The Review of Symbolic Logic
Volume 0, Number 0, xxxx 2016

PRIVATIVE NEGATION IN *THE PORT ROYAL LOGIC*

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Abstract. In this paper I argue that negation in *The Port Royal Logic* is not a failed or incoherent approximation of Boolean complementation as maintained by Sylvain Auroux and Marc Dominicy, but is rather a version of privative negation from medieval logic, and that as such it has a perfectly coherent semantics. The discussion reviews the critiques of Auroux and Dominicy as well as the semantics of privative negation as found in Aristotle, Proclus, Ockham, Buridan, Descartes, and Arnauld.

§1. Introduction. This paper investigates negation in the *Port Royal Logic* of Arnauld and Nicole.[1] It is part of a larger project of assessing to what degree the seventeenth century work is innovative. A full treatment of the topic would determine to what extent the *Logic* carries forward ideas from pre-Cartesian logic, the innovations it introduces to accommodate Descartes' metaphysics and epistemology, and the ways it anticipates modern logic. This paper focuses on one part of this effort, the concept of negation and its contribution to the "algebra" of ideas in the *Logic*. Of particular concern is whether Boolean complementation is among the "algebraic" operations that give "structure" to ideas.

Two important studies of the *Logic*'s idea structure—Dominicy (1984) and Auroux (1993)—maintain that its account anticipates nineteenth century logic in major ways.[2] In particular, they interpret this structure as approaching modern Boolean algebra. On their reading the "containment" relation on ideas is a partial ordering.[3] In addition they abstract meet and join operations in the modern sense from the *Logic*'s mental operations of restriction and abstraction. They read the *Logic*'s idea of *existence* as a minimal element

The author wishes to acknowledge the research support of the Charles Phelps Taft Fund of the University of Cincinnati.

[1] Arnauld and Nicole, *La Logique ou l'Art de Penser* (abbreviated *LAP*) in Arnauld (1813) (abbreviated *KM*). The English translation is Arnauld (1996) (abbreviated *B*).

[2] Enriques (1926), Auroux (1993), Dominicy (1984).

[3] The paper assumes the following algebraic concepts. $<B,\wedge,\vee,-,0,1>$ is a *Boolean algebra* if and only if it is a structure satisfying the following conditions. Let x, y and z be arbitrary members of B.

(1) $<B,\wedge,\vee>$ is a *lattice*, i.e., $\wedge$ and $\vee$ are closed binary operations on B, $x \wedge y = y \wedge x$; $x \vee y = y \vee x$; $(x \wedge y) \wedge z = x \wedge (y \wedge z)$; $(x \vee y) \vee z = x \vee (y \vee z)$; $x \wedge x = x = x \vee x$; $x \vee (x \wedge y) = x = x \wedge (x \vee y)$.

(2) $<B,\leq>$ is a *partially ordered structure*, i.e., $\leq$ is a binary relation on B that is reflexive, transitive and antisymmetric or, equivalently, $x \leq x$, $x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y$,$(x \leq y$ & $y \leq z) \Rightarrow x \leq z$, $(x \leq y$ & $y \leq x) \Rightarrow x = y$.

(3) $<B,\wedge,\vee>$ is *distributive*, i.e., $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$; $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. 0 and 1 are, respectively, the *least* and *greatest element* of B in $<B,\wedge,\vee,0,1>$, i.e.,
$0 \leq x \leq 1$; $1 \wedge x = x$; $1 \vee x = 1$; $0 \wedge x = 0$; $0 \vee x = x$.

(4) $-$ is a *complementation operation* on B i.e., $-$ is a closed monadic operation on B, $x \wedge -x = 0$, $x \vee -x = 1$, $- - x = x$, $-1 = 0$, $-0 = 1$; $x \leq y \Leftrightarrow x \wedge -y = 0 \Leftrightarrow -y \leq -x \Leftrightarrow -x \vee y = 1$, $-(x \wedge y) = -x \vee -y$, and $-(x \vee y) = -x \wedge -y$.

doi:10.1017/S175502031600023X

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relative to the containment relation and suggest that there should also be maximal idea that contains all modes, even contradictory ones.

These interpretations are not without difficulties. Unlike a Boolean join operation, which is dyadic, abstraction is most straightforwardly read as monadic: the soul abstracts an idea not from of a pair of ideas, but from the definition of a single idea or perception. Contrary to their suggestion, *existence* does not seem to be contained in every idea because some ideas like *golden mountain* are fictional. Nor is there any explicit text to the effect that there is a maximal contradictory idea. Abstraction does not appear to be distributive over restriction.

On the other hand, both authors agree that the *Logic*'s term negation falls short of Boolean complementation and, indeed, when viewed overall, is incoherent. In this paper I will argue that the *Logic*'s term negation is a version of medieval privative negation, that it is perfectly coherent, and that it is inappropriate to evaluate it within a Boolean framework. This reading goes against attempts to see in the *Logic* an early version of nineteenth century class theory.[4] I suggest, rather, that it should be understood as a conservative work. Its goal was less to develop a new branch of abstract mathematics than to render consistent with Descartes' new metaphysics and epistemology central parts of traditional logical doctrine.

§2 addresses Auroux's critique, and §3 Dominicy's. §4 explains the logical properties of privative negation and how it differs from Boolean complementation. §5 argues that the *Logic*'s term negation is a perfectly coherent non-Boolean variety of privative negation.

§2. Auroux's reconstruction and critique. Auroux's analysis of the *Logic*'s account of negation develops over time. In his earlier work he reconstructs the *Logic*'s theory of the structure of ideas axiomatically and in the process lays out various axioms that characterize negation. Idea structure here is characterized as approximating a lattice in which negation is a kind of proto-complementation. In his later work Auroux argues that the *Logic*'s negation is inconsistent with other parts of its metatheory, particularly its truth theory. I will argue that although Auroux's axioms fit the *Logic*'s intentions, the features he attributes to negation do not characterize Boolean complementation, and that when negation is understood as privative, it is perfectly consistent with the *Logic*'s wider theory.

Both in his axiomatization and in his later discussion, Auroux understands "structure" in the algebraic sense as consisting of an ordering relation and operations on the set of ideas. These are defined in terms of "comprehension," a technical concept new to the *Logic*. An idea's comprehension consists essentially of the set of properties that define it, and the containment relation that gives order to ideas arises from the set inclusion relation on comprehension-sets. To explain the details and their implications for negation, it is necessary to explain the role of comprehension more fully.

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[4] Not all commentators understand the *Logic* as an anticipation of Boolean algebra. See, for example, Conimbricenses (1617), pp. 318–20, and Pariente (1995), p. 246:

> L'originalité du livre ne réside pas, il est vrai dans ses innovations formelles. Arnauld et Nicole ne sont pas des inventeurs sur le plan du calcul logique. Rien n'est plus éloigné de leur style de réfléxion que les efforts diversifiés et inlassables d'un Leibniz pour mettre sur pied un formalisme efficace et rationnel.

Russell Wahl writes, "It is a mistake, I believe, to read into the *Logic* a prelude to set theory." Wahl (2008), p. 673.

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The Cartesians' rejection of Aristotelian body-mind causal interaction entailed a rejection of medieval theories of signification that depended on it. In earlier theory a concept was said to "signify"—today we say "refer to"—an object if it was causally linked to it by sensation and abstraction. Rejecting this causal link, the Cartesians explained an idea's signification by recourse to *objective being*, a concept appropriated from medieval logic. In their version, God associates with each idea a list of defining properties or *modes*. Today we would call this list an idea's *intentional content*; the *Logic* calls it its *comprehension*:

> I call the *comprehension* of an idea the attributes that it contains in itself, and that cannot be removed without destroying the idea. For example, the comprehension of the idea of triangle contains extension, shape, three lines, and the equality of these three angles to two right angles, etc.... none of these attributes can be removed without destroying the idea, as we have already said, whereas we can restrict its extension by applying it only to some of the subjects to which it conforms without thereby destroying it.[5]

Signification is then explained in terms of this content without appealing to body-mind causation: an idea *signifies* all those actual objects that satisfy the modes in its comprehension. Signification in turn is used to define a special sense of extension, and extension to define truth.

