Leibniz's Conceptions of Modal Necessity

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ABSTRACT

An inconsistency between two notions of necessity in Leibniz is explored: necessity as true in all cases, as mooted in his logic, and necessity as having a finite proof or “analysis”, as found in the correspondence with Clark, the *Monadology*,and *Theodicy*. Suppose, for the sake of argument, that the two concepts of necessity are coextensive. The logical sense suggests an S5 modal necessity, and entails that for all p, $\Diamond\neg p \models \Box\Diamond\neg p$. The finite proof concept allows three possibilities: a proposition has a proof and is necessary, its negation has a proof and is impossible, or neither. It follows that for some p, $\Diamond\neg p$ and $\neg\Box\Diamond\neg p$. The contradiction is resolved by proposing that the intended notion is provability rather than having a proof, and that such a notion coincides with the concept of completeness in an S5 modal system. The paper concludes by showing that the S5 notion of necessity coincides with provability as completeness in an extension of Chris Swoyer’s intensional logic for Leibniz to an S5 modal system.

1. INTRODUCTION

Leibniz’ account of modality is of interest in two ways. First, there are problems in the interpretation of *necessity* and *provability*. Second, given what he seems to mean, there are problems of consistency. At different times and places he seems to characterize the same idea differently.

What most *logicians* know about Leibniz is that we trace to him the “possible world” truth-conditions for the modal operators. What most *philosophers* know is that Leibniz is a species of compatibilist: he holds that a fact of human action is determined at creation by God but that “fact” is not necessary in the sense that it cannot be finitely proven. What *students of Leibniz’ logic* have observed is that these two accounts of necessity are difficult to reconcile. The unqualified notion that $\Box p$ is true at w iff p is true at every w' appears to determine an S5 modal logic, but the property of “finitely proven” does not conform to S5 metatheorems for necessity.

2. POSSIBLE WORLD TRUTH-CONDITIONS AND FINITE PROOF

Leibniz’s statement of the possible words account is indeed suggestive of a simple S5 type theory.[1] He says, for example,

[1] For discussions of the texts supporting the possible world definitions see H. Posser, *Zur Theorie der Modallebegriffe bei G. W. Leibniz* (Steiner: Wiesbaden, 1969) and Wolfgang Lenzen, “Leibniz’s Logic” in vol. 2 of the *Handbook of the History and Philosophy of Logic,* Dom Gabbay

> The possible is whatever can happen or whatever is true in some in some case;
> The necessary is whatever can not happen or whatever is true in every case.[2]

If in the first sentence "possible" is understood an operator on sentences and if "the possible" is understood as short for truth in the actual world of a sentence governed by this operator, and if "case" is understood as a possible world, then the formula reads like the S5 truth conditions of $\Diamond p$. If similar assumptions are made for the second sentence, it suggests the S5 conditions for $\Box p$.[3] The S5 reading is suggested by the fact that here truth of the modal proposition in a given world is evaluated by reference to all worlds whatever without qualification. In the jargon of modern logic, it is an S5 system because the implied "accessibility relation" is the identity relation, and identity meets the characteristic requirements of an S5 accessibility relation because it is an equivalence relation (reflexive, symmetric, and transitive). In the special case of identity the equivalence relation partitions the set of worlds into a single equivalence class and the S5 truth conditions can be stated without an explicit mention of the accessibility relation:

$V_w(\Box p)$=T iff for any possible world w', $V_{w'}(p)$=T
$V_w(\Diamond p)$=T iff for some possible world w', $V_{w'}(p)$=T

The alleged interpretive difficulty arises from the fact that if his account is really an S5 sort, Leibniz should subscribe to the usual S5 metatheorems. One is troublesome. It is usually stated in terms of the material conditional, but since Leibniz's logic does not use the material conditional[4], it may be stated using entailment. Let $p \models q$ iff, for any set of possible worlds W and any w in W, if $V_w(p)$=T then $V_w(q)$=T. Then the following is an S5 metatheorem:

