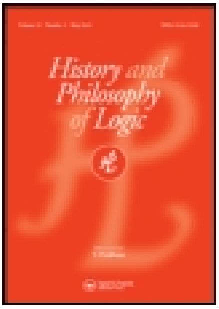

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Łukasiewicz's Many-valued Logic and Neoplatonic Scalar Modality

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This paper explores the modal interpretation of Łukasiewicz's n-truth-values, his conditional and the puzzles they generate by exploring his suggestion that by 'necessity' he intends the concept used in traditional philosophy. Scalar adjectives form families with nested extensions over the left and right fields of an ordering relation described by an associated comparative adjective. Associated is a privative negation that reverses the 'rank' of a predicate within the field. If the scalar semantics is interpreted over a totally ordered domain of cardinality n and metric θ, an *n-valued Łukasiewicz algebra* $<C, \wedge, \vee, \Rightarrow, \theta, e>$ is definable. Privation is analysed in terms of non-scalar adjectives. Any Boolean algebra of 2^n 'properties' determines an $n+1$ valued Łukasiewicz algebra. The Neoplatonic 'hierarchy of Being' is essentially the order presupposed by natural language modal scalars. Łukasiewicz's $\sim$ is privative negation, and $\rightarrow$ proves to stand for the extensional (antitonic) dual *if* … *then* for scalar adjectives, especially modals. Relations to product logics and frequency interpretations of probability are sketched.

1. The Problem

The many-valued logic of Łukasiewicz is doubly puzzling: modal and non-modal ideas seem to be conflated into the same 'truth-values', and the conditional is difficult to motivate in a principled manner.

Łukasiewicz conflates modal with non-modal ideas, for example, in explaining that three-valued logic's 0, $\frac{1}{2}$, and 1 simultaneously represent three seemingly non-equivalent families:

false, neither-true-nor-false, true
impossible, possible, necessary
determinately-true, neither-determinately-true-nor-determinately-false, determinately-false

Likewise he explains the values in the interval [0,1] as 'degrees of probability corresponding to various possibilities',[1]

In standard possible worlds semantics, however, truth and falsity are the prior terms used to define concepts of modality and determination. The usual understanding is that Łukasiewicz is not using terms in the standard way, and that by 'necessity' he means something other than 'truth in all (relevant) worlds'. What modal concepts he intends, however, remains an open question.

The mystery is deepened by the connectives. In applications of three-valued semantics there are two standard sets of truth-tables usually employed in large part because they can be motivated by clear principles. These are called by Kleene the *weak* and *strong* connectives respectively:

1 1920, 87; 1917, 113, 123, 124; 1922/23, 130.

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	$\sim$	$\wedge$	0	$\frac{1}{2}$	1	$\vee$	0	$\frac{1}{2}$	1	$\rightarrow$	0	$\frac{1}{2}$	1
0	1		0	$\frac{1}{2}$	0		0	$\frac{1}{2}$	1		1	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0		0	$\frac{1}{2}$	1		1	$\frac{1}{2}$	1		0	$\frac{1}{2}$	1

Kleene's weak matrix

	$\sim$	$\wedge$	0	$\frac{1}{2}$	1	$\vee$	0	$\frac{1}{2}$	1	$\rightarrow$	0	$\frac{1}{2}$	1
0	1		0	0	0		0	$\frac{1}{2}$	1		1	1	1
$\frac{1}{2}$	$\frac{1}{2}$		0	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	1		$\frac{1}{2}$	$\frac{1}{2}$	1
1	0		0	$\frac{1}{2}$	1		1	1	1		0	$\frac{1}{2}$	1

Kleene's strong matrix

The classical truth-values will be understood here to be 0 and 1. The three relevant principles are:

1. Weak classical conservativeness: if the values of the parts are all classical, then the value of the whole is determined by the classical truth-table.
2. Contagion of the non-classical: if any part is non-classical then so is the whole.
3. Strong classical conservativeness: if the value of one part is sufficient for determining the value of the whole in all classical cases (when the parts are all 0 or 1), then it remains so in the non-classical cases (when some parts receive values other that 0 or 1).

The first two rules determine the weak matrix, and are argued to be plausible in various applications in which $\frac{1}{2}$ is interpreted as *meaningless*, *non-sense*, *incoherent*, or *paradoxical*.[2]

If the third principle is also adopted so that it takes precedence over the second in cases of conflict, then the three principles together determine the strong matrix. This matrix is often applied when the third value represents non-determinateness or lack of knowledge. It is plausible to argue that if it is determined that P is false or if known that it is so, then it is determined or known on the basis of this information alone that $P \wedge Q$ is false, even if Q is neither determined nor known. Similarly, given this reading, it is plausible to hold that $P \vee Q$ is true if either part is, and that $P \rightarrow Q$ is true if P is false or Q is true.

Łukasiewicz's tables are very like those for the strong connectives.

	$\sim$	$\wedge$	0	$\frac{1}{2}$	1	$\vee$	0	$\frac{1}{2}$	1	$\rightarrow$	0	$\frac{1}{2}$	1
0	1		0	0	0		0	$\frac{1}{2}$	1		1	1	1
$\frac{1}{2}$	$\frac{1}{2}$		0	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	1		$\frac{1}{2}$	1	1
1	0		0	$\frac{1}{2}$	1		1	1	1		0	$\frac{1}{2}$	1

Lukasiewicz's L_3

They depart, however, in the evaluation $\frac{1}{2} \rightarrow \frac{1}{2} = 1$ which violates the second principle in a way not justified by the third.

2 They have also been so used by Halden, Åqvist, and others. The weak connectives are what Bochvar's calls the *internal* connectives and are used by him to interpret semantic paradoxes. The strong connectives have been applied by Kleene, Kripke and others. See any standard introduction for a review, e.g. Bolc and Borowik, 1992 and Rescher 1969.

What justifies this special assignment? Though Łukasiewicz makes general remarks about truth-values and the inferential properties desirable in a logical system, he says very little about the individual 'lines' of the truth-tables, and nothing specifically about his assignment of 1 to $\frac{1}{2}\rightarrow\frac{1}{2}$.

Most have assumed the motivation is grounded in the goal of a plausible logic. Łukasiewicz lays down the condition of adequacy for a logical system that the theorem set be the non-empty closure of a set of axioms under a set of rules that includes detachment and substitution.[3] The logics he investigates are all systems of this sort. They also conform to other logical standards he does not specifically mention but which are relevant. Relative to an axiom system, let A be the set of axioms and let σ range over substitution assignments. Let $X\vdash P$ (*deducibility*) iff there is a proof of P from X; $\vdash P$ (*theoremhood*) iff $A\vdash P$; and $X\models P$ (*provability*) iff ((for all $Q\in X$, $\vdash Q_\sigma$) only if $\vdash P_\sigma$)). Further, $\vdash$ is a *consequence relation* iff it is reflexive, transitive and monotonic, and is *classically constrained* iff ($X\vdash P$ only if the argument from X to P is classically deducible (and hence classically valid)).

In intuitionistic logic and Łukasiewicz's systems, but not in the weak and strong matrices, $\models$ is a consequence relation and $\vdash$ is classically constrained. The reason is that semantic entailment, which is co-extensional with $\vdash$, is defined in matrix logic as a relation transmitting 'designated values', and tautologies (co-extensional with theorems) as sentences that are always designated. In three-valued applications moreover usually only 1 is designated. There are then no tautologies among the weak and strong connectives because for any sentence there is the valuation in which it is non-designated because all its parts are non-classical. The resulting provability relation cannot be a consequence relation nor the deducibility relation classically constrained except in a trivial sense. Since Łukasiewicz was developing axiom systems designed to characterize sets of tautologies, these defects would be decisive. He avoids the problem by adopting as primitives the weak negation (same as the strong) and a version of the strong conditional in which the case $\frac{1}{2}\rightarrow\frac{1}{2}$ is changed to equal 1. A non-empty set of tautologies is then axiomatizable as a deductive system, deducibility is classically constrained, and provability a consequence relation.

The choice has two additional desirable consequences. It allows for strong conjunction and disjunction to be introduced by definition, and it corrects the implausible anomaly of both the weak and strong matrices that $P\rightarrow P$ is not a tautology.

When truth-values are interpreted modally, however, Łukasiewicz's conditional remains puzzling, even paradoxical. It may be granted that it is plausible to say $P\rightarrow P$ is a tautology and should therefore be 1, even when P is $\frac{1}{2}$. But the situation is less clear when the antecedent and consequent are distinct. Indeed, it is odd to say that it follows as a truth of semantics from the fact that the distinct sentences P and Q both have the value $\frac{1}{2}$, that $P\rightarrow Q$ is necessary, or even that it is true. Consider *Socrates is human* as a typical sentence true in the actual world but neither necessary nor impossible. By assumption both the sentences *Socrates is human* and $\sim$(*Socrates is human*) are possible. Then by the truth-table of the conditional *Socrates is human*$\rightarrow\sim$(*Socrates is human*) is actually true. By detachment, then, $\sim$(*Socrates is human*) is also true in the actual world, which is absurd. By *reductio* it follows that *Socrates is human* is not true. This argument

3 Łukesiewicz and Tarski, 1956.

may be generalized to show that there could be no non-necessary truths. (The same argument applies *mutatis mutandis* if the third value is explained in terms of 'determination'.)

The argument's fallacy lies no doubt in its implicit understanding of modalities in terms of possible worlds. It was already evident from Łukasiewicz's collapsing of modal with non-modal ideas that possible world analysis does not fit his account. But if he has some other notion in mind, what is it?

At one point Łukasiewicz explains the third truth-value in these words:

> If we make use of philosophical terminology which is not particularly clear, we could say that ontologically there corresponds to these sentences neither being nor non-being but possibility. Indeterminate sentences, which ontologically have possibility as their correlate, take the third truth-value.[4]

The philosophy being alluded to is not that of Carnap or Kripke; it is rather part of classical metaphysics, which Łukasiewicz knew well. This is the tradition of the Platonists and Neoplatonists, of St Augustine and Thomas Aquinas, with notions of modality rather foreign to today's metalogic. Modal ideas are not defined in terms of possible worlds. Rather, 'necessity' is collapsed with 'truth in the highest sense' and with 'Being itself', all three ideas being variable concepts admitting of 'degrees.' Their ranks are described by philosophers of the period using what linguists today call scalar adjectives. Scalar adjectives in turn are explainable in a semantic theory of variable degrees. These degrees fall in a total ordering (the 'chain of Being'), with algebraic operations in terms of which the connectives may be interpreted. These include a negation operation $\sim$ that is antitonic around a midpoint of the ordering, a minimum operation, and a maximum. Thus if P corresponds to at least n degrees of Being, $\sim P$ would represent $-n$ degrees. If P represented at least n degrees of Being and Q at least m degrees, then $P \wedge Q$ would stand for the minimum of $\{n,m\}$ and $P \vee Q$ the maximum. An operation for the conditional is also definable that insures a straightforward interpretation of the conditional: $P \rightarrow Q$ asserts that the consequent corresponds to at least as much Being as the antecedent. If the consequent does have at least as much Being as the antecedent, then the conditional is true without qualification; it represents the totality of Being. There are however degrees of failure. The lower the being represented by the antecedent relative to that represented by the consequent, the less true is the conditional and the lower the degree of Being it represents.

Given this reading of the truth-values, the motivation immediately follows for the many-valued truth-tables, including the values they assign to classical tautologies. Let there be three values $\{0,\frac{1}{2},1\}$. Since $\frac{1}{2}$ is the midpoint of the scale, if P is $\frac{1}{2}$, so is $\sim P$. Thus, if $\wedge$ represents the minimum operation and $\vee$ the maximum, $P \wedge \sim P$, $\sim(P \wedge \sim P)$, and $P \vee \sim P$ will all be $\frac{1}{2}$. However, $P \rightarrow P$ will be 1 because the antecedent possesses at least as much Being as the consequent.