Because a comprehension is a series of defining modes, it determines an ordering. The more complex idea contains the less. Because comprehensions are in effect sets of modes, their ordering is essentially set inclusion on these sets. Auroux, and as we shall see Dominicy as well, assume that even if the *Logic*'s authors do not say so in so many words, they held that comprehension-sets form in effect a partial ordering (a reflexive, transitive, and antisymmetric relation), and that the ordering on comprehensions determine a corresponding containment relation on ideas: idea A is *contained in* idea B if and only if the comprehension of A is a subset of that of B. Moreover, the relation between an idea and its comprehension is so tight that an idea's identity conditions are defined by comprehension: idea A is *identical to* idea B if and only if their comprehensions are identical. In short, the *Logic* assumes what we would today call a 1-1 isomorphic mapping between ideas as partially ordered by containment and comprehension-sets as partially ordered by set inclusion.

Moreover, as understood by Auroux, comprehension-sets are the basis not only of order but of structural operations as well. On his view, the *Logic*'s mental operations of abstraction and restriction are suggestive of meet and join lattice operations in the modern sense. Here the topic is negation, and it is not appropriate to go into much detail about abstraction and restriction. Sufffice it to say that both are mental operations understood in the medieval sense, and both generate new ideas from old.

Abstraction forms a simpler idea from a prior more complex idea by removing a mode from its comprehension.[6] As we will see shortly, Auroux regards abstraction as essentially a kind of meet operation on ideas that corresponds to intersection on comprehension-sets. If abstraction were to fully correspond to set intersection, it would map a pair of ideas as arguments to that idea defined by the intersection of the arguments' comprehension-sets.

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5 *LAP* I:6, *KM* V, 144, *B* 39.

6 *LAP* I:5, *KM* V 143, *B* 38, and *Des vraies et des fausses Idées* 6, *KM* I, 207–208, Arnauld (1990 [1683]), 74–75.

Restriction also generates new ideas. It does so by combining the comprehensions of prior ideas.[7] Auroux sees in it a kind of join operation on ideas. If restriction were to fully correspond to set union, it would map two ideas as arguments to that idea that has as its comprehension-set the union of comprehensions of the argument pair. In sum, Auroux reads the *Logic* as attributing to ideas a structure similar to a lattice in the modern sense. It is in this context that he discusses negation, the topic of this paper.

Auroux formulates his earlier view in an axiom system that details the formal properties of idea containment, abstraction, restriction, and idea negation. To appreciate how the treatment of negation approximates that of Boolean algebra, it is necessary to see how it functions within this full set of relations.

Auroux's axiom set is given below. In the notation, E represents the set of ideas, $<$ is an ordering relation on E, $\sqcap$ and $+$ are binary operations on E, and $-$ is idea negation. Auroux's understanding seems to be that $<$ corresponds to strict set inclusion on comprehension-sets. $\sqcap$ and $+$ represent abstraction and restriction, respectively, which are interpreted as set intersection and union, respectively, on comprehension-sets.[8]

Auroux's structure of ideas is $<E,<,+,\sqcap,->$ such that

1*. $<$ is a strict partial order (irreflexive, asymmetric, and transitive).

2*. $\exists x, y, \ldots, q \in E.\ x \sqcap y \sqcap y \sqcap \cdots = q.$

3*. $\exists x, y, \ldots, z \in E.\ x - y - \cdots = z.$

4*. $\exists x, y, \ldots, z \in E.\ x = y + \cdots + z.$

5*. $\forall \mathrm{v}, \mathrm{w}, x, y, z \in E$
$\mathrm{v} = \mathrm{w} + x + y \wedge z = x + y. \rightarrow \mathrm{v} = \mathrm{w} + z$
$\mathrm{v} = \mathrm{w} + z \wedge z = x + y. \rightarrow \mathrm{v} = \mathrm{w} + x + y.$

6*. $\exists x, y, z, \ldots, q \in E.\ x - y - \cdots - q = z \leftrightarrow x = x + y + \cdots + q.$

7*. $\forall x, y, z, \ldots, q \in E.\ q \notin \{x, y, z\} \rightarrow.$
$x \sqcap y \sqcap z \cdots = q \leftrightarrow.\ x = q + r \wedge y = q + t.$

8*. $\forall x, y, z, \ldots, q \in E.\ x, y, z, \ldots, q$ are all distinct $\rightarrow$
$\mathrm{w} = x + y \cdots + z \rightarrow \mathrm{w} < x \wedge \mathrm{w} < y \wedge \ldots.$

9*. $\forall x, y, z, \ldots, q \in E.\ x \sqcap y \sqcap z = q \rightarrow x < q \wedge y < q \wedge z < q \wedge \ldots.$

10*. $\forall x, y.\ x < y \leftrightarrow x + y = x.$

11*. $\sim \exists y \forall x.\ x - x = y.$

It is helpful to consider each axiom briefly. The overall intention seems to be to characterize a lattice-like structure in which $<$ is a strict partial ordering. $\sqcap$ and $+$ have some but not all the properties of greatest lower bound and least upper bound and operations, respectively, and $-$ has some of the properties of Boolean relative complementation.

We shall see, however, that the properties fall short of a lattice. The axioms are not sufficient to insure that the structure is complemented, or that $\sqcap$ and $+$ are associative and commutative. Moreover, since the structure need not be complemented, idea negation, which is represented by $-$, need not be relative complementation. On the other hand, the axioms impose some lattice structure.

Axioms 1*, 2*, 4*, 5*, and 10* determine a minimal ordering and would also impose some lattice structure if $x \leq y$ is defined $x + y = x$ and $x \sqcap y = y$. Axiom 1* insures that

[7] *LAP* I:8, *KM* V, 151–52, *B* 44–45.

[8] Auroux (1982), p. 89.

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$<$ is a strict partial order. Axioms 2* and 4* say in effect that the operations $\sqcap$ and $+$ are nonempty. Axiom *5 is unnecessary because as formulated it holds by the substitutivity of identity. Axiom 10* is formulated in terms of a strict order. As it stands, it would be false in a lattice because $x + x = x$ yet $\sim (x < x)$. What is probably intended is a version of the lattice truth $x \leq y \leftrightarrow x \wedge y = x$. This axiom confirms the direction of the order in the sense that a more restricted idea is lower in the order. Restriction, which is represented by $+$, would then be a meet operation if the structure were a lattice. A unique minimal idea, i.e., a most restricted idea, if it existed, would be a least element.

Axioms 8* and 9* also impose some lattice structure. 8* requires that the value of $+$ is a lower bound (but not necessarily a greatest lower bound) and 9* that the value of $\sqcap$ is an upper bound (but not necessarily a least upper bound).

What interests us here, however, is negation, which is mentioned or implicit in Axioms 3*, 6*, 7*, and 11*.

Axiom 3* insures that $-$ is nonempty.

Axiom 11* appears to say that there is no idea with an empty comprehension.

Axiom 6* and 7* are especially relevant to the interpretation of negation. Both affirm a structural thesis developed in Book V of the Logic that mandates that there is an abstraction to z from x if and only if there is a parallel deconstruction by restriction to x from z. Auroux does not explain which texts he is intending to represent but the doctrine is spelled out at length in Book IV's discussion of "synthesis" and "analysis."[9] Synthesis is the process of progressively adding modes to the comprehension of a simpler idea. Analysis is the process of progressively removing the modes from a complex idea. Throughout the early sections of the book the authors explain that there is a construction of a complex idea from a simpler one by synthesis if and only if the simpler idea can be abstracted from the complex one by analysis. Suppose, for example, that comprehensions are ordered "in steps" as follows: {*existent*} $\subseteq$ {*material, existent*} $\subseteq$ {*animate, material, existent*} $\subseteq$ {*rational, animate, material, existent*}. Synthesis consists of progressive restriction from the first to the last, and abstraction of progressive simplification from the last to the first.
The intent seems to be to formalize the rule that there is an abstraction generating one idea from another if and only if there is a corresponding series of restriction simplifying the second idea to the first.