(0) $\Diamond p \models \Box\Diamond p$, equivalently $\Diamond\Box p \models \Box p$

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and John Woods, eds. (Amsterdam: Elsevier/North Holland, 2004). Leibniz stresses the provability account especially the *Theodicy*. For the argument that the first is essentially an S5 reading of the modalities and that the two accounts are incompatible see Robert Merrihew Adams, "Leibniz Theories of Contingency", in M. Hooker, *Leibniz: Critical and Interpretive Essays* (Univ. of Minnesota Press: Minneapolis, 1982)

[2] *Possible est quicquid potest fieri seu quod verum est quodam … casu, …. Necessarium est quicquid not potest fieri seu quod veru est omni … casu.* See the discussion in Lenzen, p. 46.

[3] Strictly speaking, S5 truth-conditions are formulated in terms of a reflexive, transitive, and symmetric accessibility relation:

$V_w(\Box p)$=T iff for any w′ such that wRw', $V_{w'}(p)$=T
$V_w(\Diamond p)$=T iff for some w′ such that wRw', $V_{w'}(p)$=T

But in the trivial case in which R is the identity relation the simpler formulations are equivalent.

[4] His conditional is closest to Lewis's strong implication. See Lenzen's reconstructions.

The difficult arises in reconciling this metatheorem with the alternative account of necessity in terms of finite proof. Though Leibniz has rather quite precise views about syntax and proof theory, the alleged difficulty arises quite generally from the notion of proof, and turns on the existence of "open questions." Suppose for the sake of argument that a proposition is necessary if and only if it has a proof. It seems that there are some propositions *p* that are neither proven nor refuted. By refutation of *p* let us mean a proof that $\neg p$. That is, for some *p*, the following are true:

(1) $\neg\Box p$ (equivalently $\Diamond\neg p$), and
(2) $\neg\Box\neg p$ (equivalently $\Diamond p$)

Moreover, it also seems possible that open questions may extent to metatheoretic knowledge about these propositions as well. That is, for some *p* both (1) and (2) are open questions. It follows that for some *p,* neither (1) nor (2) is proven:

(3) $\neg\Box\neg\Box p$ (equivalently $\neg\Box\Diamond\neg p$), and
(4) $\neg\Box\neg\Box\neg p$ (equivalently $\neg\Box\Diamond p$)

But (2) and (4) entail

(5) $\Diamond p$ and $\neg\Box\Diamond p$

and (1) and (3) entail

(6) $\Diamond\neg p$ and $\neg\Box\Diamond\neg p$

which are inconsistent with (0).

The standard reconciliation of open questions with necessity is to distinguish between issues of truth and knowledge, but this move will not work here because on the standard account it is perfectly possible for a proposition to be true, even necessary, without our knowing it is so. *A fortiori,* it is possible for a proposition to be necessarily true without our having a proof of it. But this possibility is precluded here because necessity is identified with having a proof.

Modern logic, however, suggests another relatively straightforward reconciliation. This is to construe proof possession dispositionally, not as "having a proof" or as "having been proven", but as "possibly proven" or "provable." There are two ways in which modern semantics captures provability in this sense.

The more familiar is also the more relevant to Leibniz: to capture provability by means of a completeness results. In logistic system that are statement complete those formulas that are logical truths, understood as those true under every interpretation, are exactly those that are provable as theorems of the system. They all in principle have finite proofs, although only a relatively few have been actually proven. Leibniz's notion of necessity may be understood as finitely provability in this sense. Leibniz in fact construes all facts about the actual worlds and all facts about individuals as non-necessary or, in his usage, as contingent. Not only do humans not possess finite proofs of these propositions, they are unprovable in principle. The problem is that they are composed of subject terms that name infinite concepts. As we shall see below, it is impossible in Leibniz's logical systems for such propositions to be to proven from (in Leibniz terminology, "analyzed" to) identity propositions in the way he envisages.