The remainder of the paper sets out the semantics for scalar adjectives, especially modal ones, and employs essentially Łukasiewicz's readings of the truth-values to explain how they determine his truth-tables. Section 2 presents the theory and its motivation informally. Section 3 defines the ideas precisely and sets forth the key results.

4 1970b, 126.

2. Informal Analysis of Scalar Necessity

2.1. *Scalar adjectives*

Natural languages like English possess comparative adjectives (phrases) which function conjointly with associated families of monadic scalar adjectives. For example the comparative *happier than* works in combination with the adjective series *ecstatic*, *happy*, *content*, *so-so*. Semantically the comparative stands for an ordering relation over individuals. The union of the left and right fields of this relation constitutes 'a field of comparison' within which individuals may be understood as possessing, to various degrees, a common 'background property' like happiness. The associated scalar adjectives then are interpreted over this field, and take it as their 'range of significance' in the sense that they are meaningful for precisely the elements that fall in the ordering. Their extensions are subsets of the field; they are 'not true' of objects in the field outside their extensions, and are undefined for objects entirely outside the field.

The various scalars moreover are understood as threshold concepts. Something falls under a scalar predicate if it possess the background property to sufficient degree. The notion of degree is unpacked in its most fundamental sense not in terms of a metric or measure, but by means of the set inclusion ordering relation on predicate extensions. This ordering 'nests' the extensions of the predicates in a linear (total) ordering. Objects falling under P have the background property 'to a higher degree' than those under Q iff the extension of P is a subset of that of Q. For example, the set of objects for which it is true to say that they have sufficient happiness to fall under *ecstatic* is a subset of that for which it is true to say that they are happy enough to fall under *content*. That is, the extension of (*at least*) *ecstatic* is included in that of (*at least*) *content*. Adjectives in a family are conventionally listed in a series (row) with a predicate of narrower extension to the left of one with a broader extension. In natural language the background ordering itself is usually described by the comparative form of one of the monadic adjectives in the series; e.g. *happier than* describes an ordering over the significance range of the happiness predicates, which includes as a subset the extension of the non-comparative *happy*. Below are some examples from English of scalar families and their associated comparatives:

ecstatic, happy, content, so-so	*is happier than*
miserable, sad, down, so-so	*is sadder than*
boiling, hot, warm, tepid	*is hotter than*
freezing, cold, cool, tepid	*is colder than*

Horn has proposed what he calls 'test frames' as a criterion for identifying the inclusion of P_1 to the left of (less inclusive, higher in the order than) P_2 in a scalar family:[5]

X is not only P_2, but P_1.
X is at least P_2, if not (downright) P_1.

5 See Horn 1989 for a full discussion of the background syntax and distribution of scalar adjectives and their negations, as well as evidence for the scalar properties of modal predicates. Horn, however, argues for the superiority of pragmatic rather than matrix or model theoretic explanations of scalar inference.

X is P_2, {or/possibly} even P_1.
X is not even P_2, {let alone/much less} P_1.
X is P_2 , and is {in fact/indeed} P_1.
X is P_1, or at least P_2.

If speakers naturally say *somebody is not only happy but ecstatic*, this is evidence for the semantic hypothesis that the predicate *ecstatic* has a extension included in that of *happy*. If this frame and others hold for a series of monadic adjectives with an associated comparative, there is evidence that the series forms a scalar family. Once this fact has been established, it follows as a prediction of the semantics theory that the inference from *x is (at least) ecstatic* to *x is (at least) happy* is valid.[6]

Scalar lists, moreover, come in pairs related to one another as opposite perspectives on the same ordering, the predicates in the second list ranking objects using the converse relation of that in the first. The *happier than* comparative of the first list, for example, is the converse of the *is sadder than* ordering of the second list. The first list nests from left to right predicates with increasingly broader sets of less happy individuals. The second list, if reversed, would then continue this order of increasingly broader sets of less happy individuals.

Horn proposes arraying a list and its 'opposite' as a single series, ordered by the relation appropriate to the former. The four lists then become two.

ecstatic, happy, content, so-so, down, sad, miserable
boiling, hot, warm, tepid, cool, cold, freezing

He observes that the joined lists sometimes meet at a common or 'mid-point' term—at *so-so* in lists one and two, at *tepid* in lists three and four. In addition there is frequently a semantic evaluation associated with the over-all comparison. This evaluation often allows one extreme to be identified as 'good' or 'positive', and the other as 'bad' or 'negative'. Being ecstatic and warm, for example, is 'better than' being miserable and cold. In the manner of Horn, the purported 'positive' pole is placed to the left and the negative to the right. We return to this difficult idea below.

Scalar adjectives are also associated with a family of logically rich negation operations. One variety is a 1-1 antitonic mapping on truth-values. It is expressed in English by various affixes including the prefix *un-*. Though it is not lexically acceptable with every scalar adjective, when it is grammatical, it converts a scalar into an complex adjective of the corresponding 'rank' on the opposite pole of the order. Thus *un-happy* is roughly synonymous with *sad*, the predicate of the 'same position' on the opposite side of the happiness list.

A restriction observed by Horn is that privative negation appears to be acceptable to speakers when affixed to adjectives from the positive (left) pole of the full ordering, but not with those on the negative (right). Though *unhappy* is acceptable, *unsad* is not; though discontent is acceptable, *un-down* and *un-glum* are not.[7] He suggests that this asymmetry

6 Horn explains many other kinds of supporting evidence that help identify scalar famlies, including the acceptability of various negative affixes, some of which are described below. For details on the relational properties that the comparative must intuitively support see Åqvist 1981, 1–26, and on logical relations that scalars together with the comparative must intuitively support Martin 1995, 169–196.

7 In addition, when a series has a midpoint adjective, it too, as a rule, is not open to this negation. We do not say *un-tepid* or *un-so-so*. The reason may be parsimony because a midpoint privative is equivalent to a Boolean negation.

in distribution may be used as a linguistic criterion for identifying the direction of the scalar order: the 'positive' (higher, left) side is that open to privative negation.

There is also classical Boolean negation, expressed in English by sentence negations *it is not the case that* and adjective affixes like *non*-, that transforms a scalar into a compound that stands for its Boolean complement within the domain of scalar comparison, e.g. *non-happy* is true of *e* iff *happy* is false of *e*.

There is also *hyper-negation*. This is the *alpha intensivum* of classical Greek, regularized in later philosophical Greek by the prefix *hyper* (*super* in Latin), expressed in English by some uses of *not*:

Its **not** *hot, its boiling.*
He's **not** *(just) active, he's hyperactive.*
It's **not** *(just) a conductor but a superconductor.*

It transforms a scalar into a compound synonymous to a scalar higher in the series with a narrower extension, e.g. *super-happy* is synonymous to *ecstatic*. What we know of God, say the Neoplatonists, are not positive propositions like *God is* (*merely*) *good*, but negatives like *God is hyper-good*.

Scalar usage also makes use of an operator that is the inverse of hyper-negation, known in traditional grammar as the *alpha privatum*. Its semantic function is to convert a predicate to one at the next stage lower in the scalar ordering. It is represented in classical Greek by the alpha prefix and also by *hypo*-. In Latin it is expressed by *sub*-, and in English by affixes like *sub*- and -*less*, as in *subhuman* and *sightless*.[8]

Neoplatonism treats modal predicates as scalar adjectives in this sense that their semantics presupposes a scalar order.[9] Though details vary within the tradition, it is fair to say that its standard metaphysics assumes the universe of existing things is ordered by a relation variously described as marking degrees of reality, 'ontic generation', or causation. It is also referred to as a moral and aesthetic order, and as one of perfection or, viewed oppositely, of privation. Though Neoplatonic philosophy is often technical, it is typically formulated using adjectives that in natural language form scalar families with associated comparative adjectives describing their fundamental order. For example, associated with *is more real than* is the scalar series of technical terms:

absolute, substantial, subsistent, unreal

Comparatives describing substantiality, understanding, beauty, and goodness—all viewed in the tradition as different ways of talking about the same underlying order—are likewise associated with scalar families:

adamantine, hard, solid, firm, tangible, soft/weak, wispy/evanescent
strong/firm/solid, self-supporting, weak/rickety/wobbly, insecure/dangerous
riveted, attentive, awake, wandering, dreamy, asleep
incisive, lucid, cognizant, scatter-brained, dotty, demented

8 This is closely related to the *negatio privatum* of mediaeval logic. See the discussion below and Martin in press (*Synthese*).

9 The historical account here is necessarily much simplified. For a fuller discussion of both historical and logical issues see Martin 1995.

brilliant, smart, pedestrian, dull, stupid
all, most, some, rare, unheard of
eternal, occasional/intermittent, never

In particular, Horn and others have amassed a good deal of evidence for the scalar nature of the modal series associated with comparatives like *is more real than, is more certain than, is truer than*:[10]

necessary, likely/probable, possible, unlikely/improbable, impossible

The list meets Horn's criteria for a scalar series in which *impossible* is the 'opposite' (antitonic reflection) of *necessary*. In philosophical usage, which as Łukasiewicz remarks is not particularly clear, the series may be viewed as describing degrees of being or truth. The most real and most true is the necessary or determined. Somewhat less real, having less being, and being somewhat less true is the possible. The least real, having no being or truth at all, is the impossible or the determined-to-be-false. Necessity, possibility and impossibility are not to be understood as derivative semantic concepts defined in a more primitive vocabulary within possible worlds semantics. Rather, the Platonic tradition attaches modalities directly to the background ordering governing reality.

Since the subset relation holding among scalar extensions is a finite linear ordering, it invites an idealized description in terms of a metric. Once metrizied, moreover, it allows for the definition of operations that form a Łukasiewicz algebra, which is a generalization of Łukasiewicz's many-valued matrix. Let the set of truth-values in the finite n-valued case be $n = \{\frac{0}{n-1}, \frac{1}{n-1}\frac{2}{n-1}, \ldots, \frac{n-1}{n-1}\}$ and ω in the denumerable case.

Definition 1.[11] An n-valued *Łukasiewicz algebra* relative to a metric θ is a structure $\mathrm{L}_{|n|} = <\mathrm{C}, \wedge, \vee, \Rightarrow, -, e>$ such that $|\mathrm{C} \models n$, θ is a 1-1 onto mapping from C to $\{\frac{i}{n} | \ 0 \leqslant i < n\}$ if $|\mathrm{C}| = n \leqslant \omega$, or to ω if $|\mathrm{C}| = n = \omega$, such that:

$\theta(e) = 1;$
$\theta(-a) = 1 - \theta(a);$
$\theta(a \wedge b) = \min\{\theta(a), \theta(b)\};$
$\theta(a \vee b) = \max\{\theta(a), \theta(b)\};$
$\theta(a \Rightarrow b) = \min\{1, 1 - \theta(a) + \theta(b)\}.$

In such a structure the dual $\Leftarrow$ of $\Rightarrow$ is defined: $a \Leftarrow b = -(-a \Rightarrow -b)$. Hence,

10 Here *unlikely* and *improbable* are privatives that obey the predicted regularities by attaching to predicates of a corresponding position on the opposite pole of the ordering. Horn remarks on additional regularities that explain why *impossible* is formed from the midpoint predicate though it is the semantic privative of the supremum *necessity*.

