CLASSICAL INDETERMINACY, MANY-VALUED LOGIC, AND SUPERVALUATIONS*

John Martin
Department of Philosophy
University of Cincinnati
Cincinnati, Ohio

I. Introduction

The problem of representing 3-valued supervaluational languages by 2-dimensional product languages is pursued in [3], [8], and [4]. A solution is presented here that attempts to incorporate into such a representation familiar and plausible intuitive principles from traditional many-valued theories. In particular, it is argued in Sections II and III that the concept of classical indeterminacy, which is the key motivational idea behind supervaluation, also underlies the matrices of Jan Łukasiewicz and of S.C. Kleene's strong connectives and that indeterminacy is of quite general importance to the understanding of one tradition in many-valued logic. In Section IV the kinship between supervaluations, on the one hand, and the 3-valued matrices of Łukasiewicz and Kleene, on the other, is developed both in intuitive discussion and in a formal characterization of the former in terms of the latter. In Section V the four-valued semantics of Hans Herzberger [3] is interpreted in terms of classical indeterminacy and is used to express in two dimensions and four values the ideas of the Kleene and Łukasiewicz theories. Then by the characterization of Section IV supervaluations are represented in two dimensions. Finally, in Section VI this representation is shown to be co-extensive to that of Herzberger in [4] and to provide, in effect, a defense of his theory in terms of traditional conceptual foundations.

II. Indeterminacy and Projection by Classical Completion

In his first serious effort to provide an interpretation for the third truth-value, Łukasiewicz writes:

> I maintain that there are propositions which are neither true nor false but <u>indeterminate</u>. All sentences about future facts which are not yet decided belong to this category.... If we make use of philosophical terminology which is not particularly clear, we could say that ontologically there corresponds to these sentences neither being nor nonbeing but possibility. ([7] p. 37)

In addition, he says of such indeterminate propositions that we should "suspend our judgment" (p. 35) about whether they are true. Causal indeterminacy, possibility, and ignorance may all be used to interpret the third value and to explain 3-valued projections of truth-values onto molecular sentences because of a common property: each of these concepts implies that though we may now be barred from a determination of truth or falsity, a determination of truth or falisty is nevertheless possible in principle. If the barrier is epistemological, for example, we may not <u>know</u> whether a sentence is true, but it is true or false nevertheless. If the barrier consists of causal indeterminacy, a sentence describing the indeterminate future may not now be determinate, but the fact it describes either will or will not come about, and in that sense the sentence is either true or false. Similarly if the barrier is metaphysical, a possibility either will or will not be realized.

These readings of the third value go a long way towards determining the projection of values onto molecular sentences. Consider conjunction and disjunction. The resolvability of indeterminacy into determinacy sometimes suffices for assigning a determinate value to a whole with determinate parts. Though one conjunct be indeterminate and the other false, the whole conjunction is false because however the indeterminacy is resolved, in either case the whole conjunction is false. Likewise, no matter how the indeterminacy of a sentence is resolved, whether into truth or falsity, its disjunction with determinately true sentence is always true. The reasoning process behind this projection may be broken down into the following steps. (1) Divide possible cases according to how the indeterminate parts are resolved into classically determinate values. (2) In each of these cases the parts with classical determinate values originally will continue to have these values. (3) Within each case apply classical truth-tables to determine the classical value for the whole within that case. (4) If the whole has the same value in every case, then it has that value determinately; otherwise it is indeterminate. This reasoning process can be summarized by means of the concept of "classical completion" which may be informally understood as any resolution of the indeterminate parts of an expression into determinate values. The third value is then assigned according to the following rule: a sentence is determinate if all its classical completions are and is indeterminate otherwise. Interestingly, the concept of classical completion may be developed in various ways and, as I shall endeavor to show, these developments characterize a family of familiar manyvalued projections.

This family of projections falls along a scale of how much its concept of classical completion recognizes the internal grammatical structure of the parts of sentences to be evaluated. To explain this phenomenon some conventions are needed. A <u>syntax</u> consisting of a set of sentences <u>S</u> constructed from a set of atomic sentences <u>A</u> by syntactic

*I would like to thank Hans Herzberger for comments on an earlier draft of this paper and especially for remarks leading to the definition of a <u>projection</u> in Section V. I would also like to thank Calvin Normore for stimulating discussion on the nature of indeterminacy, particularly that variety concerned with time.

operations Ω may be understood as a structure consisting of A followed by S followed by the various Ω. (A more definite syntax will be stipulated below.) Let A, B, and C range over S, and p and q over A. Let Ω_i $(A_1,\ldots,A_n)$ be the result of applying the i-th operation to sentences $A_1,\ldots,A_n$. As usual, one sentence may occur at more than one place in $\Omega(A_1,\ldots,A_n)$ and may therefore occur more than once in the series $A_1,\ldots,A_n$. Let C be the set of all classical valuations c such that c maps S into {T,F} and conforms to the classical truth-tables for each operation of the syntax. Let v range over 3-valued valuations mapping S into {T,F,N}. Nothing I shall say will be inconsistent with the reading of 'v(A)=N' as 'v is undefined for A'.