Leibniz' collapse of necessity as "true in all cases" to the finitely provablity, then, may be construed as an anticipation of the modern characterization of valid formula (true in all models) by finite proof systems. Moreover, if the language contains an S5 necessity operator, this operator expresses within the object language itself this semantic sense of necessity.[5]

In what remains of this paper I would like to show how an S5 operator can be introduced into Leibniz own logic in a way that captures both the possible world and probability sense of necessity.

3. Leibniz's Logic of Conceptual Inclusion

There have been several modern reconstructions of Leibniz's logic. Though they reconstruct different texts and emphasize different logical ideas, they all set forth an account of the core common to Leibniz's logical work: that proof is in some sense a finite reduction to "identities." These are very useful in setting out in a clear way – clearer than Leibniz himself – the sort of finite proof theory Leibniz had in mind. The proof theory moreover is open to characterization semantically. Chris Swoyer has provided a model theoretic completeness result in which the provability relation, which is unpacked in terms of a natural deduction system, is shown to coincide with a semantic validly relation defined in the modern manner across "possible interpretations". [6] Swoyer's semantics, moreover, provides the foundation on which to analyze *necessity* in the modern manner as "truth across worlds." If Leibniz really had in mind by necessity what we mean in modern logic, namely truth across possible worlds where possible worlds serve as indices of model theoretic interpretations,

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[5] To do so, (1) relativize valuations to worlds in a model structure consisting of a set of possible worlds and an accessibility relation, (2) require accessibility to be an equivalence relation, (3) use the truth-conditions for $\Box p$ of the previous note, and (4) define $\models p$ as true in all worlds in all model structures. It follows that $\Box$ obeys the S5 axioms, and that the following are equivalent: (1) $\models p$, (2) for every structure and every world w in the structure, $V_w(p)$=T, and (3) for every structure and some world w in the structure, $V_w(\Box p)$=T.

[6] Swoyer, Chris, "Leibniz on Intension and Extension", *Nous*, 29:1 (1995) pp. 96-114, and "Leibniz's Calculus of Real Addition", *Studia Leibnitiana,* 26:1 (1994), pp. 1-30. See also Wolfgang Lenzen, "On Leibniz's Essay '*Mathesis rationis*' (Critical Edition and Commentary)". *Topoi* 9, 1990, 29-59; *Das System der Leibnizschen Logik* (Berlin: de Gruyter, 1990) (Grundlagen der Kommunikation und Kognition); and "Leibniz's Logic, " *op. cit.,* and *Calculus Universalis - Studien zur Logik von G. W. Leibniz*. (Paderborn: Mentis, 2004).

then it should be possible to analyze necessity by "cross worlds" necessity operator within Swoyer's semantics.

To make concrete the relevant senses of finite proof and "possible interpretation", we begin by setting out the core elements of Swoyer's reconstruction. The syntax is constructed from a set of atomic terms, a term operator ⊕ (Leibniz' notation) for conceptual composition, a predicate ≤ for conceptual inclusion, a predicate = for conceptual identity, and the sentential negation operator ~. Here *t≤t′* is read *t is included in t′*, which is read more naturally in its converse or "extensional idiom" as *t′ is t*. (This reversal is similar to Aristotle's syllogistic jargon in which what is more naturally said as *S is P* he says in the converse form *P is said of S.*) We shall retain the usual syntactic distinction based on word order, and call *t* the subject and *t′* the predicate of *t≤t′.* As in the Port Royal Logic singular propositions are viewed as a special case of universal affirmatives. Accordingly, *t≤t′* may be read extensionally as *Every t′ is t.*[7] For simplicity, the structure of concepts, which Swoyer treats abstractly as a meet semi-lattice, will be defined here as a algebra of set.[8]

DEFINITION. A *conceptual syntactic structure* is any <*AConTrmsConTrms,ASProp,Sprop*, ≤,=,⊕,~> such that

1. *AConTrms* (the *atomic concept terms*) is a finite set of spoken words;
2. ⊕ (the *conceptual addition* operator) is a binary operation on spoken words is defined for *AConTrms* and that does not generate loops, either finite or infinite.
3. *ConTrms* (the *finite concept terms*) is the closure of *AConTrms* under ⊕;
4. ≤ (the *conceptual inclusion* predicate) and = (the *conceptual identity* predicate) are binary operations whose domain is set of pairs from Con but with ranges that are sets of words that do not intersect each other or *ConTrms*.
5. *ASProp* (the *atomic spoken propositions*) is the union of the domains of ≤ and =.
6. ~ is a unary operation on *ASProp* with a range that does intersect *AConTerms, ConTrms* or *ASProp,* and that does not generate loops.
7. *Sprop* (the *spoken propositions*) is the closure of *ASProp* under ~.