11 This generalization of the algebra supporting Łukasiewicz's many-valued logics is different from that of the same name developed in Bolc and Borowik 1992. There $\Rightarrow$ is defined in terms of a Heyting pseudo-complement (as in intuitionistic semantics) with a resulting many-valued truth-table that is not the same as that of Łukasiewicz's conditional. In the case of four-values, for example, the table reads:

$\Rightarrow$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
0	1	1	1	1
$\frac{1}{3}$	0	1	1	1
$\frac{2}{3}$	0	$\frac{1}{3}$	1	1
1	0	$\frac{1}{3}$	$\frac{2}{3}$	1

$\theta(a \Leftarrow b) = 1 - \min\{1, 1 - \theta(b) + \theta(a)\} = \max\{0,\ \theta(b) - \theta(a)\}$.

It remains to be explained how the operations of this algebra fit the natural language semantics of negations, conjunctions and conditionals formed by scalar adjectives, especially those from the modal series. The explanation calls for a deeper account of privation.

2.2. *Privation*

Consider Aristotle's paradigms of privative predicates, *blindness* and *toothlessness*. A simple model for sight privation posits a two-element property set $\mathfrak{p} = \{\boldsymbol{R}, \boldsymbol{L}\}$ in which $\boldsymbol{R}$ represents sightedness-in-the-right-eye and $\boldsymbol{L}$ sightedness-in-the-left. There are then four possible property combinations or 'compound properties': $\varnothing$, $\{\boldsymbol{R}\}$, $\{\boldsymbol{L}\}$, and $\{\boldsymbol{R}, \boldsymbol{L}\}$. The power set $\mathcal{P}\mathfrak{p}$ forms a four element Boolean algebra, and sight privation may be analyzed as the inhering in an individual of some compound other than $\mathfrak{p}$ itself. Semantically this property structure is intensional and classical. It would provide intensional interpretations for the full set of sentences formed from the sentence variables {*x can see in the right eye, x can see in the left eye*} by the connectives $\sim$, $\wedge$, $\vee$ and $\rightarrow$.

A finite Boolean algebra of n sets, moreover, has the property that it may be partitioned into $n+1$ ranks determined by the cardinality of its elements. The four element algebra of sight, for example, may be partitioned into three ranks: $\{\varnothing\}$, $\{\{\boldsymbol{R}\},\{\boldsymbol{L}\}\}$, and $\{\{\boldsymbol{R},\boldsymbol{L}\}\}$. A rank may be viewed as a disjunction of compound properties. For example $\{\{\boldsymbol{R}\},\{\boldsymbol{L}\}\}$ represents the property of being either right- or left-eyed. Moreover, these sets correspond to the adjectives in a scalar series:

fully sighted, monocular, blind

and as such they may be called *scalar properties*. Since the sets in the partition fall into a linear ordering, they determine a Łukasiewicz algebra. Indeed, it is a general truth that the ranks of a finite Boolean algebra determine a n-valued Łukasiewicz algebra.

The elements of both the Boolean structure and its corresponding Łukasiewicz algebra are formal proxies of properties, of Boolean monadic properties in the first structure and of scalar properties in the second. Within semantic theory, then, they are properly thought of as intensions. As determined by the metatheory of standard intensional logic, to these intensional structures correspond antitonic extensional algebras. Following the tradition in mediaeval logic, let the *signification* of a property be defined as the set of all possible objects that posses that property.[12] Corresponding to the Boolean algebra of compound properties $\{\varnothing, \{\boldsymbol{R}\}, \{\boldsymbol{L}\}, \{\boldsymbol{R},\boldsymbol{L}\}\}$ is a Boolean algebra of signification sets (hereafter 'significations') that reverses the original order:

$\varnothing \subseteq \{\boldsymbol{R}\} \subseteq \{\boldsymbol{R},\boldsymbol{L}\}$, but

$\{x \mid x$ is possible & $\forall \boldsymbol{P} \in \{\boldsymbol{R},\boldsymbol{L}\}, \boldsymbol{P}$ inheres in $x\} \subseteq \{x \mid x$ is possible & $\forall \boldsymbol{P} \in \{\boldsymbol{R}\}, \boldsymbol{P}$ inheres in $x\}$
$\subseteq \{x \mid x$ is possible & $\forall \boldsymbol{P} \in \varnothing, \boldsymbol{P}$ inheres in $x\}$.

A signification in addition determines an extension relative to a world, namely the set theoretic intersection of the signification with that world's domain. These Boolean structures then provide the usual framework for a classical intensional semantics of a sentential language. Sentences are interpreted intensionally by a homomorphism from syntax to the properties in the Boolean structure. These in turn are mapped homomorphically into significations, which in turn are mapped homomorphically into extensions. A logical matrix may be defined by specifying as the designated value some maximal ideal of a Boolean algebra. The resulting logic is classical.

If finite, the Boolean algebra of significations also determines as its rank reduction a Łukasiewicz algebra of possible object sets. Each such rank of the Boolean algebra determines a set of possible objects: those that fall under at least one of the compound properties of the corresponding rank in the Boolean property structure. We may group these sets into the universe of a second algebra in which its elements, ordered by the subset relation, represent the possiblia at each privational rank.

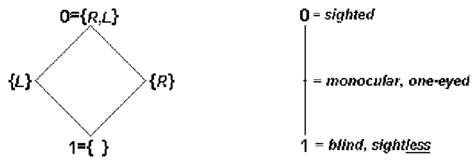
Starting, then, with the initial intensional Boolean algebra of properties, two series of three structures is determined: (1) the intensional Boolean algebra of properties, its intensional Łukasiewicz rank reduction and its antitonic significational Łukasiewicz algebra; and (2) the intensional Boolean algebra of properties, its antitonic significational Boolean algebra and the Łukasiewicz rank reduction of this significational structure. If the intensional Łukasiewicz algebra is understood with is ranking reversed in the usual manner so that extensions are genuinely antitonic to extensions, then the third structure of the first series is the same as that of the second. The same set is simultaneously the signification of a scalar property and the union of the significations of the various conjuncts in a compound Boolean property defining that scalar.

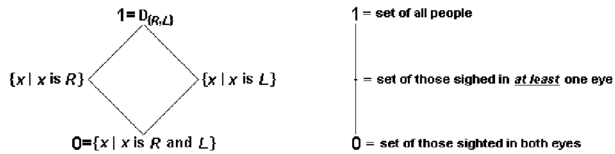
A logical matrix may be defined from an n-valued Łukasiewicz algebra by specifying the structure's maximal element as the only designated value. The entailment relation then is characterized by Łukasiewicz's n-valued logic.

The full algebraic situation is diagramed below. The intensional structure of properties are on the top, with their antitonic significational structures beneath them. The Boolean structures are on the left with their Łukasiewicz rank reductions to their right. Note that to preserve the conceptual paradigm that intensional is antitonic to extensional structure, the enumeration in the two structures is reversed. Since in extensional logic, 1 traditionally represents the highest degree of 'truth', and 0 the lowest, here 0 is used as the intensional correlate of extensional 1 and 0 as that of extensional 1.

Aristotle's other paradigm of privation, toothlessness, is open to a similar analysis. Humans have 32 teeth, each of which has a number: $t_1,\ldots,t_{32}$. To these corresponds a set $\mathfrak{P}$ of primitive properties of the form *having tooth* t_i, for $i \leqslant 32$. There are then 32^2

12 In fourteenth-century termist logic, *signification* is the relation that holds between a term and the possible objects that it could stand in for some true proposition. It is distinguished from the related idea of *supposition* and *ampliation* which are defined relative to special contexts: supposition is defined relative to a true sentence and is the relation holding between a term and the objects it stands for in that sentence, and ampliation is defined relative to a true sentence in which the copula is modified by by an ampliative modifier like *possibly* or *will be* and is defined as the relation holding between a term and the *possiblia* that the term stands for in the context of that sentence. Signification and ampliation are related in modal cases, e.g. in true sentences in which *possibly* modifies the copula, a term supposits for the objects it signifies. In this paper signification is used because it has the properties required by the analysis: signification determines a 1–1 mapping from a term's intension to set of objects that fall under the term, and does so in a manner that is well defined independently of the truth of propositions relative to worlds.

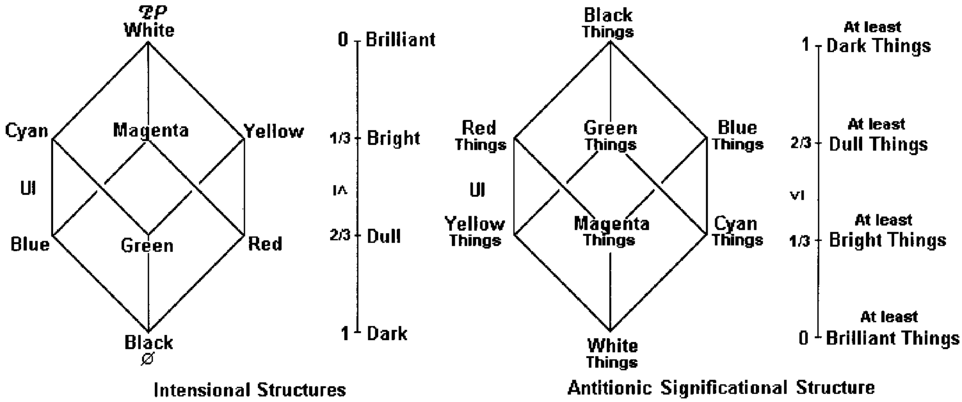




different combinations of teeth represented by the set $\mathfrak{P}\mathfrak{p}$ of compound properties. This set includes the property *having no teeth* represented by $\varnothing$, and the property *having all 32 teeth* represented by $\mathfrak{p}$ itself. The relation $\subseteq$ over $\mathfrak{P}\mathfrak{p}$ is then a partial order, and $<\mathfrak{P}\mathfrak{p},\cap,\cup,-,\varnothing,\mathfrak{p}>$ is a Boolean algebra. Antitonic to this structure is an algebra of significations, and coordinate to each of these Boolean algebras are rank reductions which are mutually antitonic Łukasiewicz algebras. In principle, except perhaps in technical contexts, natural language could then have predicates naming the compound 'tooth' properties of the Boolean structures, and scalar predicates for the nodes of their reduction.

One of the best examples of privation relevant to this discussion is the algebra of light—light, moreover, is probably the most frequently cited metaphor used by Neoplatonists attempting to explain the structure of reality. Colored light forms a Boolean algebra ordered by a relation of physical privation. By using subtractive filters, white light yields an eight element algebra of black, red, green, blue, cyan, yellow, magenta and white. By applying one of three appropriate filters to white light, it may be reduced to cyan, magenta or yellow. By pairwise overlaying these filters, blocking out yet more light, red, blue and green are produced. Overlaying all three filters blocks out all light and yields black, the utter deprivation of light. Antitonic to the structures of light properties are those of their significations.[13]

13 There is actually much more to the analysis of privation. The Neoplatonic account used here is expounded in detail in Martin 2001, 187–240. Aristotelian privation is closely related. Let $\{R,L\}$ be a set of privative properties for 'full sightedness' that hold naturally of a natural kind K understood as the extension of $\{R,L\}$, e.g. the set of humans. The set $\{L\}$ is then a privation set relative to $\{R,L\}$ in the Neoplatonic sense. It represents a degree lower in the privational hierarchy since any entity in the extension of $\{L\}$ but not in that of $\{R,L\}$ suffers a privation. This fact is described slightly differently in the medieval tradition. Following Aristotle, it observes that objects in the extension of $\{R\}$ are 'naturally apt' to be fully sighted, to be in K, to possess both properties R and L and that being just R is a privation. To express membership in the extension of $\{L\}$ privatively, the medievals then exploited the fact that the extension of $\{L\}$ is the relative complement the extension of $\{R\}$ relative to K, and interpreted a privative negation operator *non-* in terms of this complementation, defining it by exposition. For example, *S is left-blinded* or *S is non-left-sighted* is analyzed as *S is naturally a human (in the extension of* $\{R,L\}$*) and* $\neg(S$ *is* $R)$. That is, *S is non-L* means *S is in K and* $\neg(S$ *is* $L)$. This is logically equivalent to saying that S is in the Neoplatonic privation of the extension of $\{R,L\}$ with respect to L. For a discussion of this sense of privation and a related sense in modern linguistics see Martin in press.