To see how classical completions may respect more or less of the syntactical structure of sentential parts, consider the case in which v assigns N to all the parts of a sentence to be evaluated. Let v(A)=v(B)=N. Then what is v(A&B)? Should any assignment of T or F to members of {A,B} be a classical completion to v? If yes, then v(A&B)=N. But what if B=~A? Should there be a classical valuation that ignores the internal structure of B and assigns A and B the same value? Or suppose A contains an atomic part p that is determinate in the original 3-valued valuation v. Should there be a classical completion c that assigns a value to p other than that assigned by v? Various projections have characteristic answers to these questions depending on how much of the internal structure of parts is "sealed-off." Each projection, then, will employ its own sense of sealing-off. The only condition on acceptable sense of sealing-off is that it identify at least one sequence $B_1,\ldots,B_n$ of sentences as a sealing-off for any sequence $A_1,\ldots,A_n$. (This concept of sealing-off is similar to one employed in [5] in connection with Bochvar's external connectives.) If $B_1,\ldots,B_n$ is a sealing-off in the relevant sense for $A_1,\ldots,A_n$, then each B_i $(i=1,\ldots,n)$ will be understood to stand as a proxy for the occurance A_i in the classical calculations for the truth-value of $\Omega(A_1,\ldots,A_n)$. Let i be understood to range over $\{1,\ldots,n\}$ and the qualification $i=1,\ldots,n$ will generally be omitted. Notice that under these very weak restrictions, it is possible for there to be a sense of sealing-off such that $B_1,\ldots,B_n$ may seal-off $A_1,\ldots,A_n$ yet different sentences stand proxy for the same sentence occurring at more than one place in $A_1,\ldots,A_n$. That is, it is possible that $A_i=A_j$ yet $B_i\neq B_j$.

Definition If $B_1,\ldots,B_n$ is a sealing-off of $A_1,\ldots,A_n$ in sense X, then by a classical completion in sense X of v for $A_1,\ldots,A_n$ with sealing-off $B_1,\ldots,B_n$ is meant any $c\in C$ such that if v $(A_i)\in\{T,F\}$, the $c(B_i)=v(A_i)$.

Thus, a sealing-off identifies proxies for the occurances of a sentence's parts and a classical completion resolves indeterminacies by assigning classical values to these proxies. We are now in a position to identify 3-valued projections that conform to the rule of assigning determinate values if all classical completions do. A set V of 3-valued valuations will be said to confirm a sentence A iff $\forall v\in V$, v(A)=T, said to falsify A iff $\forall v\in V$, v(A)=F, and said to neutralize A otherwise.

Definition A language with a projection by classical completion in sense X (briefly, X-val) is any set of valuations v such that v maps A into {T,F,N} and
(1) if for some sealing-off $B_1,\ldots,B_n$ of $A_1,\ldots,A_n$ in sense X and any classical completion c of v for $A_1,\ldots,A_n$ with sealing-off $B_1,\ldots,B_n$, c confirms $\Omega(B_1,\ldots,B_n)$, then $v(\Omega(A_1,\ldots,A_n))=T$,
(2) if for some sealing-off $B_1,\ldots,B_n$ and any classical completion c as in (1), c falsifies $\Omega(B_1,\ldots,B_n)$, then $v(\Omega(A_1,\ldots,A_n))=F$,
(3) $v(\Omega(A_1,\ldots,A_n))=N$ otherwise.

Before proceeding to examples of such languages, let us limit the discussion to a particular syntax PCS=<A, S, ~, &, v, →> for the sake of perspicuity.

III. Kleene and Łukasiewicz

The first classical completion language we shall consider is that which coincides with the valuations generated by Kleene's strong connectives ([6], pp. 334-5). Kleene's projection is characteristically blind to all internal structure of parts and seals them off completely.

Definition A sealing-off in sense K of $A_1,\ldots,A_n$ is any sequence $p_1,\ldots,p_n$ of different atomic sentences.

Thus, not only does the proxy p_i of A_i have no internal structure, no two occurrences of the same sentence can be assigned the same proxy: $p_i\neq p_j$ even if $A_i=A_j$. The motivation for this version of sealing-off derives from examples with logically independent parts. Consider A&B in which neither A nor B logically implies the other in classical logic and $v\in$K-Val. Then if v(A)=v(B)=N, there is, as there intuitively should be, one classical completion c for sealing-off p,q such that c(p&q)=T and another c' such that c'(p&q)=F. Hence v(A&B)=N. Likewise if v(A)=v(B)=N, v(AvB)=N. The reader may easily check that any set K-val projects values in accordance with Kleene's strong matrix K:

	~	&	T	F	N	v	T	F	N	→	T	F	N
T	F		T	F	N		T	T	T		T	F	N
F	T		F	F	F		T	F	N		T	T	T
N	N		N	F	N		T	N	N		T	N	N

The Matrix K

Another example of a notion of classical completion that generates a matrix language is one inspired by Łukasiewicz's treatment of the conditional. In contrast to the strong connectives and his own tables for & and v, Łukasiewicz's table for the conditional takes <N,N> into T. We may speculate that his primary motive was to render A→A always T. A similar intuition concerns conjunction and disjunction. If A→A is always T, we would expect A&~A to be always F and Av~A to be always T.