Axiom 7* expresses this relation directly. As it stands it seems to be missing existential quantifiers. The intention seems to be:

$$\forall x, y, z, \ldots, q \in E.\ q \notin \{x, y, z, \ldots\} \rightarrow .\ \exists r, t, s, \ldots (x \sqcap y \sqcap z \cdots = q \leftrightarrow .\ x = q + r \wedge y = q + t \wedge z = q + s \ldots).$$

Axiom 6* describes essentially the same relation but in terms of negation:

$$\forall x, y, \ldots, z \in E.\ x, y, \ldots, z \text{ are distinct} \rightarrow (x - y - \cdots = z \leftrightarrow x = y \sqcap \cdots \sqcap z).$$

Here $x - y$ appears to represent the idea formed from x by removing the modes definitive of y. On this reading, however, $x - y$ would be the same as $x + y$. Perhaps Aurox has in mind the Boolean definition of relative complementation: $x - y = x \wedge -y$. If ideas formed a Boolean algebra and complementation were well-defined, the ideas necessary for Axiom 6* would be provided by complementation: $(x \wedge y) = z \leftrightarrow x = z \vee -y$.

What is relevant to this paper is the use here of what seems to be a species of relative complementation. Although the full axiom set is not sufficient to insure all the properties

[9] *LAP* IV:2, *KM* V, 362–367, *B* 233–238; IV:6, *KM* V, 379, *B* 248.

of a Boolean algebra, or even of a complemented lattice, it is consistent with both.[10] In sum, it is fair to say that Auroux in his early account attributes to ideas a pre- or proto-Boolean structure. The particular interpretive issue for this paper, then, is the degree to which the properties of term negation in the *Logic* are to be appraised by comparison to Boolean complementation. Below in §5, I argue that the *Logic*'s negation is not Boolean but rather a species of medieval privative negation, which has logical properties quite different from Boolean complementation. We shall see, however, that although it is non-Boolean, it does support principles like Axioms 6* and 7*. Moreover, it does so without requiring the maximal and minimal elements that must accompany Boolean complementation.

In his later and more extended work, *La Logique des Idées,* Auroux's views evolve. He argues there that the *Logic*'s notion of negation is incoherent.[11] He continues to assume without the formal details that ideas are partially ordered by containment, that the order is induced by an isomorphic subset relation on comprehension-sets, and that abstraction and restriction, which he calls (following Enriques) *addition* and *comparison*, are parallel to intersection and union on comprehension-sets.[12] He also makes explicit that the order has a minimal element, which he calls *Monde*, and a maximal element, *Être*. Quoting Guniot (1778), he understands *Être* to contain no "qualités particulières d'un individu jusqu'à ce qu'il soit dépouillé de toutes, même de l'existence." That is, it corresponds to the empty set of modes (unlike Axiom 11* above).

Auroux argues, however, that the *Logic*'s account of negation is defective. He writes:

> One can easily show: i) That the intensional interpretation leads to logical errors (cf. Auroux (1979), pp. 140–143); ii) That negation does not behave symmetrically on extension and comprehension (cf. Auroux (1978), p. 5). We will return to these questions in chapters 3 and 4. The axioms show that PR was aware of the problem and that it is forced to resolve it without leaving the theory of ideas. In addition, they indicate clearly that the thesis according to which existential considerations are absent from classical logic is erroneous. We will show in 4.4.4 that an intensional theory of negation is impossible.[13]

In the passages cited in the above quotation, he explains his objections in more detail.

First, he offers a general criticism of the so-called intensional interpretation of the concept of extension, arguing that it gives the wrong truth-conditions to categorical propositions. Identifying the correct definition of extension is important because extension is used in the *Logic* to define truth. Recall that in the *Logic* the terms of a proposition are ideas, which have extensions. A universal affirmative is *true* if and only if the extension of the subject term is a subset of the extension of the predicate, and a universal negative is *true* if and only if the intersection of their extensions is empty.[14]

[10] An example would be family of nonempty sets partially ordered by a relation $\leq$ such that $x \leq y$ only if $x \subseteq y$, $x \wedge y$ is a $\leq$-lower bound but not in every case a greatest lower bound, $x \vee y$ is a $\leq$-upper bound but not in every case a least upper bound, and $\wedge$ and $-$ are identical.

[11] pp. 93–100.

[12] p. 94. Enriques (1926), p. 125.

[13] Auroux (1993), p. 81, author's translation.

[14] The *Logic* uses the equivalent formulation: the universal affirmative true iff the extension of the predicate restricted by that of the predicate is identical to that of predicate.

The issue that concerns Auroux is the definition of "extension." An interpretation he rejects, which we may call *the intensional reading*, holds that an idea's extension is determined by its comprehension alone: *the extension* of idea A (*in the intensional sense*) is the set of all ideas B such that the comprehension of B is a subset of that of A. This reading, however, makes truth entirely conceptual. A universal affirmative turns out to be true if the subject's intension includes that of the predicate, making a universal affirmative true by definition if true at all. Auroux's point is that this reading has unacceptable consequences for the universal negative. *No cow is a horse* would be false because the set of ideas defined by *cow* and that defined by *horse* do not intersect.

The alternative Auroux favors is what we may call *the referential reading*. It defines an idea's extension as the set of ideas that signify only objects in the world that the idea itself signifies: *the extension* of A (*in the referential sense*) is the set of all ideas B that signify only objects that A signifies.[15] Equivalently, the extension of A is the set of all ideas B such that if all the modes in the comprehension of B are true of x, then all the modes in the comprehension of A are true of x.

Auroux is correct to argue in favor of the extensional interpretation and to point out its implications for the truth-conditions of categorical propositions.[16] Unfortunately, the truth-conditions of negative categoricals do not speak directly to the nature of term negation, the topic of this paper. The *Logic* is silent on the relation between negation as a sentence operator and negation as a term marker. For example, it does not discuss traditional syllogistic rules like obversion or *de omni et nullo*, which relate positive and negative categorical propositions to term negation.[17] I shall argue in §5 that the *Logic*'s term negation is privative negation and that its semantics is perfectly consistent with the referential truth-conditions of the negative categoricals.

Auroux has a second, more algebraic criticism of the *Logic*'s term negation. As he puts it, intensions and extensions of ideas are not "symmetric," and hence the structure of ideas is not "dual" to that of extensions. Auroux is in part correct. The issue may be simplified if the four structures are carefully compared. These structures are those of ideas, comprehensions, significance ranges, and extensions.

Let $\mathrm{Comp}(A)$ be the comprehension of an idea A, a set of modes. Let $\mathrm{Sig}(A)$ be its *significance range*, the set of things that A signifies or, equivalently, the set of all things that satisfy all the modes in its comprehension. Let us adopt the referential interpretation, and let $\mathrm{Ext}(A)$ be the extension of A, i.e., the set of ideas B such that the modes in $\mathrm{Comp}(B)$ are true of only the objects in $\mathrm{Sig}(A)$.[18] There are then four partially ordered structures: (1) the set of ideas ordered by idea-containment, (2) the set of comprehension-sets ordered by set inclusion, (3) the set of significance ranges ordered by set inclusion, and (4) the set of extensions ordered by set inclusion. We may now clarify the degree to which each can be mapped into the other.

First, (1) is isomorphic to (2) because, as explained above, set inclusion on comprehension sets induces a corresponding containment ordering on ideas.

Second, there is an into homomorphism from (2) to (3). More precisely, there is an antitonic mapping from (2) to (3). Assume $\mathrm{Comp}(A) \subseteq \mathrm{Comp}(B)$. Hence, for any x,

[15] Auroux (1993), 135.

[16] See the discussion in Martin (2011).

[17] For example, the consequence from *no S is P* to *every S is non-P,* and conversely.

[18] An idea's significance range is what in modern logic is called its extension, which is also the usage of Leibniz and Frege. For discussions of the historical origin of Arnauld and Nicole's earlier usage see Cronin (1966) and Martin (2012).

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if all the modes in the Comp(B) are true of x, all the modes in Comp(A) are true of x, i.e., Sig(B) $\subseteq$ Sig(A). Moreover, the mapping is into because it is possible that two distinct ideas, which are distinct because they have different comprehensions, have the same significance range. This happens if as a matter of fact modes in their comprehensions are contingently true of the objects they signify. C*ows in the pasture* and *Jack's favorite animals* have distinct comprehensions, but they may signify the same things. Indeed, it is to be expected of a referential theory that two distinct ideas may pick out the same objects contingently.