DEFINITION. A *finite conceptual structure* is any <*ACon,FCon,*{T,F},{T,F},,$\supseteq^{CF}$,$=^{CF}$,∪,*Neg*> such that

1. *ACon* (called the *finitary atomic concepts*) is a finite set of mental modes;
2. <*FCon,* ⊆, *ACon,* ∅> is the Boolean algebra of subsets of *ACon.* (*FCon* is called the family of *finite concepts*; the singleton {*a*} of an atomic concept *a* is also called an *atomic concept*; and when there is no possibility of confusion, {*a*} is identified with *a.*)

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[7] The syntax as it stands cannot express the full syllogistic because it cannot express particular affirmatives. The best it can manage is a stronger expository form. In those cases in which a super concept *M* of both terms in known, *Some S is P* is equivalent to *All M is P* and *All M is P.*
[8] The points made here in terms of sets are straightforwardly generalized into the more abstract framework.

3. $\supseteq^{CF}$ (the *conceptual inclusion* operation) and $=^{CF}$ (the *conceptual identity* operation) are respectively the characteristic functions (from $FCon^2$ to {T,F}) of the relations $\supseteq$ (the converse of set inclusion $\subseteq$ on *FCon*) and = (identity on *FCon*).
4. $\cup$ (the *conceptual addition* operation) is the set theoretic union operation on *FCon*.
5. *Neg* is the negation truth-function {<T,F>,<F,T>}.

Since the set of atomic concepts is finite, it follows that the set of concepts *FCon* is also finite. It also follows by definition that $c \cup c'$ is the $\subseteq$- least upper bound and $\supseteq$-greatest lower bound of $\{ c,c'\}$. Since $\cup$ is a total function defined for every pair in *FCon,* concepts are to be understood as "logically independent" in the sense that for the purposes of concept formation any two concepts can be combined. This abstractness will prove to be important later. At this point it is sufficient to remark that some of these combinations will be incoherent or "impossible" according to Leibniz' scheme for reasons other than logic. They may, for example, violate regularities of natural taxonomy by combining properties from disjoint branches, *e.g.,* a property appropriate to a camel, like on-humped, with that of a bird, like seed-eater. They may also combine natural contraries appropriate to members of a single species, like redness and blueness. They may also violate Leibniz's levels of natural organization that he believes are determined by the integral part-whole relation among substances, *e.g.,* a property appropriate to a substantial part, liker the biliousness of bile, may combine with one appropriate to its substantial whole, like the rationality of a person, to make a concept that is true of no possible single substance.

DEFINITION. A *finitary language* is any pair <*CSyn,FCStrctr*> such that *CSyn* and *FCStrctr* are respectively a conceptual syntax and a finite conceptual structure.

DEFINITION. A *finitary interpretation* relative to a finitary language <*CSyn,FCStrctr*> is any homomorphism $\Im$ is any homomorphism from *CSyn* to *FCStrctr.*

DEFINITIONS. *X semantically entails p relative to a finitary language FL* (briefly, $X \models_{FL} p$) iff for all finitary interpretation $\Im$ of FL, if for all *q* in *X,* $\Im(q)$=T, then $\Im(p)$=T; and *X finitely semantically entails p* (briefly, $X \models_{F} p$) iff for all finitary languages *FL*, $X \models_{FL} p$.

Since Leibniz does not have the material conditional, Swoyer formulates the theory as a natural deduction system. Proofs are finite arrays, which I shall treat as tree structures, each node of which is a deduction $X \vdash p$ that is either a basic deduction or that follows from the nodes above by an inference rule.