Boolean and Lukasiewicz Algebras of Light

2.3. *The Łukasiewicz connectives*

While it is true that Boolean algebras generate Łukasiewicz algebras and that these determine logical matrices with classical and Łukasiewicz logics, it remains to be shown that their operations are those appropriate for interpreting the natural language meaning of the connectives *not*, *and*, *or* and *if … then* when employed with sortal adjectives.

2.3.1. *The Strong Tables for* not, and, *and* or Łukasiewicz tables for *not*, *and*, and *or* generalize to *n*-values Kleene's strong tables. If non-bivalence, in which falsity is understood as a genuine subvariety of non-truth, is considered conceptually plausible, then principles 1–3 of §1 do seem to capture general intuitions about evaluating these connectives. As it was remarked earlier, logicians have found fault with the strong connectives less for the intuitive implausibility of their tables than for the difficulty of defining for them a plausible entailment relation—a problem Łukasiewicz avoids by introducing $\wedge$ and $\vee$ in terms of the more expressive $\sim$ and $\rightarrow$. At this level of abstraction, then, the strong tables may be assumed to have some plausibility as capturing conceptual intuitions about the relation of the truth-values to the connectives. Łukasiewicz's semantics, however, goes further by relegating the full panoply of values to a linear order not postulated by Kleene or by the three motivating rules. His negation pivots around a midpoint, conjunction is an extensional minimum operation and disjunction an extensional maximum—all features that turn on the order among values. It is the tie to order, moreover, that makes the operators suitable for interpreting the connectives conjoined to scalar adjectives.

2.3.2. *Negation* Intuitively the *un*- operator of natural language affixed to a scalar adjective reverses the 'ontic rank' of its interpretation, whether considered intensionally or extensionally. The algebraic operation on the values $\{0, \ldots 1\}$ that accomplishes this reversal is $-$ as defined in a Łukasiewicz algebra. If *x is sighted* stands for the property of having sight to at least the degree *n*, then *x* is *un-sighted* has as its intension the property of being sighted to the opposite degree, to degree $1-n$. Significationally the value of *x is un-sighted* is the set antitonic to that picked out by *x is sighted*, namely the set of all *possibilia* for which the double negation of the original argument is true. In addition the formal operation $-$ goes beyond natural language in abstracting away from its lack of nested negations of this sort, an abstraction required here if the explanatory framework is to be that of a full sentential syntax.

Consider in particular the modal interpretation of truth-values in terms of degrees of Being or Truth. If P is necessary then $\sim P$ should be impossible, and conversely. If P is possible but not necessary, then $\sim P$ should be so as well. Again, these values are those determined by the operation $-$ on intensions. Extensionally, if P is necessary and stands for the 'smallest' set of possible objects, then $\sim P$ is impossible and stands for the 'largest', and conversely. Also, P is possible and not necessary, and stands for an intermediate set, iff $\sim P$ does likewise. This is the operation $-$ defined on significations.

2.3.3. *Conjunction and disjunction* Intuitively, the conjunctive predicate *x is bright and x is dull* has as its intension the property of being either bright or dull, and as its signification the set of all possible objects that are either. Intuitions about disjunction are dual. In a formal four-valued representation within a Łukasiewicz algebra, *x is bright* would be assigned an intensional property with the value $\frac{1}{3}$ while *x is dull* would be assigned the value $\frac{2}{3}$. The compound *x is bright and x is dull* is assigned the intensional join of the two intensional parts having the value $\frac{1}{3}$, which is the intensional maximum of the values of the parts since among intensions 0 is maximal. Among significations *x is bright* is assigned a set with the rank $\frac{2}{3}$, *x is dull* one with the rank $\frac{1}{3}$, and *x is bright and x is dull* stands for their extensional meet of these two, the extensional minimum, which has rank $\frac{1}{3}$. Overall the formal pattern matches the scalar intuitions, and the situation is similar for disjunction.

2.3.4. *The conditional* Perhaps not surprisingly, it is the intensional $\Leftarrow$ that directly captures the meaning of the natural language scalar conditional. It in turn dictates in accordance with the framework of intensional semantics that the corresponding extensional operation is $\Rightarrow$.

Łukasiewicz's $\rightarrow$ is best understood as describing the intensional interpretation of *if … then* applied to modal scalars. Let it be assumed (as in the Platonic tradition) that there are degrees of being and truth, a sentence's degree of truth being a direct reflection of the degree of being it describes. In scalar semantics the intensional $\Leftarrow$ is an operation that assigns to the argument pair $<$being-**P**, being-**Q**$>$ the value consisting of the complex property being-**P**-insures-at-least-as-much-being-as-being-**Q**. Such a property is conveyed in natural language by the more or less contrived phrases:

having as much being as P insures having as much being as Q
P-ness is (at least) as real as Q-ness.
it's as true that x is P as it is that x is Q
if x is P then x is Q

That this semantics and its associated truth-table for $\Leftarrow$ correspond to intuitions about the meaning of scalar *if … then* is shown by a consideration of cases.

2.3.5. *Cases in which the intensional value of P has 'more being' than that of Q* Such cases are represented in the intensional Łukasiewicz algebra by assigning to P a value greater (closer to 0) than that of Q, and in the antitonic significational structure by requiring that the set picked out by Q includes (is 'higher than') that which is picked out by P.

Consider a typical case in a three-valued intensional semantics. Let P be possibly true of an individual i and Q necessarily true. Then in the scalar intensional interpretation, P receives the maximal intensional value 0 indicating necessity, and Q is assigned the midpoint value $\frac{1}{2}$ indicating possibility. Moreover the property of being-necessary ranks higher than that of being-possible on the relevant ontic scale. All the primitive Boolean properties that constitute being-necessary are thus included in those of being-possible. It is a truth of the metaphysics assumed by the semantics that being-necessary is sufficient for being-possible. It is then appropriate to summarize this situation in property vocabulary by saying that the following is true: the complex property 'if-necessary-then-possible' is true of i. But saying this property truly applies is not enough in a semantics in which truth has variable values. How true is it? It is trivially true, forced by the content of the concepts 'necessity' and 'possibility'. This is truth to the fullest extent. The appropriate intensional value is therefore 0. This is exactly the value of possibility $\Leftarrow$ necessity. Significationally $\{x | x \text{ is possible}\}$ has the value $\frac{1}{2}$, and $\{x | x \text{ is necessary}\}$ has 1. The antitonic mirroring dictates that its antitonic value is that of $\{x | x \text{ is possible}\} \Rightarrow \{x | x \text{ is necessary}\}$, namely 1.

This line of reasoning explains the problematic assignment of 1 to $P \to Q$ when P and Q are non-synonymous and both have the value $\frac{1}{2}$. When *if* ... *then* is meant in the scalar 'metaphysical' sense captured by the intensional Łukasiewicz algebra, the conditional must receive the maximal value because it is a trivial conceptual truth that if a subject i has an ontic value of $\frac{1}{2}$ then it has an ontic value of $\frac{1}{2}$. Another way to say the same thing is that Łukasiewicz's connectives abstract away from all of an adjective's descriptive content other than its degree of being or truth. If an antecedent and consequent pick out the same region of the ontic scale, they have the same content in this sense.

There are two important ways in which non-synonymous adjectives may represent the same ontic rank. The first is philosophically interesting because it captures the famous Neoplatonic doctrine of 'the univocity of the good'. This is the thesis that a broad family of seemingly distinct evaluative properties—being, goodness, beauty, perfection, causation, necessity, spirituality, substantiality—are claimed to describe the same single ontological order. Associated with each of these evaluative comparatives is its own family of scalar adjectives. If they all describe the same order, it is possible for a predicate from one family to describe the same portion of the ordering as one from another family. It is this fact, for example, that underlies the standard Neoplatonic claim that matter is the privation of evil. Semantically, the predicates *material* and *evil* belong to different scalar families—the morality and spirituality groups respectively—but abstractly represent the same (lowest) portion of the ontic ordering.

Even standard Boolean predicates may be understood abstractly in terms of Łukasiewicz algebras. Let us call the *abstract content* of a Boolean predicate the region assigned to its rank within the background ordering of the rank reduction of the governing Boolean algebra. This region is represented by a truth-value in the Łukasiewicz algebra. In the color example the abstract content of the predicates *red* and *blue* is the same, which is the region in the Łukasiewicz structure assigned to *dull*. Under its 'Łukasiewicz reading' then the conditional *x is red* → *x is blue* has the highest value, 0 intensionally and 1 extensionally.

Indeed, the reading encapsulates the conceptual content of Łukasiewicz's truth-table for the conditional. The conditional is extensional and multiply abstract. It is extensional because its truth-table describes the antitonic function mirroring the intensional operation appropriate to the unpacking of 'has at least as much being or truth'. It is abstract to the first degree in that among scalar predicates it subtracts any descriptive content characteristic of the particular scalar family to which the adjectives belong. In addition, among Boolean predicates it abstracts from any descriptive differences among predicates of the same rank. A full intuitive reading then of $P \rightarrow Q$ with Łukasiewicz's truth-table is:

> *If any descriptive content other than degree of being or truth is ignored and matters are considered extensionally, if P then Q.*

2.3.6. *Cases in which the intensional value of the antecedent is less than that of the consequent* Some properties have less than complete being and some sentences less than complete truth. To fit the framework of the intensional Łukasiewicz semantics, the conditional in these cases should vary, being intensionally higher (closer to 0) the more the arguments differ. Again let us consider a three point modal scale and stipulate that P is 1, Q is $\frac{1}{2}$) and R is 0. Consider the three sub-cases given the three point scale:

	Sentence	Intensional Interpretation
1.	$Q \rightarrow P$	necessity$\Leftarrow$possibility
2.	$R \rightarrow P$	necessity$\Leftarrow$impossibility
3.	$R \rightarrow Q$	possibility$\Leftarrow$impossibility

If we compare the examples, it would be natural to say they differ in 'degrees of truth.' Possibility carries half the being of necessity, and hence there is some truth to saying that being-possible insures as much being as being-necessary. It is, however, not entirely or utterly true as is the conditional in earlier cases in which the being of the antecedent equaled or exceeded that of the consequent. It is even less true to say that being-impossible, which has the lowest degree of being, insures as much being as being-necessary, because being-impossible carries along with it none of the being that makes up necessity, certainly less than being-possible. In this way it is plausible and natural to say that the degrees of truth of the three examples are respectively the highest (the intensional value 0), middle ($\frac{1}{2}$) and lowest (1). Lastly, consider the case of $R \rightarrow Q$ in which the values of both parts is less than complete, but that of the antecedent is less than that of the consequent. Since being impossible insures no degree of being, the conditional is not true in any complete sense. The situation however is less egregious than $R \rightarrow P$, which makes a bolder claim and is utterly untrue. To distinguish between the two cases using the framework of degrees of truth, it is plausible, even natural, to say that $R \rightarrow P$ is less truth than $R \rightarrow Q$. Since the former possesses no truth whatever and is more misleading than the latter, it is natural to say that the latter possesses some truth, and to represent this fact on the three point scale by assigning it $\frac{1}{2}$. The analysis is analogous whatever the number of values in the language, and may be generalized: the greater the difference between the truth-value of the antecedent and consequent, the greater the degree of truth of the conditional. Since among intensions, the value 0 ranks top, the generalization may be given the precise form:

$$\mathrm{Int}(P \to Q) = \mathrm{Int}(Q) - \mathrm{Int}(P)$$

In the earlier cases in which the antecedent is equal to or greater than the consequent, the whole is 0. Combining both sorts of case yields the global generalization:

$$\mathrm{Int}(P \to Q) = \max\{0, \mathrm{Int}(Q) - \mathrm{Int}(P)\} = 1 - \min\{1, 1 - \mathrm{Int}(Q) + \mathrm{Int}(P)\}$$

This is exactly the defining condition in the intensional Łukasiewicz algebra of a scalar language for $\mathrm{Int}(P) \Rightarrow \mathrm{Int}(Q)$.