Lastly, (3) is isomorphic to (4). First, there is an order-preserving mapping from significance ranges to extensions. Let Sig(A) $\subseteq$ Sig(B) and let C be in Ext(A). Then, C signifies only objects signified by A. But since Sig(A) $\subseteq$ Sig(B), C signifies only objects signified by B, and hence C is in Ext(B). Hence, Ext(A) $\subseteq$ Ext(B). Moreover, the mapping is 1-1. This holds first because a significance range uniquely determines an extension. It does so because Sig(A) uniquely determines $\{B|\text{Sig}(B) \subseteq \text{Sig}(A)\}$ and $\{B|\text{Sig}(B) \subseteq \text{Sig}(A)\}=\text{Ext}(A)$. Conversely, Ext($A$) uniquely determines Sig(A) because $\text{Sig}(A) = \cup\,\{\text{Sig}(B)|B \in \text{Ext}(A)\}$.

The duality issue raised by Auroux and its relation to term negation may now be settled. The truth-conditions for a universal negative like *no student is asleep* are Ext(*student*) $\cap$ Ext(*asleep*) $= \emptyset$, which may be expressed in terms of order as Ext(*student*) $\subseteq$– Ext(*asleep*). But as explained above and as Auroux suggests, this ordering on extensions (and hence significance-ranges) does not entail Comp(*student*)$\subseteq$– Comp(*asleep*), which is the corresponding intensional ordering on comprehensions (and hence ideas). Although Auroux is correct in this observation, it does not support his claim that duality fails.

In algebra today duality is defined for partial orderings. One partially ordered structure is said to be *dual* to another if there is an into homomorphism from the first to the second. Clearly there is an antitonic homomorphism from (1) to (4) because the various mediating mappings obtain: there is an isomorphism between (1) and (2), an antitonic into homomorphism from (2) to (3), and an isomorphism from (3) to (4). It follows that ideas are dual in the standard sense to extensions.

On the other hand, although duality holds for the partial orderings, a more significant mapping fails. As I shall argue in §5, term negation in the *Logic* has properties quite different from Boolean complementation, which is the "negation" operation in structures refered to in (2) and (3) above. Thus, when structured by restriction, abstraction and term negation, there is no homomorphic mapping from ideas to the power set algebras referred to in (2) and (3). It is perhaps this stronger mapping that Auroux and Dominicy had in mind in raising the question of duality.

In summary, in his early work Auroux proposes a reconstruction of the *Logic*'s structure of ideas as a kind of imperfect Boolean algebra on which are defined containment, abstraction and restriction. The structure proposed also has a "negation" operation that satisfies some of the properties of Boolean complementation. In his later work Auroux argues that the *Logic*'s referential truth conditions for negative categoricals are inconsistent with ideas' being dual to extensions. He is correct that the truth-conditions for universal affirmatives support the referential reading. The issue of duality, however, is irrelevant. As duality is usually defined, ideas ordered by containment are dual to extensions ordered by set inclusion. This duality, moreover, is consistent with the referential reading. It will also become clear below that both duality and the referential reading are also consistent with the *Logic*'s term negation, which is a variety of privative negation. Let us set aside for the moment the exact extend to which the less-than-Boolean negation captured by Auroux's axiom system fits the *Logic*.

§3. Dominicy's critique of Port Royal negation. Unlike Auroux, who bases his criticism on general features of the *Logic*'s metatheory, Dominicy argues on the basis of texts. It is true that texts by Arnauld that mention term negation are rare, but Dominicy focuses on three in which it is prominent. One is from the *Logic* itself, a second from the later work of Arnauld, and a third from Descartes' *Meditations,* which influenced the *Logic*. Dominicy argues that in these texts the authors equivocate on negation, unintentionally confusing two different senses. In this section we will review the texts cited by Dominicy and his interpretation of them.

Text I

The first text cited is from the *Logic*:

> Finally, we should note that it is not always necessary for the two differences dividing a genus both to be positive, but it is enough if one is, just as two people are distinguished from each other if one has a burden and the other lacks one, although the one who does not have the burden has nothing the other one does not have. This is how humans are distinguished from beasts in general, since a human is an animal with a mind, *animal mente praeditum*, and a beast is a pure animal, *animal merum*. For the idea of a beast in general includes nothing [63] positive which is not in a human, but is joined only to the negation of what is in a human, namely the mind. So the entire difference between the idea of an animal and the idea of a beast is that the comprehension of the idea of an animal neither includes nor excludes thought—the idea even includes it in its extension because it applies to an animal that thinks—whereas the idea of a beast excludes thought from its comprehension and thus cannot apply to any animal that thinks.[19]

Dominicy interprets (1.1), which occurs in the text above, as (1.1*),[20] using the symbol $<$ which represents both idea and comprehension containment. [21]

(1.1) a human is an animal with a mind, *animal mente praeditum*, and a beast is a pure animal, *animal merum.*

(1.1*) $\sim(\mathit{thought}<\mathit{animal})$

[19] Enfin, il faut remarquer qu'il n'est pas toujours nécessaire que les deux différences qui partagent un genre soient toutes deux positives; mais que c'est assez qu'il y en ait une, comme deux hommes sont distingués l'un de l'autre, si l'un a une charge que l'autre n'a pas, quoique celui qui n'a pas de charge n'ait rien que l'autre n'ait. C'est ainsi que l'homme est un animal qui a un esprit, *animal mente praeditum*, & que la bête est un animal pur, *animal merum*. Car l'idée de la bête en général n'enferme rien de positif qui ne soit dans l'homme; mais on y joint seulement la négation de ce qui est en l'homme, savoir l'esprit. De sorte que toute la différence qu'il y a entre l'idée de l'animal & celle de bête, est que l'idée d'animal n'enferme pas la pensée dans sa compréhension, mais ne l'exclut pas aussi, & l'enferme même dans son étendue, parce qu'elle convient à un animal qui pense; au-lieu que l'idée de bête l'exclut dans sa compréhension, & ainsi ne peut convenir à l'animal qui pense. *LAP I:*7, *KM* V 148–49, *B* 42–43.

[20] Dominicy (1984), pp. 44–45.

[21] Several pages (p. 43) earlier Dominicy suggests that the *Logic*'s structure of ideas may be reconstructed as an algebra of "Carnapian properties" (functions from possible worlds to extensions within those worlds). There he there formally defines $<$ and a negation operation as Boolean operations on these properties. In his symbolization of (1.1*), (1.2*), and below, however, he uses the symbol $<$ and the negative affix *non* more loosely. He does not define this informal use, and its meaning has to be abstracted from his comments on the examples.

But he reads (1.2), which is also from the text, as implying (1.2*):

(1.2) the idea of a beast . . . is joined only to the negation of what is in a human, namely the mind

(1.2*) *nonthought <beast*

He interprets the negation in (1.1) as what he calls *négation faible,* and that of (1.2) as *négation forte.* He grants that what is being said in both cases is true, namely that it is not the case that the idea *thought* is part of the definition of the idea *animal,* and that the idea *nonthought* is part of the definition of beast. He sees a difficulty, however, because he holds that the *Logic*'s authors mistakenly believe that the uses of negation in both cases are the same. He concludes that they equivocate on the meaning of *not*. Unfortunately, Dominicy does not explain exactly how the two negations are supposed to differ.

Some differences are obvious. Clearly they differ syntactically. Syntactically what he calls *négation faible* is negation operator on sentences in the metalanguage. It is applied to a metalinguistic assertion of idea inclusion, which is expressible in the syllogistic by an object language proposition in the form of a universal negative:

(1.1**) *no beast thinks*

Négation forte, on the other hand, seems to be a term negation. It attaches to a noun to yield a noun phrase.

Because Dominicy also holds that there is an equivocation on negation, he also believes that the two negations differ semantically. We shall return to the question of how they differ semantically shortly.