The matrix which carries through these ideas is that in which & and v are introduced by the usual definitions in terms of →. Łukasiewicz himself employs Kleene's strong tables for & and v, and it is only as a generalization of his table for → that Ł and its motivational framework can be attributed to him.

	~	&	T	F	N	v	T	F	N	→	T	F	N
T	F		T	F	N		T	T	T		T	F	N
F	T		F	F	F		T	F	N		T	T	T
N	N		N	F	F		T	N	T		T	N	T

The Matrix Ł

In order to embrace the intuition that A→A, A&~A, and Av~A should be always T, F, and T, the concept of classical completion must be defined so as to respect the grammatical structure of sentential parts. In the case of A→A, each occurrence of a sentence should be represented by the same proxy in its sealing-off. Then if both occurrences of A were represented by the same atomic p, p→p would be T in all classical completions of v, and thus v(A→A)=T. In the case of A&~A and Av~A the internal structure of parts must constrain classical completions. If A is represented by a proxy p, then one would expect ~A to be represented by ~p. Then every classical completion of v would assign F to p&~p and T to pv~p, and hence v(A&~A)=F and v(Av~A)=T. Thus on these intuitions we may conclude that at the very least sentential parts should be represented as having some structure in the analysis of a classical completion. But how much structure? The answer characteristic of Ł is the very generous principle that proxies may have any structure whatever consistent with our knowledge of classical logic and with our knowledge of the v-values of the various sentiential parts. This constraint follows directly from the intended reading of N as indeterminate. For consider what it is to say v(A)=N. It means that in the world that v describes, A is classically indeterminate. Hence A should not be either a classical tautology or a classical contradiction, for if so, then by inspection we should know its truth-value. But indeterminacy should be a matter of contingent fact. Hence we arrive at the first way in which the interpretation of N constrains the possible structure of proxies: if v(A)=N, then a proxy B for A must not be either classically valid or classically unsatisfiable. A second way in which the interpretation of N constrains the classical structure of sentential parts in that any assignment of a value to A by v must be logically consistent with v's assignment of values to other sentences. The logic applicable here is again classical logic. An inconsistency would arise, for example, if v(A)=N, v(B)=T and B classically entails A. Then we should know that v(A) was T and not N. The ability of a proxy to possess any structure consistent with these two constraints is captured in the following definition of sealing-off. If F is any set of valuations, then let A be F-valid iff ∀f∈F, f(A)=T; let A be F-satisfiable iff ∃f∈F, f(A)=T; and let A F-entail B, briefly $A \models_F B$, iff ∀f∈F, if f(A)=T, then f(B)=T.

Definition (1) B is a sealing-off of A in sense Ł for v iff if v(A)=N then B is C-satisfiable and not C-valid;

(2) B_1, B_2 is a sealing-off of A_1, A_2 in sense Ł for v iff, if either A_i ($i \in \{1,2\}$) is N in v then the other A_j ($j \neq i$) is such that

(a) if $v(A_j)=T$, then B_j is C-satisfiable, not $B_j \models_C B_i$ and not $B_j \models_C {\sim}B_i$,

(b) if $v(A_j)=F$, then ${\sim}B_j$ is C-satisfiable, not ${\sim}B_j \models_C B_i$, and not ${\sim}B_j \models_C {\sim}B_i$,

(c) if $v(A_j)=N$, then B_i and B_j are both neither C-unsatisfiable nor C-valid.

Theorem Any set Ł-Val projects values according to the matrix Ł.

Proof. The case for negation is straight forward and those for &, v, and → have similar proofs. Consider &. Suppose $\ell(A_i)=\ell(A_j)=T$. Then that $\ell(A_1 \,\&\, A_2)=T$ is shown by considering those c, p, and q such that c(p) = c(q) = T. That ℓ is functionally well defined and $\ell(A_1 \& A_2) \neq F$ is shown by the fact that for any B_1, B_2 if $c(B_1)=c(B_2)=T$, $c(B_1 \& B_2)=T$. The proof for the other bivalent cases is similar. Suppose $\ell(A_i)=N$, $\ell(A_j)=T$. Clearly $\ell(A_i \& A_j) \neq T$ for if so for some B_i, B_j and any c, $c(B_i)=T$, $c(B_i \& B_j)=T$, $c(B_j)=T$, and $B_j \models_C B_i$, absurd by 2a. Likewise $\ell(A_i \& A_j) \neq F$ for if so, for some B_i, B_j and any c, $c(B_j)=T$, $c(B_i \& B_j)=F$, $c(B_i)=F$, $c({\sim}B_i)=T$, $B_j \models_C {\sim}B_i$, absurd by 2a. Hence $\ell(A_i \& A_j)=N$. Suppose $\ell(A_i)=N$, $\ell(A_j)=F$. Then, $\ell(A_i \& A_j)=F$, for consider p, q, and c such that $c(p)=c(q)=F=c(p\&q)$. Further $\ell(A_i \& A_j) \neq T$ for if so, for some B_i, B_j and all c, $c(B_j)=F \neq T$. Suppose $\ell(A_i)=\ell(A_j)=N$. Then $(A_i \& A_j)=F$ for consider p and ~p, and any c, $c(p \& {\sim}p)=F$. Further $\ell(A_i \& A_j) \neq T$ for if so, for some B_i, B_j and any c, $c(B_i \& B_j)=T$, B_i and B_j are C-valid, absurd by (2c). QED