In this analysis the conditional's truth-values represent degrees within a modality ranking. However, any *n*-valued scalar privation ranking, regardless of whether the family of scalars is explicitly modal, determines an extensional conditional within Łukasiewicz's truth-table. More importantly, when viewed abstractly, any privation process is implicitly 'modal' in the Platonic sense because, as connoted by the term *privation*, it presumes an ordering from the fuller, more complete, and more real to the less. Pivation is not possible without such a 'subtractive' ordering. It is such an order that is described in the intensional structure of any scalar family.

Łukasiewicz semantics is most plausibly understood as implementing ideas at this level of abstraction. Truth-values represent degrees of scalar privation in which all conceptual content other than the scalar ordering has been suppressed. Łukasiewicz truth-tables then are direct descriptions of the rules characteristic of the connectives that govern the projection of such degrees from parts to wholes.

It should also be remarked that the reading of modality in a scalar language provides a simple frequentist explanation of Łukasiewicz's characterization, cited above, of the truth-values as 'degrees of probability corresponding to various possibilities'. The values are numerical representatives of ordered finite sets of the *possibilia* that the predicate is true of. These are the *significations* that form an Boolean structure of sets, antitonic to intensions. They are in effect abstractions to *possiblia* form the more usual notion of extension. It will suffice to state the finite case. The ratio of (the finite cardinality of) the signification of (*at least*) *necessary* to that of (the finite cardinality of) the set of all possible objects is 'less than', and in this sense less probable than, that of (the finite cardinality of) the signification of (*at least*) *possible* to that of all possible objects. Likewise possibility is less probable than impossibility.

Much of the plausibility of the overall analysis turns on the formal development of the main definitions and results, which is the subject of the next section. The paper will conclude with some brief remarks on the relation of *n*-value Łukasiewicz logics to classical product logics.

3. Formal Theory

3.1. *Structures*

An ordering $\leqslant$ is said to be *total* on C, if it is a partial ordering (reflexive, transitive and antisymmetric) on C and $\forall a,b \in C$, either $a \leqslant b$ or $b \leqslant a$. An *algebra* or *structure* is any $S = \langle C_1, \ldots, C_k, R_1, \ldots, R_l, f_1, \ldots, f_m, O_1, \ldots, O_n \rangle$ such that each C_i is a set, each R_i is a relation on $\mathfrak{U} = \cup\{C_i \mid 1 \leqslant k\}$ (called the *universe* of S), each f_i is a function on these elements, and each O_i is one of the elements. Two structures are of the *same character* if they have the same number of sets, relations, functions, and objects, and functions and relations of the same rank are of the same number of places. A *homomorphism* from $S = \langle C_1, \ldots, C_k, R_1, \ldots, R_l, f_1, \ldots, f_m, O_1, \ldots, O_n \rangle$ to $S' = \langle C'_1, \ldots, C'_k, R'_1, \ldots, R'_l, f'_1, \ldots, f'_m, O'_1, \ldots, O'_n \rangle$ is any mapping θ from $\mathfrak{U}$ to $\mathfrak{U}'$ such that for $1 \leqslant i \leqslant l$, $\langle a_1, \ldots, a_n \rangle \in R_i$ iff $\langle \theta(a_1), \ldots, \theta(a_n) \rangle \in R'_i$; for $1 \leqslant i \leqslant m$, $\theta(f_i(a_1, \ldots, a_n)) = f'_i(\theta(a_1), \ldots, (a_n))$; and for $1 \leqslant i \leqslant n$, $\theta(O_i) = O'_i \in \mathfrak{U}'$. An *isomorphism* is any 1-1 onto homomorphism. By the *lattice* (*determined by the ordering* $\leqslant$) is meant the structure $\langle C, o_1, o_2 \rangle$ such that C is closed under the binary operations o_1 and o_2 defined as follows: $o_1(a,b)$ (called the lattice *meet* of a and b) is the greatest $\leqslant$-lower-bound of $\{a,b\}$ and $o_2(a,b)$ (called the lattice *joint* of a and b) is the least $\leqslant$-upper-bound of $\{a,b\}$. When only one matrix is under discussion it is customary to use the infix operator $\wedge$ to refer to lattice meet, and $\vee$ to lattice join, and to the lattice as $\langle C, \wedge, \vee \rangle$. In some cases below however these operators will refer to the meet of one lattice and the join of a second, or vice versa. In all cases, however, the order in which the operations are listed in the lattice name determines which is its meet (the first) and which its join (the second).

If $\langle C, \wedge, \vee \rangle$ is a lattice, it follows directly that the operations are idempotent, commutative and associative. Further, a is called the *zero element* of a lattice iff, $a \in C$ and is the unique $\leqslant$-least element. It is called the *unit element* iff, $a \in C$ and is the unique $\leqslant$-greatest element of C. If only one lattice is under discussion its unit element is named 1 and its zero element 0. In some case below however 0 will name the zero element of one lattice and the unit element of another, and similarly for 1. A structure is *finite* if its domain is. A homomorphism θ from a lattice $\langle C, \wedge, \vee \rangle$ to another $\langle C', \wedge', \vee' \rangle$ is said to be *antitonic* iff for any $a,b \in C$, $a \leqslant b$ iff $\theta(b) \leqslant \theta(a)$. $\langle C, \wedge, \vee \rangle$ is said to be *antitonic* to $\langle C', \wedge', \vee' \rangle$ iff there is an antitonic homomorphism from the first to the second. If θ is antitonic, it follows that $\theta(a \wedge b) = \theta(a) \vee' \theta(b)$ and $\theta(a \vee b) = \theta(a) \wedge' \theta(b)$. The structures are *strictly antitonic* if the mapping θ is an isomorphism. A *congruence relation* $\equiv$ on C is any binary reflexive, transitive and symmetric relation on C, and the *equivalence class* of a under $\equiv$ (denoted $[a]$) is the set of all elements of C equivalent to a under $\equiv$. The *quotient algebra* (*determined by a lattice* $\langle C, \wedge, \vee \rangle$ and an *equivalence relation* $\equiv$ *on* C) is $\langle \{[a]\}_{a \in C}, \wedge_{\equiv}, \vee_{\equiv} \rangle$ such that $[a] \wedge_{\equiv} [b] = [a \wedge b]$ and $[a] \vee_{\equiv} [b] = [a \vee b]$. It then follows that the operation [ ] is a homomorphism from a lattice onto its quotient algebra.

3.2. *Łukasiewicz algebras*

A structure $Ł_{|n|} = \langle C, \wedge, \vee, \Rightarrow, -, e \rangle$ is an n-valued *Łukasiewicz algebra* relative to θ iff θ (called a *rank* assignment) is a 1-1 onto mapping from domain C to range $\{\frac{i}{n} \mid 0 \leqslant i < n\}$ if $|C| = n \leqslant \omega$, or to range ω if $|C| = \omega$, such that:

$$\theta(e) = 1$$
$$\theta(-a) = 1 - \theta(a)$$
$$\theta(a \wedge b) = \min\{\theta(a), \theta(b)\}$$
$$\theta(a \vee b) = \max\{\theta(a), \theta(b)\}$$
$$\theta(a \Rightarrow b) = \min\{1, (1 - \theta(a)) + \theta(b)\}$$

$Ł_{|n|}$ ranges over such algebras for $n=2,\ldots,\omega$. In Łukasiewicz algebras it follows that C is closed under its operations, that $<C,\wedge,\vee>$ is a distributive lattice with unit element 1 and zero element 0, with a total ordering relation $\leqslant$, and that:

$-1=0$ and $-0=1$
$-(-a)=a$ and $-(a\vee b)=-a\wedge -b$
$\theta(a\Rightarrow b)=1$ if $\theta(a)\leqslant\theta(b)$
$\theta(a\Rightarrow b)=1-\theta(a)+\theta(b)$ if $\theta(a)>\theta(b)$

If a Łukasiewicz algebra contains an element a such that $-a=a$, this element is unique and is called the algebra's *midpoint*. $a\Leftarrow b$ (the dual of $\Rightarrow$) is defined as $-(-a\Rightarrow -b)$. In the four-valued case, the truth-tables for $\Rightarrow$ and $\Leftarrow$ are:

$\Rightarrow$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
0	1	1	1	1
$\frac{1}{3}$	$\frac{2}{3}$	1	1	1
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1	1
1	0	$\frac{1}{3}$	$\frac{2}{3}$	1

$\Leftarrow$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
0	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{2}{3}$
$\frac{2}{3}$	0	0	0	$\frac{1}{3}$
1	0	0	0	0

Lukasiewicz's Conditional $\Rightarrow$ and its Dual $\Leftarrow$ in L_4

It follows that C is closed under $\Leftarrow$, and

$\theta(a\Leftarrow b)=1-\min\{1,1-\theta(b)+\theta(a)\}=\max\{0,\ \theta(b)-\theta(a)\}$
$\theta(a\Leftarrow b)=0$ if $\theta(b)\leqslant\theta(a)$
$\theta(a\Leftarrow b)=\theta(b)-\theta(a)$ if $\theta(b)>\theta(a)$

Note that if $Ł_{|n|}=<C,\wedge,\vee,\Rightarrow,-,1>$ is a Łukasiewicz algebra, then $<C,\wedge,\vee>$ and $<C,\vee,\wedge>$ are strictly antitonic distributive lattices, the former with zero element $0=-1$ and unit element 1, and the latter with zero element 1 and unit element $0=-1$.

If the domain C of $Ł_{|n|}$ is $\{\frac{i}{n}|0\leqslant i<n\}$ or ω, making θ the identity mapping, then $Ł_{|n|}$ is called the (*distinguished*) *n-valued Łukasiewicz algebra*, and referred to as $Ł_{2n}$. $Ł_n$ ranges over these structures. It follows in $Ł_n$ that:

$-a=1-a$;
$a\wedge b=\min\{a,b\}$;
$a\vee b=\max\{a,b\}$;
$a\Rightarrow b=\min\{1,1-a+b\}$;
$a\Leftarrow b=1-\min\{1,1-b+a\}$.

Clearly $Ł_n$ is unique and isomorphic to $Ł_{|n|}$. It will be convenient to identify the two. Moreover, it is clear that there is an antitonic isomorphism to $Ł_{|n|}$ (and hence also to $Ł_n$) that reverses the order of $Ł_n$. This antitonic image plays a role in the intensional semantics below and it is constructed as follows. The operation $-$ on C of $Ł_n=<C,\wedge,\vee,\Rightarrow,-,1>$ is such that:

$a\leqslant b$ iff $-b\leqslant -a$;
$--a=a$;
$-(a\wedge b)=-a\vee -b$;

$-(a \vee b) = -a \wedge -b$;
$-(a \Rightarrow b) = -a \Leftarrow -b$.