DEFINITION. A *deduction of p from X* (briefly $X \vdash p$) is any pair <*X,p*>. A deduction $X \vdash p$ is *basic* iff $p \in X$, *p* is an identity proposition of one of the following forms:

$t \oplus t' = t \oplus' t$
$t \oplus t = t$
$(t \oplus t') \oplus t'' = t \oplus (t' \oplus t'')$

DEFINITION. The *natural deduction rules* =+,=–,≤+,≤– and ~ are defined by display as follows, where $p(t/t')$ is like p except for containing one or more occurs of t' where p contains t, and where p^* ranges over p and $\sim\sim p$, the contradictory opposites of $\sim p$:

=+ $\dfrac{X \vdash p \quad Y \vdash t = t'}{X \vdash p(t/t')}$ =– $\dfrac{X \vdash p \quad Y \vdash t = t'}{X, Y \vdash p(t'/t)}$

≤+ $\dfrac{X \vdash t \oplus t' = t''}{X \vdash t \leq t''}$ ≤– $\dfrac{X, t \oplus t' = t'' \vdash p \qquad Y \vdash t \leq t''}{X, Y \vdash p}$

~ $\dfrac{X, p \vdash q \quad Y \vdash q^*}{X, Y \vdash p^*}$ Th $\dfrac{X \vdash p}{X \cup Y \vdash p}$

DEFINITION. A single node containing a deduction is a proof tree; any tree that results from extending the root nodes of (one or two) proof trees by one of the deduction rules is a proof tree; nothing else is a proof tree.

Definition. $Y \vdash q$ is derivable from $X_1 \vdash p_1, \ldots, X_n \vdash p_n$ iff there is a proof tress with $X_1 \vdash p_1, \ldots, X_n \vdash p_n$ as its leaf nodes and $Y \vdash q$ as its root node. $X \vdash p$ *is finitely provable* or a *theorem* iff it is derivable from nodes that are all basic.

By reference to this core theory it is now possible to explore ways in which necessity and finite provability may be understood as coinciding. Let us here consider one familiar in modern logic: completeness.

4. COMPLETNESS

In the reconstruction necessities are either "identities", *i.e.* basic deductions, or are propositions proven from/reduced to basic deductions in finite proof trees. Below is an example proving the proposition asserting that a concept is included in any complex concept composed from it.

$$\dfrac{\dfrac{\dfrac{\varnothing \vdash t_1 \oplus (t_2 \oplus t_3) = (t_1 \oplus t_2) \oplus t_3 \text{ (basic)} \qquad \varnothing \vdash t_1 \oplus (t_2 \oplus t_3) = (t_1 \oplus t_2) \oplus t_3 \text{ (basic)}}{\varnothing \vdash (t_1 \oplus t_2) \oplus t_3 = t_1 \oplus (t_2 \oplus t_3)}\,{=}{-}}{\varnothing \vdash t_1 \oplus (t_2 \oplus t_3) = (t_1 \oplus t_2) \oplus t_3}\,{=}{-}}{\varnothing \vdash t_1 \leq (t_1 \oplus t_2) \oplus t_3}\,{\leq}{+}$$

This proof is easily generalized to show that $\varnothing \vdash t_1 \leq (t_1 \oplus \ldots \oplus t_n) \oplus t_{n+1}$. The proof is typical. It possible to show proof theoretic theorems of idempotence, associativity, and commutivity for =, thus justifying the dropping of parentheses when convenient. In general $\varnothing \vdash t_1 \oplus \ldots \oplus t_n \leq t_1 \oplus \ldots \oplus t_m$ for $n \leq m$. Also useful are the trivial results that show that = could be eliminated in favor of ≤:

$t \leq t'' \vdash t \oplus t' = t''$

$t \oplus t' = t'' \vdash t \leq t''$

The reconstructed system is argument sound and complete.