IV. Supervaluations

Supervaluations have always been explained by reference to an intuitive concept of classical indeterminacy. Thus in his original paper [12], Bas van Fraassen explains how a supervaluation S assigns T or F, respectively, to A in a situation if all classical valuations for that situation that assign references to non-refering singular term all assign T or F, respectively, to A. The crucial idea is that a state of affairs may not determine a unique classical valuation but rather a set of valuations which agree on determinate sentences and disagree about sentences we do not care about. These "don't cares" must be given T or F in a classical valuation, but since they do not describe a determinate fact various classical valuations capturing what is determined in the situation will disagree about their value. A 3-valued supervaluation represents the situation by dividing sentences into those determinately true, those determinately false, and those which are neither by recording whether all classical valuations assign a sentence T, whether all classical valuations assign it F, or whether some assign it T and others F. I shall first present the usual theory of supervaluations and then go on to show how superlanguages may be developed within the framework of classical completions.

Though in some developments of the theory (as in [12] and [10]) a concept of a partial model representing a state of affairs is first defined and from it a set of classical valuations describing it, I shall follow the more abstract cause of [11], p. 95 and [12], and represent possible situations directly by sets of classical valuations. Since in a given interpretation of supervaluations not every set of classical valuations corresponds to some possible situations, a language will be built up from a subset C*of the set C of all possible classical valuations. A set V of 2-valued valuations will be said to <u>establish</u> a 3-valued valuation f iff f(A)=T if V confirms A, f(A)=F if V falsifies A, and f(A)=N otherwise. If V establishes f, we let $V=E_f$.

<u>Definition</u> A <u>superlanguage</u> for PCS is any <C*,<u>S</u>> such that C*⊆C and there is some family <u>F</u> of subsets of C* such that <u>S</u> is the set of all 3-valued valuations established by members of <u>F</u>.

We say <C*,<u>S</u>> is <u>generated from</u> <u>F</u> when, as in the definition, <u>S</u> is the set of valuations established by members of <u>F</u>.

It is now possible to show that in a very strong sense of equivalent an equivalent theory results if the notion of an establishing set is replaced by a set of classical completions. The relevant sense of classical completion is evident from the fact that all the structure of a sentence's parts and only that structure is admitted in calculating the various classical truth-values of a sentence. Parts stand for themselves and no other sentence may stand for them.

<u>Definition</u> A sealing-off of $A_1,\ldots,A_n$ in sense <u>S</u>* of $A_1,\ldots,A_n$ is $A_1,\ldots,A_n$ itself.

As the proof of the following result shows, sets of classical completions are maximal establishing classes. By the maximal establishing class MEs for s let us mean the set of classical valuations c such that if s(A)ε{T,F}, then s(A)=c(A). Also let $CC_s(A_1,\ldots A_n)$ be the set of classical completions for $A_1,\ldots,A_n$ and and s in sense s*. Also let CC_s be the set of all classical completion <u>simpliciter</u> of s in sense s*: $CC_s=\{CC_s(A_1,\ldots,A_n):\ A_1,\ldots,A_n$ are sentences of PCS}.

<u>Theorem</u> For every superlanguage <C*,<u>S</u>> there exists a set <u>S</u>*-val, and conversely for every set <u>S</u>*-val there exists a superlanguage <C*,<u>S</u>> such that <u>S</u>=<u>S</u>*-val.