Let $\text{Ł}_{|n|^{\smile}} = <\mathrm{C}, \vee, \wedge, \Leftarrow, -, 0>$ be called the *antitonic reflection* of $\text{Ł}_{|n|}$ (and likewise $\text{Ł}_{\breve{n}} = <\mathrm{C}, \vee, \wedge, \Leftarrow, -, 0>$ is that of Ł_n). Then $-$ is an antitonic isomorphism form $\text{Ł}_{\breve{n}}$ to Ł_n, and more generally, from $\text{Ł}_{|n|^{\smile}}$ to $\text{Ł}_{|n|}$. $\text{Ł}_{\breve{n}}$ and $\text{Ł}_{|n|}$ range over antitonic images of this sort. In the intensional semantics below $\text{Ł}_{|n|^{\smile}}$ and $\text{Ł}_{\breve{n}}$ function as an intensional structure that are mirrored in 'reverse' by the significational structure $\text{Ł}_{|n|}$ and Ł_n.

3.3. *Boolean Algebras*

A Boolean algebra is any $<\mathrm{C}, \wedge, \vee, -, 0, 1>$ such that:

1. $<\mathrm{C}, \wedge, \vee>$ is a lattice,
2. $\wedge$ and $\vee$ are distributive,
3. 0 (the first distinguished element listed) is its zero element and 1 (the second distinguished element) is its unit element, and
4. C is closed under $-$, $a \wedge -a = 0$ and $a \vee -a = 1$ ($-$ is the *complementation* operation on C).

A homomorphism θ from one Boolean algebra $<\mathrm{C}, \wedge, \vee, -, 0, 1>$ to another $<\mathrm{C}', \wedge', \vee', -', 0', 1'>$ is *antitonic* iff if it is antitonic from $<\mathrm{C}, \wedge, \vee>$ to $<\mathrm{C}', \wedge', \vee'>$. If it is isomorphic then the two Boolean algebras are *strictly antitonic*. Clearly, the operation $-$ is an antitonic isomorphism from $<\mathrm{C}, \vee, \wedge, -, 1, 0>$ onto the Boolean algebra $<\mathrm{C}, \wedge, \vee, -, 0, 1>$. In the classical intensional semantics below a Boolean algebra of 'intensions' is employed as the structure in terms of which a syntax is interpreted. This structure is then mirrored in 'reverse' by its strictly antitonic Boolean algebra of sets of possible objects determined by the intensions.

3.4. *Determination of Łukasiewicz by Boolean algebras*

A Boolean algebra determines a Łukasiewicz algebra. By a *branch* of a finite Boolean algebra $<\mathrm{C}, \wedge, \vee, -, 0, 1>$ is meant a series $a_1, \ldots, a_n$ such that $a_1 = 0$ and for each $i \leqslant n$, a_i is a $\leqslant$-immediate predecessor of a_{i+1}.

The *rank* of an element a relative to some n-tuple $a_1, \ldots, a_n$ is i (briefly $\mathfrak{r}(a) = i$) iff $a = a_i$. The notion of rank is extended to Boolean and Łukasiewicz algebras: $\mathfrak{r}(\mathrm{a}) = i$ for an element a of a finite Boolean algebra B iff $\mathfrak{r}(\mathrm{a}) = i$ relative to some branch $a_1, \ldots, a_i$ of B; and $\mathfrak{r}(a) = i$ for an element a of a n-valued Łukasiewicz is algebra Ł iff $\mathfrak{r}(a) = i$ relative to the n-tuple $a_1, \ldots, a_n$ formed by the lattice ordering $\leqslant$ on Ł. If (i) $\mathfrak{U}$ is a family of subsets of some set D, (ii) $\cap$, $\cup$, and $-$ are the usual set theoretic operations on $\mathfrak{U}$, and (iii) $\varnothing$ is the empty set, it follows that $<\mathfrak{U} \subseteq \mathcal{P}\mathrm{D}, \cap, \cup, -, \varnothing, \mathrm{D}>$ is a Boolean algebra.

By a *(rank) equivalence* is meant the equivalence relation $\equiv$ that holds between all elements of the same rank: $a \equiv b$ iff $\exists i$, $|a| = |b| = i$. Let $<\{[a]\}_{a \in \mathrm{C}}, \wedge_{\equiv}, \vee_{\equiv}, -_{\equiv}, [0], [1]>$ be the quotient algebra determined by a finite Boolean algebra $<\mathrm{C}, \wedge, \vee, -, 0, 1>$ and its rank equivalence $\equiv$. It follows that (i) if the cardinality of C is n, then the cardinality of $\{[a]\}_{a \in \mathrm{C}}$ is $n+1$; (ii) that $\wedge_{\equiv}$ and $\vee_{\equiv}$ are the set theoretic operations $\cap$ and $\cup$ restricted to $\{[a]\}_{a \in \mathrm{C}}$; (iii) the ordering $\leqslant$ on $\{[a]\}_{a \in \mathrm{C}}$ (defined $[a] \leqslant [b]$ iff $[a] \wedge_{\equiv} [b] = [a]$ iff $[a] \vee_{\equiv} [b] = [b]$) is such that $[a] \leqslant [b]$ iff for some c such that $a \equiv c$, $c \subseteq b$, and (iv) $\leqslant$ is total.

Theorem

If

(1) $B = \langle C, \wedge, \vee, -, 0, 1\rangle$ is a finite Boolean algebra of cardinality n with rank function $\mathfrak{r}$;
(2) $\langle \{[a]\}_{a\in C}, \wedge_{\equiv}, \vee_{\equiv}, -_{\equiv}, [0], [1]\rangle$ is the quotient algebra of $\langle C, \wedge, \vee, -, 0, 1\rangle$;
(3) θ is a 1-1 function on $\{[a]\}_{a\in C}$ such that $\theta([a]) = \mathfrak{r}(a)$;
(2) $[a] \Rightarrow_{\equiv} [b] = \{c\in C \mid \mathfrak{r}(c) = \min\{1, 1-\mathfrak{r}(a)) + \mathfrak{r}(b)\}\}$;

then

(1) $-_{\equiv}[a] = [-a] = 1-\mathfrak{r}(a)$ and $[a] \Leftarrow_{\equiv} [b] = \{c\in C \mid \mathfrak{r}(c) = 1-\min\{1, 1-\mathfrak{r}(b) + \mathfrak{r}(a)\}\}$; and
(2) $\langle \{[a]\}_{a\in C}, \wedge_{\equiv}, \vee_{\equiv}, \Rightarrow_{\equiv}, -_{\equiv}[1]\rangle$ is a Łukasiewicz algebra of cardinality $n+1$.

By the *Łukasiewicz reduction* (*of cardinality* $n+1$) *determined by an* 2^n*-valued Boolean algebra* B (briefly, $ŁB_{|n+1|}$) is meant this quotient algebra of B under rank equivalence with $\Rightarrow_{\equiv}$. It follows from the definition that such a reduction is an $n+1$-valued Łukasiewicz algebra.

In the case in which the Boolean algebra is an algebra of sets $\langle \mathcal{P}D, \cap, \cup, -, \varnothing, D\rangle$ with $|D| = n$, its reduction $Ł_{|n+1|}$ on C′ may be further simplified. Each equivalence class $[a]$ of rank $\mathfrak{r}(a)$ in C′ may be uniquely represented, in a way that preserves order, by the set of objects in D that are elements of any set in some family of that rank. That is, the function θ from C′ into $\mathcal{P}D$ such that $\theta([a]) = \{x \mid \exists y,\ y\in[a]$ and $x\in y\}$ is a 1-1 mapping such that $[a] \leqslant [b]$ iff $\theta[a] \subseteq \theta[b]$. Below $[a]$ is identified with $\theta[a]$ in cases in which θ is well defined.

3.5. *Syntax*

By a (*classical sentential*) *syntax* is meant a structure $\mathrm{Syn} = \langle \mathrm{Sen}, \chi_{\wedge}, \chi_{\vee}, \chi_{\rightarrow}, \chi_{\sim}\rangle$ such that Sen is the closure of a denumerable set of sentential variables *Var* under syntactic operation $\chi_{\wedge}, \chi_{\vee}, \chi_{\rightarrow}, \chi_{\sim}$ (which are called $\sim, \wedge, \vee, \rightarrow$ when there is no possibility of confusion).

3.6. *Extensional Semantics*

By a (*logical*) *matrix* for the syntax $\mathrm{Syn} = \langle \mathrm{Sen}, \chi_{\wedge}, \chi_{\vee}, \chi_{\rightarrow}, \chi_{\sim}$ is meant a structure $\langle C, D, o_1, \ldots, o_4\rangle$ such that (i) $\langle C, o_1, \ldots, o_4\rangle$ is of like character to Syn, (ii) $\langle C, o_1, o_2\rangle$ is a lattice and (iii) D (called the set of *designated values*) is a non-empty subset of C. Here each o_i is understood as the truth-function used to interpret the grammatical operation χ_i. Thus o_1 interprets conjunction and o_2 disjunction. When there is only one lattice or matrix under discussion it is convenient to refer respectively to $,o_1, \ldots, o_4$ by $\wedge, \vee, \Rightarrow$ and $\sim$.

By a *valuation* $\mathfrak{v}_{M}^{Syn}$ relative to a syntax $\mathrm{Syn} = \langle \mathrm{Sen}, \wedge, \vee, \rightarrow, \sim, \rangle$ and a matrix $M = \langle C, D, \wedge, \vee, \Rightarrow, -\rangle$ is meant any homomorphism from Syn to $M = \langle C, \wedge, \vee, \Rightarrow, -\rangle$. The set of all such valuations is referred to by $\mathfrak{Val}_{M}^{Syn}$.

A homomorphism θ from one matrix $M = \langle C, D, \wedge, \vee, \Rightarrow, -\rangle$ to another $M' = \langle C', D', \wedge', \vee', \Rightarrow', -'\rangle$ of like character is said to be a *matrix homomorphism* and to *preserve designation and non-designation* iff for any $a\in C$, $a\in D$ iff $\theta(a)\in D'$. The two matrices $M = \langle C, D, \wedge, \vee, \Rightarrow, -\rangle$ and $M' = \langle C', D', \wedge', \vee', \Rightarrow', -'\rangle$ are *antitonic* iff there is a homomorphism θ from $M = \langle C, \wedge, \vee, \Rightarrow, -\rangle$ to

$M' = <C', \wedge', \vee', \Rightarrow', -'>$ in terms of which $<C, \wedge, \vee>$ and $<C', \wedge', \vee'>$ are antitonic lattices and θ preserves designation and non-designation. The structures are *strictly antitonic* if θ is an isomorphism.

By an *extensional language* L is meant a structure $<\text{Syn},\text{M}>$ such that Syn is a sentential syntax and M is a matrix for Syn. Syn ranges over syntaxes, M over matrices, and L over languages. If $\text{L} = <\text{Syn},\text{M}>$ is a language, then a set X of sentences is said to *semantically entail* a sentence P in L (briefly $X \Vdash_L P$) iff for any $v \in \mathfrak{Val}$, if $\mathfrak{v}$ assigns every sentence in X a value in D, then $\mathfrak{v}(P) \in$ D. A sentence is a *tautology* relative to L (briefly, $\Vdash_L P$) iff $\forall \mathfrak{v} \in \mathfrak{Val}$, $\mathfrak{v}(P) \in$ D.