THEOREM (Swoyer). $X \models_F p$ iff $X \vdash p$ *is finitely provable.*

It is natural therefore to read Leibniz's claim that all necessary propositions can be reduced in finite way to identifies as a statement in the idiom of his day about the completeness of his reduction rules.

Moreover from the perspective of modern logic it is a relatively simple matter to extend the language so that it would contain the capacity to express its own finitely provable validity relation by means of an S5 necessity operator on formulas. It is worth pausing to do so formally since the issue we are discussing is whether Leibniz' necessity can be understood as that of an S5 modal logic. The syntactic and semantic definitions are extended as follows:

DEFINITION. A S5 *conceptual syntactic structure* relative to a finite conceptual syntactic structure <*AConTrms,ConTrms,ASProp,Sprop*,⊕,≤,=,~> is <*AConTrms,ConTrms,ASProp,*S5*Sprop′*,≤,=,⊕,~,$\Box$> such that ~ and $\Box$ are unary operations on *ASProp* such that

1. their ranges that do not intersect *AconTerms, ConTrms* or *ASProp,*
2. their domains that do not intersect, and that do not generate loops, and
3. S5*Sprop* (the *spoken propositions*) is the closure of *ASProp* under ~ and $\Box$

DEFINITION. An S5 *world system* is a structure <*K,*≤> such that *K* is non-empty (a set of *worlds*) and ≤ is an equivalence relation reflexive, transitive, and symmetric) on *K*.

DEFINITION. A S5 *finitary language* is any pair <S5*CSyn,FCStrctr*> relative to *CSyn* such that S5*CSyn* and *FCStrctr* are respectively an S5 finite conceptual syntax relative to *CSyn* and a finite conceptual structure.

DEFINITION. A S5 *finitary interpretation* relative to a S5 finitary language <S5*CSyn,FCStrctr*> relative to *CSyn,* a world structure <*K,*≤>, and a world *k* in K, is any function $\Im_k$ mapping S5*Sprop* into {T,F} such that $\Im_k(p)$=T.

1. $\Im_k$ restricted to *Sprop* is a homomorphism from *CSyn* to *FCStrctr,* and
2. $\Im_k(\Box p)$=T iff for all k' in K such that $k \leq k'$,
3.

DEFINITIONS. *X semantically entails p relative to an* S5 *finitary language FL* (briefly, $X \models^{S5}_{FL} p$) iff for all S5 finitary interpretation $\Im_k$ of FL, if for all q in X, $\Im_k(q)$=T, then $\Im_k p)$=T; and X *finitely semantically entails p* (briefly, $X \models^{S5}_{F} p$) iff for all finitary languages FL, $X \models^{S5}_{FL} p$.

In the proof theory the set of provable deductions $\vdash$ is expanded to an enlarged set $\vdash^{S5}$ by adding the standard natural deduction rules for S5 to the previous set of rules:

$$\Box\text{+T} \quad \frac{X \vdash \Box p \qquad Y, p \vdash q}{X, Y \vdash \Box q} \qquad\qquad \Box- \quad \frac{X \vdash \Box p}{X \vdash p}$$

$$\Box\text{+S4} \quad \frac{X \vdash \Box p \qquad Y, \Box p \vdash q}{X, Y \vdash \Box q}$$

$$\Box\text{+S5} \quad \frac{X \vdash \Diamond p \qquad Y, \Diamond p \vdash q}{X, Y \vdash \Box q}$$

METATHEOREM.

1. Both $\Box p \vdash \Box\Box p$ and $\Diamond p \vdash \Box\Diamond p$ are provable.
2. $X \vdash^{S5}_{FL} p$ iff $X \vdash^{S5} p$ *is finitely provable.*

PROOF. The second result is an adaptation of standard propositional modal metatheory and will not be proven here. The first is shown by proof trees:

$$\frac{\Box p \vdash \Box p \quad \Box p \vdash \Box p}{\Box p \vdash \Box\Box p}$$

$$\frac{\Diamond p \vdash \Diamond p \quad \Diamond p \vdash \Diamond p}{\Diamond p \vdash \Box\Diamond p}$$