<u>Proof</u> Part I. Let <C*,<u>S</u>> be a superlanguage generated from <u>F</u>. Define <u>F</u>'={ME_s: E_sε<u>F</u>}. Clearly <u>S</u>'=<u>S</u>, for <u>S</u>' the set of all valuations established by members of <u>F</u>'. Claim: <u>S</u>' is a <u>S</u>*-val. (1) Clearly ∀sε<u>S</u>', s(p)ε{T,F,N}. (2) Assume $\forall c\varepsilon CC_s(A_1,\ldots,A_n)$, $c(\Omega(A_1,\ldots,A_n))$= T. Now, $ME_s \subseteq CC_s(A_1,\ldots,A_n)$. Hence $c(\Omega(A_1,\ldots,A_n))$=F. Hence, $s(\Omega(A_1,\ldots,A_n))$=F. (3) Likewise, if $\forall c\varepsilon CC_s(A_1,\ldots,A_n)$, $c(\Omega(A_1,\ldots,A_n))$ =F, then $s(\Omega(A_1,\ldots,A_n))$=F. (4) Suppose for some $c,c'\varepsilon CC_s(A_1,\ldots,A_n)$ that $c(\Omega(A_1,\ldots,A_n))$= T and $c'(\Omega(A_1,\ldots,A_n))$=F.. Define f: f(A)=c(A) if $A=A_1,\ldots,A_n$, f(A)=s(A) if s(A)ε{T,F} and $A\neq A_1,\ldots,A_n$, and f(A)ε{T,F} otherwise. Since whenever $A=A_1,\ldots,A_n$, c(A)=s(A), it follows that $f\varepsilon ME_s$. Also $f(\Omega(A_1,\ldots,A_n))=c(\Omega(A_1,\ldots,A_n))$. Likewise define f' like f except for c' in place of c. Then, $f'\varepsilon ME_s$ and $f'(\Omega(A_1,\ldots,A_n))=c'(\Omega(A_1,\ldots,A_n))$=F. Hence $s(\Omega(A_1,\ldots,A_n))$ =N.

Part II. Let S*-val be a classical completion language. Define <u>F</u>={x: for some sε S*-val, x=CC_s} and C*= <u>F</u>. Now, since ME_s= CC_s, sεS*-val iff CC_sε<u>F</u> iff ME_sε<u>F</u> iff sε<u>F</u>. QED Hence the two developments of supervaluations are equivalent in the strong sense that though the method of constructing supervaluations differs in the two theories, the set resulting from either method may be constructed from the other method. (See Herzberger [2] on this concept of equivalence for superlanguages.)

The concept of classical completion has served its purpose in showing the basic conceptual unity underlying K, Ł, and supervaluations. I would like now to set aside the motivational ideas and show that superlanguage projections can be directly characterizable in terms of the matrices K and Ł.

By totally ignoring the structure of sentential parts, K ensures that <N,N> is taken into N with the desirable result that when two sentences A and B are logically independent, A→B, Av~B, and A&~B are N. On the other hand, by allowing a large degree of structure to parts, Ł ensures that <N,N> is taken into determinate values with the desirable result that A→A, Av~A, and A&~A are T, T, and F. Unfortunately, given the constraints of truth-functionality, no matrix theory can combine both desirable features. Supervaluations manage to incorporate both by dropping truth-functionality and recognizing all and only the structure of a sentence's parts. This informal comparison of supervaluations to K and Ł can be developed into a precise characterization of superlanguage projections by stating the conditions under which K and Ł are to apply. Let M_Ω be the truth-function for operator Ω in matrix M.

<u>Definition</u> <u>A 3-valued K∘Ł modal matrix language</u> (a $K\circ Ł^3$) for PCS is any <C*, <u>K∘Ł</u>> such that C* ⊆ C and for some family F of subsets of C*, K∘Ł is the set of all valuations v on {T,F,N} such that for some C'ε<u>F</u>,

(1) for any p, v(p)=T iff C' confirms p, v(p)=F iff C' falsifies p, v(p)=N otherwise,
(2) $v(\sim A)=K_\sim(v(A))=Ł_\sim(v(A))$
(3) $v(A\&B)=K_\&(v(A),\ v(B))$ if {A,B} is C'-satisfiable, $v(A\&B)=Ł_\&(v(A),\ v(B))$ otherwise,
(4) $v(AvB)=K_v(v(A),v(B))$ if {A,B} is not C'-unassailable, $v(AvB)=Ł_v(v(A),\ v(B))$ otherwise,
(5) $v(A\rightarrow B)=K_\rightarrow(v(A),v(B))$ if A does not C'entail B, and $v(A\rightarrow B)=Ł_\rightarrow(v(A),\ v(B))$ otherwise.

The tables for $K\circ Ł^3$ given below are truth-functional except for the case of <N,N>. Conventionally, when <N,N> is assigned two values <x,y> in a table, let x be assigned when the relevant <u>Ω-condition</u> for the connective Ω holds (C'-satisfiability for &, not C'-unassailability for v, and not C'-entailment for →) and y otherwise.

	~	&	T	F	N	v	F	F	N	→	T	F	N
T	F		T	F	N		T	T	T		T	F	N
F	T		F	F	F		T	F	N		T	T	T
N	N		F	F	N,F		T	N	N,T		T	N	N,T

The K∘Ł^3 Projection

<u>Theorem</u> Every superlanguage is a K∘Ł^3 language and conversely.

<u>Proof</u> Let <C*,<u>S</u>> and <C*,<u>K∘Ł</u>> based on <u>F</u> be respectively a superlanguage and a K∘Ł language. It is shown that <u>S</u>=<u>K∘Ł</u>.