Text II

Dominicy goes on to cite a second text from Arnauld's later work, which he says, "falls into the same trap." It too, he holds, equivocates on negation:

> . . . the two members are such that the more noble contains all that is in the less noble, and such that they differ only in that the more noble has something that the other does not. It is in this way that man and beast differ. For a beast is not purely only *animal*, which man is also. But man in addition has a rational soul which a beast does not. This is why I can say when comparing a beast to a man that a beast eats, nourishes itself, walks and acts by the impression of its senses, but that a man acts by reason. And it would be an absurdity to reply to this comparison by objecting as does M. Mallet that one does not say of a beast anything [positive, nonprivative] that does not [also] apply to a man, which is true. But it suffices in these sorts of comparisons, [to point out] that what one says of the more noble member [*qua* what makes it more noble] is not found in the other.[22]

[22] Cited by Dominicy (1984), p. 45. The complete passage reads:
Mais est-ce que M. Mallet ne fait pas qu'il se fait beaucoup de comparaisons, *inter excedens & excessum*, comme on parle dans l'École; c'est-à-dire, dont les deux membres sont tels, que le plus noble comprend tout ce qui est dans le moins noble, de sorte qu'ils ne différent qu'en ce que le plus noble a quelque chose que n'a pas l'autre. C'est en cette manière que l'homme & la bête sont différents. Car la bête n'est purement qu'*animal*, ce qu'est aussi l'homme. Mais l'homme de plus a une âme raisonnable, ce que n'a pas la bête. C'est pourquoi je puis bien dire, en comparant la bête avec l'homme, qu'une bête mange, se nourrit, marche & agit par l'impression

Dominicy understands the passage as saying that *thought* is the differentia of *man* and that men are more noble than animals, and symbolizes the text in (2.1) as implying (2.1*), which he says employs *négation faible*:

(2.1) it suffices in these sorts of comparisons, [to point out] that what one says of the more noble member is not found in the other.

(2.1*) ~(*thought* < *animal*).

The object language formulation of (2.1*) would be the universal negative, which if true would be necessary:

(2.1**) *no beast thinks*.

He represents the text (2.2) as (2.2*), which he says employs *négation forte*:

(2.2) The two members are such that the more noble contains all that is in the less noble, and such that they differ only in that the more noble has something that the other does not. It is in this way that man and beast differ. For a beast is not purely only *animal*, which man is also. But man in addition has a rational soul which a beast does not.

(2.2*) *nonthought* < *beast*.

Text III

The third text he discusses traces the supposed equivocation to Descartes. In *Meditation* IV Descartes explains that the ideas *thought* and *body* are distinct:

> And surely the idea I have of the human soul is such that it is a thing that thinks, and is not extended in length and breadth and is such that it participates in nothing that appertains to body; it is incomparably more distinct than the idea of any corporeal thing.[23]

Dominicy summarizes the view as follows:

(3.1) the comprehension of the idea *thinking substance* is not included in that of the idea *body* (defined as *extended substance*).

According to Dominicy, the statement employs a sentential negation. Using the notation for what he calls *négation faible*, he says it should be represented by:

(3.1*) ∼(*thought* < *extension*),

which would be expressed in the syllogistic as

(3.1**) *no extended thing thinks*.

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de ses sens, mais que l'homme agit par raison. Et ce seroit une absurdité de trouver à redire à cette comparaison, en objectant, comme fait M. Mallet, qu'on ne dit rien [positif] de la bête qui ne convienne à l'homme, ce qui est vrai. Mais il suffit, dans ces sortes de comparaisons, que ce qu'on dit du membre le plus noble [qua noble] ne se trouve pas dans l'autre.
Arnauld (1776), Livre 5, chap. 1, pp. 351–352.

23 Et certes l'idée que j'ay de l'esprit humain en tant qu'il est une chose qui pense, & non estendue en longueur & profondeur, & qui ne participe à rien de ce qui appartient au corps, est incomparablement plus distincte que l'idée d'aucune chose corporelle. Adam (1897–1909) IXa 42, 60–61.

He contrasts this usage with that in *Meditation* VI where Descartes expresses a similar view using a variety of term negation.[24] There Descartes says:

(3.2) *a body is an extended thing that does not think* (*qui ne pense point*):

which Dominicy symbolizes as:

(3.2*) *nonthinking* < *extension*

He identifies the negation of (3.1*) as *négation faible* and that of (3.2*) as *négation forte*. Because they are distinct but not acknowledged to be so, Descartes too is judged to make the same conflation Dominicy finds in the *Logic*.

3.1. The meaning of négation faible and négation forte. Dominicy contends that Arnauld and Descartes unwittingly conflate two senses of negation, *négation faible* and *négation forte*. Although in an earlier discussion of an algebra of Carnapian "concepts" (functions from possible worlds to extensions within those worlds) Dominicy defines a Boolean complementation operation on concepts, in this later informal discussion of ambiguity, he does not define what he means by *négation faible* and *négation forte*.[25] Clearly he thinks the two are distinct syntactically. The former is a sentence negation, i.e., an operation on sentences that yields sentences. The latter is a term negation; it is defined on terms to yield terms. He also believes they differ semantically because otherwise there could be no equivocation. Although he does not define how they differ, he does provide some guidance.

The fact that the sentences identified as *négation faible* are straightforward sentence negations, which would be expressed in the object language of the syllogistic by universal negatives, suggests that *négation faible* is a standard bivalent classical negation. On the other hand, the fact that *négation forte* is semantically different from *négation faible,* suggests that *négation forte* is not classical. At a minimum, it suggests that it is non-Boolean in the sense that the significance range of *non-A* is, in general, a subset of the Boolean complement of the significance range of *A* relative to the domain of existing things.

Dominicy's terminology, moreover, is suggestive of several well-known narrow scope nonclassical term negations. By reference to these it may be possible to make more precise the distinction he has in mind. In particular, the terminology *négation faible* and *négation forte* suggests the distinction between weak and strong negation as found in the 3-valued logic of S. C. Kleene. Unfortunately, despite similarities this cannot be the distinction Dominicy intends. Although the weak and strong connectives for conjunction and disjunction are semantically different because they have different truth-tables, weak and strong negations do not differ semantically. Their truth-tables are identical. There are two other well-known narrow scope non-Boolean negations that do differ semantically from classical sentence negation.

One is Bochvar's distinction between external and internal connectives. Both types of connectives are 3-valued. External negation is a wide scope sentence operator that is classical in the sense that formulas written solely in terms of external connective have a classical validity relation. A term operator defined in terms of internal negation, on the other hand, would have narrow scope and would be nonclassical in the sense that

[24] Adam (1897–1909) IXa, 62, 96–97.

[25] For more detail on the non-Boolean negations discussed below—Kleene's strong and weak connectives, Bochvar's internal and external connectives, and Russell's negations in primary and secondary occurrence—see Martin (1987).

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the entailment relation of sentences written in internal connectives alone is nonclassical. Unfortunately, although external and internal negation clearly differ semantically, it is unlikely that this is the distinction Dominicy has in mind because their semantics is 3-valued. Nowhere in the *Logic* is the possibility of a nonbivalent semantics broached nor is it entertained by logicians of the period.[26] Nor does Dominicy himself mention the possibility that a sentence could be other than true or false.

Another distinction the discussion suggests is that of Russell between negation in secondary and primary occurrence. This is a distinction drawn in *Principia* between two ways to negate a subject–predicate sentence in which the subject term is a definite description. The subject–predicate sentence $Q(1xP(x))$, which is read "the one and only P is Q," is defined as $\exists x(P(x)\&\forall y(P(y) \rightarrow y = x)\&Q(x))$. Here the subject term $1xP(x)$ is a nonreferring part that is eliminated when the sentence is converted by the definition into primitive notation. The distinction between wide scope sentence negation and narrow scope term negation is defined as follows:

Sentence negation in "secondary occurrence:"

$$\neg\ Q(1xP(x)) =_{def} \neg\ \exists x(P(x)\&\forall y(P(y) \rightarrow y = x)\&Q(x)).$$

Term negation in "*primary occurrence*:"

$$(\sim Q)(1xP(x)) =_{def} \exists x(P(x)\&\forall y(P(y) \rightarrow y = x)\&\neg\ Q(x)).$$

By definition, negation in secondary occurrence is a wide scope sentence negation and is classically bivalent. Negation in primary occurrence, on the other hand, is by definition a narrow scope term negation, and is non-Boolean in the sense that $Q(1xP(x))$ and $(\sim Q)(1xP(x))$ are contraries but not contradictories—they cannot both be true, but they can both be false. Unfortunately, although Dominicy may have something like this distinction in mind, it is too narrow to fit his examples because it is limited to formulas containing definite descriptions. In sum, although his terminology is suggestive of familiar narrow scope nonclassical term negations, in the end we can say little more about the distinction other than that by *négation faible* he understands a wide scope classical sentence negation and by *négation forte* a narrow scope term negation that signifies, in general, a subset of the values signified by the term's Boolean complement.