Boolean algebras are used to interpret classical logic and Łukasiewicz algebras to interpret Łukasiewicz's finitary logics. The logical and proof theoretic notions $\vdash$, tautology, and $\models$ are defined in Section I. Though there are stronger results, it will be sufficient for purposes here to state characterizations of tautologies only.

3.7. *Classical Semantics*

F is a *filter* on a Boolean algebra $<C, \wedge, \vee, -, 0, 1>$ iff, $F \subseteq C$ and $\forall a,b \in C$ ($a \in F$ iff $a \wedge b \in F$); and a filter F is *maximal* iff $\forall a \in C$, not($a \in F$ and $-a \in F$). If $B = <C, \wedge, \vee, -, 0, 1>$ is a Boolean algebra, then by a *Boolean matrix* for Syn determined by B is meant any $MB_C = <C, F, \wedge, \vee, \Rightarrow, ->$ such that F is a maximal filter on B and $a \Rightarrow b = -a \vee b$. $L_C = <\text{Syn}, MB_C>$ is called a *classical language*. In general a Boolean algebra has more than one maximal filter and does not uniquely determine a Boolean matrix. A Boolean matrix, however, is uniquely paired with the Boolean algebra that determines it. In the special case in which C has the cardinality 2, the algebra is called B_{C2}, its matrix MB_{C2}, and its language L_{C2}. Let $\vdash_C$ be the classical deducibility relation as characterized by any standard proof theory. It is well known that a Boolean matrix is statement complete for classical logic.

Theorem For any classical language L_C, $\vdash_C P$ iff $\Vdash_{LC}$ P iff $\Vdash_{LC2} P$.
Proof That $\vdash_C P$ iff $\Vdash_{L_{C2}} P$ is true by induction. That $\Vdash_{L_C} P$ iff $\Vdash_{L_{C2}}$ follows from two facts: (1) the matrix property that if there is an onto matrix homomorphism from one matrix to a second (hence preserving designation and non-designation), then the entailment relations of the two matrices coincide, and (2) (depending on Zorn's lemma) for any x,y of a Boolean algebra B such that not($y \geqslant x$), x is a member of some maximal filter F of B. There is then an onto matrix homomorphism from B to the quotient algebra determined by the two equivalence classes M and its corresponding ideal (F's complement). This quotient algebra, with F designated, is isomorphic to the classical matrix MB_{C2}.

3.8. *Łukasiewicz semantics*

Though Łukasiewicz algebras are truth-functionally incomplete, their valid sentences (unit element designated) as well as those of functionally complete extensions are axiomatizable.[14] Since n-valued Łukasiewicz algebras over sets of truth-values are isomorphic, relative to some rank assignments θ, to Łukasiewicz algebras over any set, the same axiomatizations will apply to the more general notion.

14 See the summary in Rescher 1969.

If $Ł_{|n|} = <C, \wedge, \vee, \Rightarrow, -, 1>$ is a Łukasiewicz algebra then the *logical matrix of cardinality n* determined by $Ł_{|n|}$ is $MŁ_{|n|} = <C, \{1\}, \wedge, \vee, \Rightarrow, ->$. This matrix is unique, and the matrix uniquely determines the algebra that generates it. A *Łukasiewicz language* $LŁ_{|n|}$ is any $<\mathrm{Syn}, MŁ_{|n|}>$ such that Syn is a sentential syntax and $MŁ_{|n|}$ is a matrix relative to Syn and $Ł_{|n|}$. $Ł_{|n|}$ will be identified with its isomorphic image $Ł_n$, and $MŁ_{|n|}$ with the matrix $MŁ_n$.

Though the axioms will not be listed here, by $\vdash_{Ł_n}$ is meant the deducibility relation characterized by a Łukasiewicz' axiomatization for n-valued logic. Such a system is statement complete.

Theorem For any $n = 1, \ldots, \omega$, $\vdash_{Ł_n}$P iff $\Vdash_{Ł_n}$P iff $\Vdash_{Ł_{|n|}}$P.

Proof That $\vdash_{Ł_n}$P iff $\Vdash_{Ł_n}$P is well known. That $\Vdash_{Ł_n}$P iff $\Vdash_{Ł_{|n|}}$P follows from the fact that $Ł_{|n|}$ and $Ł_n$ are isomorphic.

The theorems of $\Vdash_{Ł_n}$ form a system in Łukasiewicz' sense.

3.9. *Intensional Semantics*

It is assumed as facts of nature prior to the specification of a syntax and a semantic theory that there is a set of properties (intensions), and a function κ that pairs each property π with its signification $\kappa(\pi)$, understood as the set of all possible objects in which π inheres.

Definition By an *interpreted intensional language* is meant any $L = <\mathrm{Syn}, M_{\mathrm{Int}}, M_{\mathrm{Sig}}>$ such that Syn is a sentential syntax, M_{Int} and M_{Sig} are matrices for Syn of like character to Syn (M_{Int} and M_{Sig} are respectively *intensional* and *significational structure*) and

1. M_{Int} is a structure on properties,
2. M_{Sig} is an algebra of subsets of possible objects such that κ is a strictly antitonic isomorphism from M_{Int} onto M_{Sig}.

Let $L = <\mathrm{Syn}, M_{\mathrm{Int}}, M_{\mathrm{Sig}}>$ be an intensional language. By an *intensional interpretation* of L is meant any valuation (matrix homomorphism) from Syn into M_{Int}. The intensional interpretations of L are grouped into the set $\mathfrak{Int}_L$ and Int ranges over $\mathfrak{Int}_L$. By *the significational interpretation* of L relative to Int is meant the composition function $\mathrm{Int}{\circ}\kappa$. In L each Int determines a unique significational interpretation $\mathrm{Int}{\circ}\kappa$. Let $\mathfrak{Sig}_L$ be the set of all $\mathrm{Int}{\circ}\kappa$ such that $\mathrm{Int} \in \mathfrak{Int}_L$, and let Sig range over $\mathfrak{Sig}_L$. Let $M_{\mathrm{Sig}} = <\mathfrak{U}, \mathfrak{D}, \wedge, \vee, \Rightarrow, ->$ and $o_{i_{\mathfrak{D}}}$ be defined: $o_{i_{\mathfrak{D}}}(x_1 \cap \mathfrak{D}, \ldots, x_n \cap \mathfrak{D}) = (o_i(x_1, \ldots, x_n)) \in \mathfrak{D}$. Then by $M_{\mathrm{Ext},\mathfrak{D}}$ (called the *extensional matrix determined by* M_{Sig} and $\mathfrak{D}$) is meant $<\{x \cap \mathfrak{D}\}_{x \in \mathfrak{U}}, \{x \cap \mathfrak{D}\}_{x \in D}, \wedge|\mathfrak{D}, \vee|\mathfrak{D}, \Rightarrow|\mathfrak{D}, -|\mathfrak{D}>$. Then $\theta_{\mathfrak{D}}$ such that $\theta_{\mathfrak{D}}(x) = x \cap \mathfrak{D}$ is a onto homomorphism from $= <\mathfrak{U}, \wedge, \vee, \Rightarrow, ->$ to $<\{x \cap \mathfrak{D}\}_{x \in \mathfrak{U}}, \wedge|\mathfrak{D}, \vee|\mathfrak{D}, \Rightarrow|\mathfrak{D}, -|\mathfrak{D}>$ that preserves designation (but not in general non-designation). Hence $\mathrm{Sig}{\circ}\theta_{\mathfrak{D}}$ is a valuation in $\mathfrak{Val}_{M_{\mathrm{Sig},\mathfrak{D}}}$. $\mathrm{Sig}{\circ}\theta_{\mathfrak{D}}$ ($= \mathrm{Int}{\circ}\kappa{\circ}\theta_{\mathfrak{D}}$), for any $\mathfrak{D}$, is called an *extensional interpretation* for L. All such extensional interpretations are grouped into the set $\mathfrak{Ext}_L$ and let Ext range over $\mathfrak{Ext}_L$.

Various levels of entailment for intensional languages are definable. Intensional entailment relative to an intensional interpretation Int is any inference invariant under the extensional interpretations consistent with Int. It in is this sense that *x is red* entails *x is coloured* in English. Entailment relative to a language is any inference invariant over different intensional interpretations over its intensional structure.[15]

- *Intensional Entailment in L.* $X \Vdash_{\mathrm{Int_L}} P$ iff for an extension determination $\theta_{\mathfrak{D}}$ of L ($\forall Q \in X$, $\mathrm{Int} \circ \kappa \circ \theta_{\mathfrak{D}}(Q) \in \mathrm{D_{Ext}}$) only if $\mathrm{Int} \circ \kappa \circ \theta_{\mathfrak{D}}(P) \in \mathrm{D_{Ext}}$.
- *Logical Entailment in L.* $X \Vdash_{\mathrm{L}} P$ iff, for any $\mathrm{Int} \in \mathfrak{Int}_{\mathrm{L}}$, $X \Vdash_{\mathrm{Int_L}} P$.

3.10. *Classical and Łukasiewicz* n-*valued Semantics*

Since the various semantic levels of intensional language tightly mirror one another, classical logic and n-valued Łukasiewicz logic are straightforwardly characterizable in intensional logics by means of Boolean and Łukasiewicz algebras. The theorems below follow directly from the definitions.

Definition 3 An intensional language $\mathrm{L} = <\mathrm{Syn}, \mathrm{MB_{Int}}, \mathrm{MB_{Sig}}>$ is called *Boolean* iff $\mathrm{MB_{Sig}}$ is determined by a Boolean algebra $\mathrm{B_{Sig}} = <\mathfrak{U}, \cap, \cup, 1, 0>$ of sets.

Theorem In a Boolean intensional language it follows that:

1. $\mathrm{MB_{Int}}$ is determined by a Boolean algebra $\mathrm{B_{Int}}$;
2. In $\mathrm{MB_{Sig}}$ and $\mathrm{MB_{Sig,\mathfrak{D}}}$, $\cap$ interprets $\wedge$, and $\cup$ interprets $\vee$;
3. If $\mathrm{MB_{Int}}$ is an algebra of sets, then its maximal element is $\varnothing$, and in $\mathrm{MB_{Int}}$ $\wedge$ (intersection over equivalence class members) interprets $\vee$, and $\vee$ (union over equivalence class members) interprets $\wedge$.

Definition 4 An *n-valued Łukasiewicz language* is any intensional language $\mathrm{L} = <\mathrm{Syn}, \mathrm{MŁ}_{|\breve{n}|\mathrm{Int}}, \mathrm{MŁ}_{|n|\mathrm{Sig}}>$ such that:

1. $\mathrm{MŁ}_{|n|\mathrm{Sig}}$ is the matrix determined by the significational Łukasiewicz algebra on sets $Ł_{|n|} = <\mathrm{C}, \wedge, \vee, \Rightarrow, -, 1>$;
2. $\mathrm{MŁ}_{|\breve{n}|\mathrm{Int}}$ is the matrix determined by the intensional algebra $Ł_{|\breve{n}|} = <\mathrm{C}, \vee, \wedge \Leftarrow, -, 0>$ that is the antitonic image of the significational algebra $Ł_{|n|}$.