Part I. That <u>S</u> <u>K∘Ł</u>. Let $s\epsilon S_i$; then $s\epsilon$<u>K∘Ł</u> relative to $C_s\epsilon$<u>F</u>. For clearly condition (1) for membership of atomic sentences is met. For negation only one case is not obvious. If s(A)=N, then C_s neutralizes A and divides into two classes: C'_s that confirms A and C''_s that falsifies A. Then C'_s falsifies ~A, C''_s confirms A, and C_s neutralizes ~A. For conjunction, representative cases are considered. (1) If s(A)=s(B)=T, then C_s confirms A and B, and confirms A&B; thus s(A&B)=T. (2) If s(A)=T, s(B)=N, then C_s confirms A and neutralizes B, and can be divided into C'_s that confirms B and A&B, and C''_s that falsifies B and A&B. Hence C_s neutralizes A&B, and s(A&B)=N. (3) If s(A)=s(B)=N, there are two cases. (a) If {A,B} is C_s-satisfiable, C_s cannot confirm A&B, for if so s(A&B)=T, s(A)=T, and s(B)=T. Also C_s cannot falsify A&B. Hence C_s neutralizes A&B and s(A&B)=N. (b) If {A,B} is not C_s-satisfiable, then C_s must falsify A&B and s(A&B)=F. Other cases and operators are similar.

Part II. Let $v\epsilon$K∘Ł relative to C'ε<u>F</u>. It is shown that C' completes v and is then in <u>S</u>. Proof is by induction on length of A. Clearly if A is atomic, v completes A by the definition of <u>K∘Ł</u>. It will suffice for the molecular cases to illustrate conjunction, which is done in tabular form. Let 0 mean is falsified by C', 1 is confirmed, and 2 is neutralized.

Possible Cases			By Induction Hypo.		By Matrices
A	B	A&B	v(A)	v(B)	v(A&B)
0	0	0	F	F	F
0	1	0	F	T	F
0	2	2	F	N	N
1	0	0	T	F	F
1	1	1	T	T	T
1	2	2	T	N	N
2	0	2	N	F	N
2	1	2	N	T	N
2	2	0,2	N	N	F,N

QED

K∘Ł^3 languages are called modal because the assignment of vε<u>K∘Ł</u> relative to C'ε<u>F</u> to Ω(A,B) will depend on the values of A and B throughout C'. This intuition can be made precise by dropping &,v,→ from the syntax, adding more usual matrix connectives plus a necessity operator **□**, and introducing &,v, and → by definition. Consider →. First add to the syntax the K and Ł conditionals, call them $\rightarrow_K$ and $\rightarrow_Ł$, and require v to interpret them by the K and Ł tables for the conditional. Then add Bochvar's external connectives **⌐**, **⩓**, and **⇒**. Let v(⌐A)=T if v(A)≠T, and v(⌐A)=F otherwise; v(A⩓B)=T if v(A)=v(B)=T, and v(A⩓B)=F otherwise; v(A⇒B)=F if v(A)=T and v(B)≠T, and v(A⇒B)=T otherwise; let v(□A)=T if ∀cεC', c(A)=T, and v(□A)=F otherwise. It is then possible to introduce A→B by (⌐□(A⇒B)⇒(A$\rightarrow_K$B))⩓(□(A⇒B)⇒(A$\rightarrow_Ł$B)) which has the correct truth-table as the reader may easily check.

V. Indeterminacy in Two Dimensions

Hans Herzberger in [3] has recently advanced a variety of 4-valued semantics that he has shown in [4] to be intimately related to supervaluations: each superlanguage can be represented by a certain kind of 4-valued language. It is the purpose of this section and the next to explore more deeply the kinship between these two approaches by interpreting them as theories of classical indeterminacy.

Herzberger proposes 4-valued valuations w over {T,F,t,f}. The classical truth-values T and F may be read as determinate truth and determinate falsity, and t and f species of indeterminacy. Indeterminate sentences in any situation may be divided into those that if determinate would be true and those that if determinate would be false. These receive t and f respectively. Further elaborations of the semantics ensure that there is such a c and offer an explanation of indeterminateness. On the proposed view of language, a sentence is indeterminate because it fails of presupposition. But even among the indeterminate sentences failing of presupposition we can, as explained before, grasp a distinction between those that would be true and false if determinate. In its most general sense, then, presupposition failure is merely the existence of a barrier - whether it be temporal, modal, epistemological, or semantic - between us and the determination of classical truth-value. Herzberger thus cross categorizes sentences relative to a situation first according to whether they would be true or false and second according to whether their presuppositions are satisfied. The characteristic functions of these two categories, which we shall call v and ν, fully describe the situation. They are also bivalent valuations into, say, {1,0}. It is in terms of these that a 4-valued w is defined yoking together both bits of information: w(A)=<v(A),ν(A)> such that T=<1,1>, F=<0,1>, t=<1,0>, and f=<0,0>. Herzberger further proposes that all such v conform to an underlying bivalent matrix capturing logical relations. This we shall assume to be the matrix C of classical logic, the most plausible candidate for the matrix describing logic. (Let C be understood here to range over {1,0}.) The motivating idea is plausible. Truth and falsity, whether determinate or not, conform to classical intuitions.