Dominicy's contention is that in the texts discussed Arnauld and Descartes conflate these two negations. I shall argue in the next section, however, that, on the contrary, the authors in fact had in mind a non-Boolean negation, and that they did not confuse it with classical sentence negation. Rather, they intentionally used both in the same texts to say different things.

§4. Privative negation. Aristotle introduces privative negation in *Categories* X. Throughout his work he makes comments on its properties that entered into subsequent lore as part of its definition. In his usage privative negation has a clear syntax. It is a negative affix ("marker") to a categorical term (common noun or adjective). Alternatively, it can be a lexicalized synonym of a privatively marked term. His examples include *nōda* (*toothless*), which is *odous* (*teeth*) marked by *nē* (*without*),[27] and lexicalized *typhlos* (*blind*)

[26] A contemporary exception is Toletus, who on one occasion seems to argue that privative negation is this kind of three-valued operation with existential presupposition. See Toletus (1596), *Peri hermeneias*, Liber I, Caput VII, pp. 125–126.

[27] *Categories* 112a30

and *phalakros* (*bald*).[28] Perhaps the most familiar Greek privative marker is the prefix α, the so-called *alpha-privatum* of classical philology. In Latin the same roles are played by *non* and *sub*. Semantically a privative term stands for subjects that lack a specific property, one which Aristotle says the subject "normally possesses." A privative term is true of a subject that suffers from

> a lack in a being or class of beings which normally possesses the property; for example, a blind man and a mole.[29]

Elsewhere he says that the subject possesses the property "naturally,"[30] and that its lack is an "imperfection."[31] In discussing division in the *Topics,* he says that one of the ways to define species is to divide a genus into two by choosing as the differentia of the second the privative negation of the first's.[32]

As privative negation was developed, it acquired two defining properties relevant to this discussion. First, it was viewed as standing for an operation on properties that when applied to the differentia of a species within a genus yielded the differentia of a second species within that genus. Second, the new difference was understood to stand for the fact that the second species lacked the property definitive of the first, and that therefore the second species was less perfect (literally less complete) and less "noble" than the first.

The role of privative negation in division and in determining order was elaborated by the Neoplatonic tradition. In doing so they developed quite an elaborate account of its logic, one that was adapted by medieval logic and which, as we shall see, reappears in the texts of Arnauld and Nicole. In this section we shall go into some of its details in order to show that, contrary to the claims by Auroux and Dominicy, its use in the *Logic,* far from being confused, and was part of a rich logical tradition, albeit one that has little to do with Boolean algebra.

Proclus Diadocus (411–485), who commented on the *Organon*, is the clearest. He distinguishes not one, but two non-Boolean negations to describe the hierarchy of Being. The first is privative negation, which he understood as an operation that converts a term that stands for one degree of Being into one that stands for a lesser degree of Being. His second "negation" is a case of what classical philologists call the *alpha-intensivum*. He sometimes calls it *hypernegation*.[33] It converts a term that stands for Being at one degree into one that stands for Being at a higher degree:

> . . . since not-Being has a number of senses, one superior to Being, another which is of the same rank as Being, and yet another which is the privation of Being, it is clear, surely, that we can postulate also three types of negation, one superior to assertion, another inferior to assertion, and another in some way equally balanced by assertion.[34]

According to the Neoplatonists, degrees of Being are the same as degrees of perfection, causal power, goodness, beauty, etc. The view that a privative negation stands for Being that is less "perfect" or "complete," or one that is "a part of" the original, was taken over by

28 *Categories* 11b15, *Metaphysics* 1022b22.
29 1011b23.
30 11b15.
31 1022b29.
32 109a34.
33 Proclus (1864), 1172:35.
34 Proclus (1864) 1072:28–1073:8, Proclus, p. 426.

the medieval logical tradition and will figure in the interpretation of Arnauld and Descartes in §5.[35]

In modern linguistics privative negation is classed as a part of family of linguistic devices used to express order, and that, as part of such a family, it has distinct logical properties.[36] As a general rule in natural languages, privative negations and intensifiers are accompanied by a characteristic collection of other expressions that includes a mass noun, a comparative adjective, and an ordered group of gradable adjectives. The mass noun stands for objects that possess a characteristic mass quantity in varying degrees. The comparative adjective names a relation that orders the objects possessing the mass. The associated gradable adjectives stand for sets that as a group partition the relation's ordered field into groups that possess the mass in progressively greater quantities. Formally, each adjective stands for a set in a partition of the comparative field. The sets in the partition are ordered so that the individuals in each set possess more of the mass quantity than any individual in any set prior in the order. For example, the mass noun *happiness* stands for the mass property happiness, and the comparative *is happier than* stands for the relation that orders objects according to how much happiness they possess. The gradable adjectives in the ordered series *miserable, sad, contented, happy, estatic* stand for ranked sets of objects in the field of *is happier than* in such a way that the objects in each set are at least as happy as any member of any set prior in the ordering.

Within this framework privative negation and intensifiers are understood to be operators on the gradable adjectives. A privative operation assigns to a set taken as argument a set prior to it in the ordering. An intensifier operation maps a set to a set higher in the order. Normally, when the set of predicates is finite and small, and the ordering relation is discrete, the privative operation picks out the set next lower in the order, and the intensifier the set next higher. Privative operators in English include *non*, *sub*, and *dis* and well as some uses of *not*, as in *discontent*, and *not* in the sentence *I am not sad; I'm miserable*.[37] Examples of intensifiers are *super* and *hyper*.[38]

[35] En effet, dans les réalités, les négations, à mon avis, présentent trois types particuliers; et tantôt, étant plus apparentée au principe que les affirmations, elles sont génératrices et perfectives de la génération des affirmations; tantôt, elles sont placées sur le même rang que les affirmations, et l'affirmation n'est en rien plus respectable que la négation; tantôt enfin elles ont reçu une nature inférieure aux affirmations, et elles ne sont rien d'autre que des privations d'affirmations. *Platonic Theology* II:5, in Proclus (1968–1997), 38:18–25.

[36] On the linguistic and logical properties of privative negation and associated order expressions see Horn (1989), Martin (2001), Martin (2002).

[37] The prefix *un* has different logical properties. Unlike privative negation, which essentially converts an adjective into a term that is a synonym for its next lower neighbor in the adjective group, *un* stands for a kind of pivot operation. It assumes that the adjective ordering is countable and that it has a 0 midpoint. It assigns to a set (a gradable predicate extension) at rank n above the midpoint, in the "positive direction," the set at rank $-n$ below the midpoint. There are various linguistic markers that determine which direction of the group is positive. Curiously, in natural languages pivot operators like *un* are defined only for the adjectives in the positive "half," and they cannot be double-negated. For example, while *happy* stands for a set above the midpoint, *unhappy,* which is a synonym of *sad*, stands for a set to the same degree below the midpoint. While *unhappy* is grammatical, both *un-unhappy* and *unsad* are ungrammatical.

[38] The Neoplatonist Pseudo-Dionysius the Areopagite used hypernegations like *hypergood* to fashion the "names" of God. See Martin (1995).

A first-order language may be extended to include these negations and their companion expressions as follows. Let M be a distinguished one-place *mass predicate*, R a distinguished two-place *comparative adjective predicate*, and $A_{-n}, \ldots, A_0, \ldots A_{+n.}$ a distinguished series of

Logically, privative and hypernegation have well-defined properties, which are quite different from classical Boolean negation. Let ~ be privative negation, ¬ hypernegation, $\leq$ the relation that orders the extensions of the gradable adjectives, and < the strict ordering defined in terms of $\leq$, i.e., $A<B$ iff ($A \leq B$ and $A \neq B$). A privative negation always stands for a set lower in the comparative ordering and a hypernegation for one higher: $\sim A<A<\neg A$. It follows that double negation fails. A privative double negation $\sim\sim A$, if it were grammatical in natural language, would be lower than both A and $\sim A$. A double hypernegation would be higher: $\sim\sim A<\sim A<A<\neg A<\neg\neg A$. Intensifiers are also not commutative. Rather, as Proclus points out, the order of Being at one level is replicated on a higher level by hypernegations, and on a lower level by privative negations: $\sim A<\sim B$iff $A<B$ iff $\neg A<\neg B$. While those who are happy are happier than those who are just contented, it does not follow that those who are discontented are happier than those who are not even happy, because some of those who are not happy may nevertheless be contented. As we shall see below, Arnauld and Descartes use privative negation in this sense to stand for species lower in an background order of "perfection."