Theorem In an n-valued Łukasiewicz intensional language it follows that:

1. In $\mathrm{MŁ}_{|\breve{n}|\mathrm{Int}}$, $\wedge$ interprets the connective $\vee$, $\vee$ interprets $\wedge$, and $\Leftarrow$ interprets $\rightarrow$.
2. In $\mathrm{MŁ}_{|n|\mathrm{Sig}}$ and $\mathrm{MŁ}_{|n|\mathrm{Sig},\mathfrak{D}}$, $\cap$ interprets $\wedge$, $\cup$ interprets $\vee$, and $\Rightarrow$ interprets $\rightarrow$.
3. For any element a of C in $Ł_{|n|}$ and any $[a]^{\cap}\mathfrak{D}$ and $[b]^{\cap}\mathfrak{D}$ of $\mathrm{MŁ}_{|n|\mathrm{Sig},\mathfrak{D}}$, $\mathfrak{r}(a) = \mathfrak{r}([a]^{\cap}\mathfrak{D})$ and $[a]^{\cap}\mathfrak{D} \Rightarrow_{\mathfrak{D}} [b]^{\cap}\mathfrak{D} = ([a]^{\cap}[b])^{\cap}\mathfrak{D}$, and hence $[a]^{\cap}\mathfrak{D} \Rightarrow_{\mathfrak{D}} [b]^{\cap}\mathfrak{D} = \{c | c \in \mathfrak{D} \;\&\; \mathfrak{r}([c]^{\cap}\mathfrak{D}) = \min\{1, 1 - \mathfrak{r}([a]^{\cap}\mathfrak{D}) + \mathfrak{r}([b]^{\cap}\mathfrak{D})\}\}$, and $Ł_{|n|,\mathfrak{D}} = <\{[c]^{\cap}\mathfrak{D}\}_{\mathfrak{D} \subseteq \mathrm{C}}, \cap\mathfrak{D}, \cup\mathfrak{D}, \Rightarrow\mathfrak{D}, -\mathfrak{D}, 1^{\cap}\mathfrak{D}>$ is a Łukasiewicz algebra which determines $\mathrm{MŁ}_{|n|\mathrm{Sig},\mathfrak{D}}$.
4. $\mathrm{MŁ}_n$ is strictly isomorphic to $\mathrm{MŁ}_{|n|\mathrm{Sig}}$ and $\mathrm{MŁ}_{|n|\mathrm{Sig},\mathfrak{D}}$; and in $\mathrm{MŁ}_{|n|\mathrm{Sig}}$, $\mathrm{MŁ}_{|n|\mathrm{Sig},\mathfrak{D}}$ and $\mathrm{MŁ}_n$, $\Rightarrow$ is an n-valued conditional.

Since $\vdash_{Ł_n}$ and $\vdash_{\mathrm{C}}$ are co-extensive respectively with the matrix entailments of arbitrary n-valued Łukasiewicz and Boolean matrices, the logics of the significational matrices are the same as those of their corresponding intensional matrices.[16]

15 Though it is natural to define entailment at yet a further level of abstraction as that which holds under all intensional languages of a certain type, e.g. Boolean or Łukasiewicz languages, this abstraction is not needed here since each instance of a Boolean or Łukasiewicz intensional language is alone sufficient for determining classical or Łukasiewicz n-valued entailment.

Theorem If L is a Boolean intensional language, then $\vdash_{C} P$ iff $\Vdash_{L} P$; and if L is an n-valued Łukasiewicz intensional language, $\vdash_{\text{Ł}_n} P$ iff $\Vdash_{L} P$.

3.11. *Scalar Languages*

The formal exposition is completed by defining a language that incorporates both classical and scalar predicates. The former are interpreted over a 2^n-valued Boolean intensional structure and the latter over its $n+1$-valued Łukasiewicz reduction. The intensional and significational structures of the Boolean language each determine its corresponding Łukasiewicz algebra. Further, since the two Boolean algebras are isomorphic, so are their reductions. Unlike the Boolean algebras, however, which are antitonic, their reductions are both Łukasiewicz algebras and by definition have 1 as their maximal element. Hence as defined they preserve each other's order and are not antitonic. In order for the combined languages to count as intensional, however, the two Łukasiewicz algebras must be antitonic as well. This feature is insured in the extended language by substituting for the intensional reduction $\text{Ł}_{|n+1|}$ its antitonic isomorphic image $\text{Ł}_{|n+1|^{\smile}}$, which as required is strictly antitonic to the Łukasiewicz reduction of significational structure. Though algebraically it matters not at all which of the two antitionic Łukasiewicz structures is viewed as having 1 on top and which 0, the definition given preserves the convention that it is the 'extensional' that has 1 as its maximal value.

Definitions Let n be finite. By an $n+1$*-valued scalar (intensional) language* is meant any $L = \langle L_1, L_2 \rangle$ such that:

1. Syn and Syn′ are disjoint sentential syntaxes;
2. $L_1 = \langle \text{Syn}, MB_{Int}, MB_{Sig} \rangle$ is an 2^n-valued Boolean intensional language;
3. $L_2 = \langle \text{Syn}', M\text{Ł}(B_{Int})_{|n+1|^{\smile}}, M\text{Ł}(B_{Sig})_{|n+1|} \rangle$. (Here $\text{Ł}(B_{Int})_{|n+1|}$ is the Łukasiewicz reduction of B_{Int}, and $\text{Ł}(B_{Sig})_{|n+1|}$ that of B_{Sig}.)

Theorem In an $n+1$-valued scalar (intensional) language:

1. the Łukasiewicz reductions $\text{Ł}(B_{Int})_{|n+1|}$ and $\text{Ł}(B_{Sig})_{|n+1|}$ are strictly antitonic, and (hence)
2. L_2 is an $n+1$ valued intensional Łukasiewicz language.

$M\text{Ł}(B_{Sig})_{|n+1|}$ is identified with its isomorphic image $M\text{Ł}(B_{Sig})_{n+1}$, and similarly $M\text{Ł}(B_{Sig})_{|n+1|}$ and $M\text{Ł}(B_{Sig})_{|n+1|\mathfrak{D}}$ with $M\text{Ł}(B_{Sig})_{n+1}$. Thus, $\langle\langle \text{Syn}, MB_{Int},$

16 For intensional languages it is not true in general that the matrix entailment of M_{Sig} coincides with that of $M_{Ext,\mathfrak{D}}$ because it is not generally the case that there is an onto matrix homomorphism preserving non-desigation from M_{Sig} to $M_{Ext,\mathfrak{D}}$, nor is there is the case of Boolean or Łukasiewicz languages. However, in both Boolean and Lukasiewicz intensional languages, significational and extensional entailments do coincide because each language type is such that the set of designated values of $M_{Sig,\mathfrak{D}}$, meets the conditions for a matrix of that type. In a Łukasiewicz intensional language $\{x \cap \mathfrak{D}\}_{x \in D}$ automatically contains only the $\subseteq$-maximal element of $M\text{Ł}_{Sig,\mathfrak{D}}$ and hence fits the definition for a Łukasiewicz matrix. In a Boolean intensional language $\{x \cap \mathfrak{D}\}_{x \in D}$ is provably a maximal filter on the Boolean significational structure $\langle \{x \cap \mathfrak{D}\}_{x \in U}, \cap_{\mathfrak{D}}, \cup_{\mathfrak{D}}, \Rightarrow_{\mathfrak{D}}, -_{\mathfrak{D}}, U \cap \mathfrak{D}, \varnothing \rangle$.
Theorem If $MB_{Sig,\mathfrak{D}}$ is an extensional structure determined by MB_{Sig} in a Boolean intensional language, and $L = \langle \text{Syn}, MB_{Sig} \rangle$ and $L' = \langle \text{Syn}, MB_{Sig,\mathfrak{D}} \rangle$ are extensional languages, then
$$X \Vdash_{L} P \text{ iff } X \Vdash_{L'} P \text{ iff } X \Vdash_{L_{C2}} P.$$
If $M\text{Ł}_{Sig,\mathfrak{D}}$ is an extensional structure determined by $M\text{Ł}_{Sig}$ in an n-valued Lukasiewicz intensional language, and $L = \langle \text{Syn}, M\text{Ł}_{Sig} \rangle$ and $L' = \langle \text{Syn}, M\text{Ł}_{Sig,\mathfrak{D}} \rangle$ are extensional languages, then
$$X \Vdash_{L} P \text{ iff } X \Vdash_{\text{Ł}_n} P \text{ iff} X \Vdash_{\text{Ł}}.$$

MB_{Sig}>,<Syn′, MŁ$(B_{Sig})_{|n+1|^{\smile}}$, MŁ$(B_{Sig})_{n+1}$>> is an $n+1$ valued scalar language, with MŁ$(B_{Sig})_{n+1}$ being the (distinguished) Łukasiewicz n-valued matrix.

Lastly, Łukasiewicz interpretations are extended to Boolean expressions. Each full branch (containing maximal and minimal elements) of the Boolean property tree is isomorphic to the intensional Łukasiewicz algebra MŁ$^{B_1}_{|n+1|^{\smile}}$. If P is a Boolean expression of L_1, Int_1 is an interpretation of L_1, and Int_2 is an intensional interpretation of L_2, the *Łukasiewicz intensional interpretation of P* relative to Int_1 (briefly $Int_1 \circ \breve{\mathfrak{r}}$) is defined as $1-\mathfrak{r}(Int_1(P))$. P is then given significational and extensional values in L_2 as determined by the intension $Int_1 \circ \breve{\mathfrak{r}}$.

3.12. *Classical Product Logics*

Intensional and scalar languages as defined here are related to Jaśkowski's product logics[17] inasmuch as classical product algebras that are defined so that each dimension conforms to the classical bivalent matrix also meet the defining conditions for a Boolean property structures as developed in this study. Let $M_1,\ldots,M_n$ be matrices of like character, and $<C_i,D_i,o_{1,i},\ldots,o_{4,i}>$ be the i-th matrix in the series. Then $M_1 x\ldots x M_n$ (called M^n if all M_i are identical) is $<C_1 x\ldots x C_n, D_1 x\ldots x D_n, o_1,\ldots,o_4>$ such that $o_i(a_1,\ldots,a_m) = <o_{i,1}(a_1,\ldots,a_m),\ldots, o_{i,n}(a_1,\ldots,a_m)>$.

Recall that M_{C2} is the classical Boolean matrix on $2=\{0,1\}$. Let $L = <L_1,L_2>$ be a scalar language as defined above. Then M_{Int} of L_{C2} is isomorphic to the product structure M^n_{c2}. Proof: to each atomic property y in P, assign a unique n-tuple position, and a characteristic function f from $\mathfrak{P}\mathfrak{p}$ into 2 indicating whether it is an element of any given conjunctive property in $\mathfrak{P}\mathfrak{p}$ taken as the argument of f.

Define a mapping ϕ from the domain $\mathfrak{P}\mathfrak{p}$ of M_{Int} to that of M^n_{c2} as follows: to each compound property π in $\mathfrak{P}\mathfrak{p}$, let $\phi(\pi)$ be $<a_1,\ldots,a_n>$ such that for $i=1,\ldots,n$, $a_i = f(y)$ where y is the primitive property with n-tuple position i and characteristic function f. It follows that ϕ is an isomorphism.

Of special relevance to an understanding of the conceptual motivation for the three-valued Łukasiewicz algebra with matrix MŁ$_3$ is the product matrix M^2_{C2}. M^2_{C2} is isomorphic to the intensional Boolean property structure of any scalar language generated by two primitive properties, and MŁ$_3$ is then the significational/extensional structure of the scalar ordering it determines. For example, MŁ$_3$ is the resulting scalar structure determined by the Boolean intensions of the two primitive properties 'truth/falsity' and 'determined/ undetermined', which have the 'four-valued logic' described in M^2_{C2}. Hence, Łukasiewicz' three-valued logic and its conditional are dictated as the extensional representation of the scalar order of these properties understood as independent Boolean classifications. This framework thus provides an 'explanation' of how both the three-valued ideas in MŁ$_3$ (and its $\Rightarrow$) and four-valued classical product logics have a source in bivalent determinations of truth and determination.

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