The behavior of the second, presuppositional dimension will depend on the particular theory of presupposition employed in defining the 4-valued language. Perhaps the simplest policy, similar to that of Kleene's weak connectives, is to allow presupposition failure of any part to affect the whole; then ν will conform to a matrix: $\nu(\Omega(A_1,\ldots,A_n))=1$ iff $\nu(A_1)=\ldots=\nu(A_n)=1$. But the theories of presupposition failure based on classical indeterminateness - those of K, Ł, and superlanguages - all allow that a whole with some parts failing of presupposition may itself be determinate.

In [3], Herzberger outlines a method for drawing into two dimensions and four values, the presuppositional theories of some 3-valued matrices. (See [9] also.) What I propose to do here is to apply these methods to K, Ł, and their hybrid K∘Ł³. By doing so I will construct a bridge over K∘Ł³ from superlanguages characteristic of K∘Ł³ to their four-valued representation. To draw out the presuppositional theories of K and Ł, ν will assign 1 exactly when K and Ł would assign a determinate value. Let T*=1, F*=1, N*=0, $\overline{\overline{T}}$=T, $\overline{\overline{F}}$=F, and $\overline{\overline{t}}$=$\overline{\overline{f}}$=N.

<u>Definition</u> For any C*⊆C, any family <u>F</u> of subsets of C*, any C'εF, and any cεC', ν <u>is projected from</u> C' and C iff ν is a valuation on {1,0} such that

(1) for any p, ν(p)=1 iff C' confirms or flasifies p,

(2) for any $\Omega(A_1,A_n)$, n≤2,

(a) if the Ω-condition holds, then $\nu(\Omega(A_1,A_n))=[K_\Omega(\overline{\overline{\langle c(A_1),v(A_1)\rangle}}, \overline{\overline{\langle c(A_n),v(A_n)\rangle}})]^*$, and

(b) $\nu(\Omega(A_1,A_n))=[Ł_\Omega(\overline{\overline{\langle c(A_1),v(A_n)\rangle}}, \overline{\overline{\langle c(A_1),c(A_n)\rangle}})]^*$ otherwise.

<u>Definition</u> A <u>4-valued K∘Ł modal matrix</u> language (a <u>K∘Ł⁴</u>) for PCS is any <C*,<u>W</u>> such that C*⊆C and for some family <u>F</u> of subsets of C*,<u>W</u> is the set of all valuations into {T, F, t, f} such that for some cεC*, C'ε<u>F</u> containing c, and ν projected from C' and c, w(A)=<c(A),ν(A)>.

The readily calculable tables for a K∘Ł⁴ read as follows. The tables for the second dimension are given first (the first dimension obeys C) and then those for <u>W</u>.

	~	&	T	F	t	f
T	1		1	1	0	0
F	1		1	1	1	1
t	0		0	1	0,1	0,1
f	0		0	1	0,1	0,1

v	T	F	t	f	→	T	F	t	f
T	1	1	1	1		1	1	0	0
F	1	1	0	0		1	1	1	1
t	1	0	0,1	0,1		1	0	0,1	0,1
f	1	0	0,1	0,1		1	0	0,1	0,1

Projection of the Presupposition Dimension

	~	&	T	F	t	f
T	F		T	F	t	f
F	T		F	F	F	F
t	f		t	f	t,F	f,F
f	t		t	f	f,F	f,F

v	T	F	t	f	→	T	F	t	f
T	T	T	T	T		T	F	t	f
F	T	F	t	f		T	T	T	T
t	T	t	t,T	t,T		T	f	t,F	f,T
f	T	f	t,T	f,T		T	t	t,T	t,T

The K∘Ł⁴ Projection

These tables illustrate that the importance of supervaluation theory is to make sentences which are logically decidable also semantically determinate. By doing so, it allows for an analysis of logical entailment that is stated in terms of determinate truth and is conservative in its emendations of classical inference. It is also clear that $\models_S$ is conservative with respect to classical logic in the sense that if A entails B in C (and hence in C*) it also <u>S</u>-entails B. The K∘Ł⁴ languages follow superlanguages in rendering determinate what is logically decidable. The exact extent to which classical logic is reflected in T in K∘Ł⁴ will be pursued in Section VI. But, in general, 2-dimensional languages do not require dependence on T alone for the purposes of logical theory, in the sense that an adequate logical entailment relation capturing classical logic need not be defined by designating T alone. Indeed, an entailment relation for any K∘Ł⁴ defined by designating both T and t coincides exactly with that of C*. Hence K∘Ł⁴ languages are distinguished from the bulk of 2-dimensional languages by the way in which like supervaluations they require logic to be reflected in T alone.

I now proceed to the mapping of K∘Ł⁴ languages into K∘Ł³ languages.

<u>Definition</u> For any wε<u>W</u> of a K∘Ł⁴ language <C*,<u>W</u>>, the <u>collapse</u> of w is that valuation cw on {T,F,N} such that cw(A)=T if w(A)=T, cw(A)=F if w(A)=F, and cw(A)=N otherwise.