Proclus also develops the role of privative negation in division. According to gradable adjective theory, an ordering on Being would follow directly from the fact that Being is a mass quantity. As such it would have degrees expressible by a comparative adjective and associated gradable adjectives. Proclus, however, defines this order indirectly by appeal to privative negation. He accepts Plotinus' teaching that Being emanates from the One in levels or "hypotheses," and he describes the process as conforming to a tree structure.[39]

The details are straightforward. The One occupies the root node of the emanation tree; from it and each subsequent node there branches a well-defined finite set t of immediate descendants. A node's immediate descendants possess less Being than their parent. The group of a node's immediate descendants is called a *taxon*, and its parent is called a *monad* relative to that taxon. He often describes a division as forming a triad of thesis, antithesis, synthesis (a view later appropriated by Hegel). For example, he divides the monad *Intelligible-Intellectuals* into the triad *hyperouranios* (*super-celestial*), *ouranios*

gradable adjectives (one-place predicates). Let the set of *gradable predicates* T be the set of gradable adjectives closed under the one-place operators ~and ¬. Let *un* be an one-place operator on gradable adjectives. A first-order structure is any $<D, \Im>$ such that

(1) D is a nonempty set and $\Im$ an interpretation function on predicates;

(2) $\Im(M) \subseteq D$;

(3) $\Im(R) = \leq \subseteq D^2$ is a connected preordering (reflexive and transitive);

(4) there is a distinguished partition $\{\Im(A_{-n}), \ldots, \Im(A_0), \ldots, \Im(A_{+n})\}$of $\Im(M)$ such that $y<x$ for any m such that $-n<m \leq +n$, for any $x \in \Im(A_m)$, and for any $y \in \Im(A_{m-1})$;

(5) $\Im(\sim T) < \Im(T)$ and $\Im(T) < \Im(\neg T)$; and

(6) for any m such that $0 < m$, $\Im(un\text{-}A_m) = \Im(A_{-m})$.

[39] Formally, a (*rooted*) *tree Tr* is defined as a partially ordered structure $< T,\leq,0>$ such that for any $t \in T$, $\{x|x \leq t\}$ is well-ordered (i.e., $\{x|x \leq t\}$ is totally ordered by $\leq$, and every subset of $\{x|x \leq t\}$ has a $\leq$-first element) and 0 is the unique $\leq$-least element in T (the *root* of *Tr*). An element t of T is *a leaf node* of *Tr* iff for any element $t\prime$ of T, if $t \leq t\prime$, then $t = t\prime$. For any nonleaf node of t of *Tr*, *the set of immediate descendants* (or *children*) of t, briefly $c(t)$, is $\{t\prime \in T|t<t\prime \;\&\sim \exists t\prime\prime \in T(t<t\prime\prime<t\prime)\}$. c is *well-defined* iff for any $t,t\prime \in T$ such that $t<t\prime$, $\exists t\prime\prime \in T(t<t\prime\prime \leq t\prime)$ and $\sim\exists t\prime\prime\prime \in T(t<t\prime\prime\prime<t\prime\prime)$. For any $t\prime$ in $c(t)$, *the parent of* $t\prime$, briefly $p(t\prime)$, is t. The set of *$\leq$-descendants of* $t \in T$, briefly d(t), is $\{t\prime \in T|t \leq t\prime\}$. In Proclus' terminology $c(t)$ is a *taxon* and t is its *monad*. *Tr* is *finite* if T is finite; *Tr* is *finitely branching* if for every t, $c(t)$ is finite. By definition a rooted tree is a meet semilattice, and its root is a minimal element.

(*celestial*), and *hypouranios* (*sub-celestial*).[40] Because for any node the set of immediate descendants is well-defined, the branchings of the tree are denumerable. It follows that the notion *nodes at a given level* is also well-defined, and that the number of levels is denumerable. The set of nodes at a given level constitutes a *hypothesis*.

What is important for our purposes is the relation between privative negation and order: the emanation tree together with privative negation with taxa determine a linear order of perfection over all the nodes of the tree. Proclus' construction is easily formulated in modern terms. There is a finitedly branching tree in which for every node n, the set t of its immediate descendants is well-defined; t is called a *taxon* relative to n, the *monad* of t. Privative negation is defined within a taxon. For any taxon t with monad m, there is an one-place operation $\sim_t$, *privative negation*, with these properties:

1. for a unique $n \in t$, the domain of $\sim_t$ is $(t \cup \{m\})$-$\{n\}$ (here n is called the *least element* of t);
2. the range of $\sim_t$ is t;
3. $\sim_t$ is 1-1; and
4. there are no "loops" in $\sim_t$, i.e., there is no series $n_1, \ldots, n_n$ such that for all i and j, $n_{i+1} = \sim_t n_i$ and $n_1 = \sim_t n_n$.

Since every node of the tree is in a taxon headed by a monad, it follows that a privative negation operation relative to a taxon is well-defined for every node in the tree. It is then possible to define a privative negation operation ~ over all the nodes of the tree; it is the union of all $\sim_t$ such that t is a taxon. A linear order (connected partial order) on all the nodes of the tree, the so-called Great Chain of Being, is then defined as the transitive closer of $\leq$ such that $x \leq y$ iff, $x = y$ or $x = \sim y$.[41] Hypernegation $\neg$ is the inverse operation of ~ and is defined for all nodes in the tree except the One, the tree's root node.

Medieval logicians took over Neoplatonic privative negation adapting its metaphysics to one that was more Aristotelian. They reverted to the more Aristotelian view that division was of genera into species, but retained the view shared by the Neoplatonists that division conformed to a tree structure—the so-called Tree of Porphyry. As Porphyry's commentarors describes it, the tree is finite and usually binary branching. What is important is that they also retained a version of Proclus' view that the descendants of a node, now understood to be species under a genus, are ordered by privative negation. When a genus is described as branching into two species, the difference definitive of the second species is often said to be the privative negation of the difference of the first, and that for this reason the second is regarded as less perfect or noble than the first. As we shall see, this is essentially the notion of privative negation used by Descartes and Arnauld. The accounts of William of Ockham (ca.1285-1347/49) and John Buridan (1295/1300-ca.1358) are representative.

Ockham in the *Summa Logica* explains privative negation as follows:[42]

> . . . affirmative propositions that contain privative terms not equivalent to infinite terms have more than two exponents. Hence, the proposition

[40] Proclus (1968–1997), 18:12–16.

[41] It is curious that the conversion into a line of the tree together with its ordered taxa is exactly the conversion that is used to determine the line of succession for the British crown from its family tree combined with each generation's age order.

[42] See Martin (2003).

> ‘He is blind’ has these exponents: ‘He is something’, ‘By nature he should have sight’, ‘He will never be able to see naturally’. But it is not possible to give firm rules for such propositions, for because of the variety of such terms the propositions in which they occur have to be expounded in different ways. Hence, ‘Socrates is blind’ has the exponents that have been mentioned. But the proposition ‘Socrates is foolish’ has these exponents: ‘Socrates is something’ and ‘Socrates does not have the wisdom which he ought to have’. Still, this is consistent with its being the case that he is able to have wisdom... even though a privative term occurs in each.[43]

Infinite negation is the technical term for what I have been calling Boolean term negation. It transforms a term into one that stands for the term’s complement within the domain of existing things. In the text Ockham says that when a privative term is not equivalent to infinite negation, its predication of a subject should be “expounded to” (i.e., understood in mental language as) a conjunction of propositions. One proposition says that the subject falls in a genus, and the second that it fails to have a property natural to that genus. His example is *Socrates is foolish*, which contains the lexicalized privative *foolish*. It asserts first *Socrates is an animal*, and second *Socrates is not wise*, which employs a privative negation. Thus, privative negation is used to distinguish a part of a genus that lacks a property humans would have naturally.

