of B^*, the negated version of some $c \in C$ is a conjunct of D'.

For example, $\bar{p} \vee (q\bar{q}p)$ and $\bar{p}$ taken together are $\mathfrak{K}$-pernicious. (For the strengthened normal form of the former is $pq\bar{q} \vee \bar{p}q \vee \bar{p}\bar{q}$ and that of the latter is $\bar{p}q \vee \bar{p}\bar{q}$; the contradictory disjunct is $pq\bar{q}$; and the subset C of conjuncts is $\{p\}$.) Notice that in classical logic, $\bar{p}$ is deducible from $\bar{p} \vee (q\bar{q}p)$. The argument is a form of *reductio*. However, it fails in $\mathfrak{K}$, for let $v(p) = 1$ and $v(q) = 1/2$. Then, $v(\bar{p} \vee (q\bar{q}p)) = 1/2$.

Theorem. $A \Vdash_{\mathfrak{C}} B$ *and* $\langle A, B \rangle$ *is not* $\mathfrak{K}$-*pernicious, iff* $A \Vdash_{\mathfrak{K}} B$.

Proof. "If" Part. Assume $A \Vdash_{\mathfrak{C}} B$ and not $(A \Vdash_{\mathfrak{K}} B)$. We show $\langle A, B \rangle$ is $\mathfrak{K}$-pernicious. Since not $(A \Vdash_{\mathfrak{K}} B)$, there is an $\mathfrak{K}$-valuation v such that $v(A) \in \{1, 1/2\}$ and $v(B) = 0$. We show first that $v(A) = 1/2$. For suppose on the contrary that $v(A) = 1$. Define a $\mathfrak{C}$-valuation as follows: for any atomic sentence p,

$$v'(p) = 1 \quad \text{if} \quad v(p) \in \{1, 1/2\}, \qquad v'(p) = 0 \quad \text{otherwise}.$$

Then, for some disjunct D of dnf(A), $v(D) = 1$ and, hence, all conjuncts of D are assigned 1 by v. Hence, v' also assigns 1 to all conjuncts of D and $v'(A) = 1$. Hence, $v'(B) = v'(\text{dnf}(B)) = 1$, and, therefore, for all conjuncts c of some disjunct D' of dnf(B), $v'(c) = 1$. Likewise, $v(c) \in \{1, 1/2\}$ for all these conjuncts of D'. But since $v(B) = 0$, we know for all D' of dnf(B), there is a conjunct c of D' such that $v(c) = 0$, which is a contradiction.

Since $v(A) = 1/2$, there is at least one disjunct D of A^* such that $v(D) = 1/2$. Now, for any of these D such that $v(D) = 1/2$, there is some non-contradictory conjunct c

of D such that $v(c) = 1$. For clearly none of these conjuncts c of D is assigned 0 by v since $v(D) = 1/2$. Further, not all conjuncts c of D are assigned $1/2$ by v. For if so, since all the atomic sentences of B are in D, $v(B) \doteq 1/2$, which is impossible. Hence for some c of D, $v(c) = 1$. Clearly, since $v(D) = 1/2$, not both c and its negated version appear in D. Hence, there is some largest, non-empty, consistent set C of conjuncts of D. Note that one of its members is assigned 1 by v and none are assigned 0.

Suppose, now, that all the disjuncts D of A^* such that $v(D) = 1/2$ are consistent. Let D be one such. Then define a $\mathfrak{C}$-valuation as follows: for any atomic sentence p,

$$v'(p) = v(p) \quad \text{if} \quad v(p) \in \{1, 0\};$$

$$v'(p) = 1 \quad \text{if} \quad v(p) = 1/2 \quad \text{and} \quad p \text{ is a conjunct of } D;$$

$$v'(p) = 0 \quad \text{if} \quad v(p) = 1/2 \quad \text{and the negated version of } p \text{ is a conjunct of } D.$$

Then, since D is consistent, $v'(D) = v'(A) = 1$, and, thus, $v'(B) = 1$. Then, there is some disjunct D' of B^* such that for any conjunct c of D', $v'(c) = 1$. But since $v(B) = 0$, we know for any D' of B^*, there is some conjunct c of D' such that $v(c) = 0$. But, then, for some conjunct c of some disjunct D' of B^*, $v'(c) = 1$ and $v(c) = 0$, which is absurd. Hence, one such D is contradictory. Let us call it D.

Consider now a largest, consistent, non-empty set C of non-contradictory conjuncts of D. Suppose all $c \in C$ are conjuncts of some disjunct D' of B^*. Then, since $v(c) \in \in \{1, 1/2\}$ and all the other atomic sentences of D' are contradictory conjuncts of D and are assigned $1/2$ by v, we know $v(D') = 1/2$. Hence, $v(B) \neq 0$ which is absurd.

"Only If" Part. That $A \Vdash_{\mathfrak{K}} B$ entails $A \Vdash_{\mathfrak{C}} B$ holds by Theorem 1. Assume that $A \Vdash_{\mathfrak{K}} B$ yet $\langle A, B \rangle$ is $\mathfrak{K}$-pernicious. Then there is some disjunct D of A^* as specified in the definition. Define an $\mathfrak{K}$-valuation: for any atomic sentence p,

$v(p) = 1/2$ for all contradictory conjuncts of D;

$v(p) = 1$ if p is not a contradictory conjunct of D and is a conjunct of D;

$v(p) = 0$ if p is not a contradictory conjunct of D and its negated version is a conjunct of D.

Then $v(D) = 1/2$ and $v(A) \in \{1, 1/2\}$. But for any disjunct D' of B^*, $v(D') = 0$ and, hence, $v(B^*) = v(B) = 0$, which is absurd.

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