John Buridan makes clear the role of privative negation in the division as follows:

> For if a term that is dividing and a finite term that is divided are related to each other by univocal predication, then one of them will be a genus and the other its species or difference, or one will be a species and the other its individual. The infinite term, however, will be taken for the other single species or the other several species, for the other difference or differences, or for the other individual or individuals.
>
> This happens sometimes because some species or difference does not have a positive name imposed on it, as when we say that of sounds some are utterances and others are nonutterances. And that a name is not imposed sometimes occurs on account of our not knowing the species or difference. It is because of this mode of division that sometimes a species is defined by means of its genus and the negations of another species, or several other species, or their differences, as when Porphyry says that an accident is a predicable that is neither a genus, nor a species, nor a difference, nor a property, or if we said that brute is a nonrational animal. Oftentimes in such divisions we use a privative in place of an infinite term as when we say, ‘of substances some are corporeal, others incorporeal’ and ‘of corporeal substances some are animate, others inanimate’.[44]

Buridan says that when a division is made by applying the Latin negative *non* to a species term or its difference, the negation should not be understood as referring to an infinite (i.e., Boolean) negation, but rather as transforming it into a term that stands for its relative complement within its genus. In other words, it is a privative negation. He goes on to say

43 *Summa Logicae,* Part II, 12–13, William_of_Ockham (1980 [ca. 1323]), pp. 119–122, See also Part I, 36, William_of_Ockham (1974 [ca. 1323]), p. 117.

44 *Summulae* 8.1.8, Buridan (2001), p. 628.

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that we make use of an explicit privative negation in this way when we do not already have a lexicalized privative term that names the species or difference. He says that this is the way in which *nonrational* should be understood. It distinguishes the species *brute* from *man* within the genus *animal*. He remarks that when the language contains a privative term marked by an affix, the term names the species directly. His examples are *incorporeal* within the genus *substance,* and *inanimate* within the genus *corporeal substance*. He is also clear elsewhere that this division is such that the one species is more "perfect" or "noble" than the other: "we say that [a man] is a more perfect, or more noble, animal than a horse."[45]

In sum, the medieval notion of privative negation is of a non-Boolean negation that has as one of its uses the division of genera into species as represented in the Tree of Porphyry. Negation is applied to the differentia of one species to form the differentia of a second. The difference formed in this way indicates that the objects it signifies lack the differentia true of members of the first species, and that this is a property that members of the genus would naturally possess. Because its difference indicates a privation, the first species is more perfect and noble than the second.

This concept of negation became standard in medieval logic and continued to be part of logical lore until the time of Descartes and Arnauld. It is commonly mentioned in philosophical treatises of the day and was taught in the logic textbooks used at the Jesuit Colleges that Descartes and Arnauld attended.[46]

§5. Conclusions

5.1. Dominicy's critique. It can now be shown that in the texts cited above by Dominicy, Arnauld and Descartes are intentionally using negation in two senses. They are describing, as did Aristotle and Buridan in the texts cited, the division of a genus into two species in such a way that the second species is defined by the privative negation of the differentia of the first. They first make use of a universal negative to say that no member of the second species has the differentia of the first. They then go on to say that the ideas formed as the privative negation of the first is the differentia of the second or, to use the *Logic*'s technical terms, that this privative idea is part of the comprehension of the second. These are standard points, which were familiar to logicians of the day. They betray no confusion. We briefly consider each of the texts.

Text I

In Text I the *Logic* says,

> a human is an animal with a mind, *animal mente praeditum*, and a beast is a pure animal, *animal merum*,

Dominicy symbolizes this by means of idea inclusion and Boolean sentence negation as

(1.1*) $\sim(\textit{thought}<\textit{animal})$,

On the contrary, the authors should be understood as making the metalinguistic claim that the idea *thought* is not part of the comprehension of the idea *animal*. This claim makes use

45 *Summulae* 3.2.8, Buridan (2001), p. 161.

46 See Eustachio-De-S.-Paulo (1648), Logica Part I, De categoriis Tract II, Postpredicamenta paragraph XI, p. 87; Raconis (1651).
Physics, sec II, pp. 44–45; and Fonseca (1964 [1575]), Bk II, chap. 17, p. 128.

of sentential negation in the metalanguage. Arnauld could equally express the claim in the object language by the necessarily true negative universal *no animal thinks*. Arnauld goes on to say

> . . . the idea of a beast in general includes nothing positive which is not in a human, but is joined only to the negation of what is in a human, namely the mind. . . . [B]east excludes thought from its comprehension . . .

which Dominicy symbolizes as

(1.2*) *nonthought* <*beast.*

It is true, as Domincy says, that Arnauld is here using negation in a second sense, and he is also right to imply that it is non-Boolean, in a way reminiscent of Russell's secondary and primary distinction. He is incorrect, however, in his claim that Arnauld does not realize that he is using negation in a second sense, and that therefore he is unintentionally equivocating The second sense is privative negation. Arnauld uses it intentionally to say that the species *brute* is defined by a privative negation of the idea *thought*. As the earlier quotation from Buridan illustrates, this claim was not unusual—it had been regarded as a truism more or less since Aristotle.[47]

Text II

The readings of Text II, which Dominicy symbolized in (2.1*) and (2.2*), should be understood similarly. It is clear that the negation in the text symbolized by (2.2*) is intentionally privative negation because Arnauld invokes the traditional view that the privative species *brute* is less "noble" than *man*. He says,

> the more noble [*man*] contains all that is in the less noble [*brute*], and such that they differ only in that the more noble has something that the other does not.

Text III

Descartes makes similar points in Text III using negation in the same two senses. In the text that Dominicy symbolizes

(3.1*) $\sim$(*thought* < *extension*).

Descartes is saying that the idea of extension is not part of the comprehension of *man* which contains the idea *thought*. In the text Dominicy symbolizes

(3.2*) *nonthinking* < *extension*.

Descartes is saying that the privative negation of the idea *thought* is part of the comprehension of the idea of *matter,* which contains the idea *extension*.

5.2. Auroux's critique. It is clear from Dominicy's texts that Arnauld's term negation is privative negation in the context of genus-species division. This observation makes it possible to address Auroux's criticism. It has already been explained that his later critique is not well-founded. Contrary to his concerns, the structure of ideas as partially ordered by idea-inclusion is trivially dual to extensions as partially ordered by set inclusion. Because it is now clear that term negation is privative negation, idea negation cannot be understood as

[47] Due to similarities of language, Auroux argues in Auroux (1992) that Arnauld was familiar with a then contemporary translation of Porphyry's *Isagoge*.

an early version of Boolean complementation, which has quite different logical properties. Even if it is granted that meet and join operations may be abstracted from the *Logic*'s operations of abstraction and restriction, the lack of Boolean complementation vitiates any attempt to read the *Logic* as advancing an incipient form of Boolean algebra.

Nor is Auroux's earlier attempt to axiomatize the *Logic* as advancing a kind of proto-Boolean algebra well conceived. The *Logic* envisages many different types of ideas—simple, complex, adjectives, common nouns with comprehensions that pick out accidental groupings, false and impossible ideas, and genera and species with Aristotelian essential defintitions. As a group they exhibit very little overall structure beyond the ordering induced by the subset relation on their comprehension-sets. On the other hand, the special set of ideas that count as genera and species do have additional structure, and term negation contributes to it. They conform to a version of the Tree of Porphyry. The tree also conforms to Auroux's Axiom VI inasmuch as it arrays a genus with a simpler comprehension "above" its conceptually more complex species. Moving up the tree is a form of analysis and down the tree synthesis. In addition, privative negation orders the species beneath its genus according to perfection and nobility. This ordering when combined with the tree's parent to child tree ordering suffices, in principle, to define a Neoplatonic "linear" order of perfection and nobility over the entire set of genera and species. These are genuine notions of structure and are, in an abstract sense, algebraic. It should be cautioned, however, that these views were not explained mathematically, nor were they new in the *Logic*. Similar conceptions had been commonplace since the Middle Ages. These conclusions suggest a more general lesson. Rather than trying to find Boolean algebra the *Logic*, it is more profitable to read it as attempting to reconcile with earlier logical doctrine the new Cartesian theory of ideas in which ideas are causally disjoint from matter and defined by their comprehensions.

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