Derivatively, we shall speak of <C*,{cw: wε<u>W</u>}> as the collapse of a K∘Ł⁴ language <C*,<u>W</u>>.

<u>Lemma</u> If <C*,<u>K∘Ł</u>> is the K∘Ł³ language based on <u>F</u>, a family of subsets of C*, and <C*,<u>W</u>> is the K∘Ł⁴ language likewise based on <u>F</u>, then <u>K∘Ł</u>={cw: wε<u>W</u>} and the former is the collapse of the latter.

<u>Proof</u>. It is shown, assuming the antecedent, that <u>K∘Ł</u> = {cw: wε<u>W</u>}. Part I. That <u>K∘Ł</u> {cw: wε<u>W</u>}. Let vε<u>K∘Ł</u> be defined relative to C'εF. It is shown by induction that v=cw, such that w is defined relative to C', some cεC' and some v projected from C' and c. To illustrate the atomic case consider the case in which w(p)=T. Then c(p)=v(p)=1. Hence C' either confirms or falsifies p, but the latter is ruled out because c(p)=1. Hence v(p)=T. The molecular cases will be illustrated by the case of

conjunction, the various subcases of which will be presented in a table.

Subcases			By Hypo.		By Matrix
w(A)	w(B)	w(A&B)	v(A)	v(B)	v(A&B)
T	T	T	T	T	T
T	F	F	T	F	F
T	t	t	F	N	N
T	f	f	T	N	N
F	T	F	F	T	F
F	F	F	F	F	F
F	t	F	F	N	F
F	t	F	F	N	f

Subcases			By Hypo.		By Matrix
w(A)	w(B)	w(A&B)	v(A)	v(B)	v(A&B)
t	T	t	N	T	N
t	F	F	N	F	F
t	t	t,F	N	N	N,F
t	f	f,F	N	N	N,F
f	T	f	N	T	N
f	F	F	N	F	F
f	t	f,F	N	N	N,F
f	f	f,F	N	N	N,F

It is clear that if w(A&B)=T, v(A&B)=T; if w(A&B)=F, v(A&B)=F; and (A&B)=N otherwise. Reasoning in like manner for the other operators, it is concluded $v\varepsilon\{cw: w\varepsilon \underline{W}\}$.

Part II, that $\{cw: w\varepsilon\underline{W}\} \subseteq \underline{K\circ \not Z}$. It is clear that for any w defined relative to $C'_{\underline{s}}$, cw meets the membership conditions for $\underline{K\circ \not Z}$ first because cw assigns values to atomic sentences according to whether C' confirms, falsifies, or neutralizes them, and secondly because identifying t and f in the tables of $K\circ \not Z^4$ yields those of $K\circ \not Z3$.

QED

<u>Theorem</u> Every collapse of a $K\circ \not Z^4$ language is a superlanguage.

<u>Proof</u>. Let <C*,<u>W</u>> on <u>F</u> be a $K\circ \not Z^4$ language and <C*,{cw: wε<u>W</u>}> its collapse. Now define the $K\circ \not Z^3$ language <C*,$\underline{K\circ \not Z}$> on <u>F</u>, which is also a superlanguage. By the lemma {cw: wε<u>W</u>}=$\underline{K\circ \not Z}$.

<u>Theorem</u> Every superlanguage is the collapse of some $K\circ \not Z^4$ language.

<u>Proof</u>. Given a superlanguage <C*,<u>S</u>>=<C*,$\underline{K\circ \not Z}$> on <u>F</u>, define the $K\circ \not Z^4$ language <C*,<u>W</u>> on <u>F</u>. By the lemma, <C*,<u>S</u>> is identical to the collapse <C*,{cw: wε<u>W</u>}> of <C*,<u>W</u>>.

VI. Herzberger's Representation

In this section it is shown that Herzberger's representation of superlanguages presented in [4] is in fact co-extensive to that of section V. This result is somewhat interesting because it is not readily apparent that Herzberger's representation can be explained by a traditional motivation. It is shown, on the contrary, that Herzberger's representation can be used in conjunction with earlier results to elaborate the manner in which $K\circ \not Z^4$ languages follow superlanguages in reflecting classical logic.

<u>Definition</u> For any C*⊊C, family <u>F</u> of subsets of C*, C'ε<u>F</u>, and cεC', C' and c are said to <u>establish</u> v iff v is a valuation on {T,F,t,f} such that
(1) v(A)=T if C' confirms A,
(2) v(A)=F if C' falsifies A,
(3) v(A)=t if C' neutralizes A and c(A)=1,
(4) v(A)=f if C' neutralizes A and c(A)=0.

<u>Definition</u> An <u>H-expansion</u> language for PCS is any <C*,<u>H</u>> such that C*⊊C and there exists a family <u>F</u> of subsets of C* such that <u>H</u> is the set of all valuations established by C'ε<u>F</u> and cεC'.

<u>Theorem</u> The H-expansion language <C*,<u>H</u>> defined relative to <u>F</u> is identical to the $K\circ \not Z^4$ language <C*,<u>W</u>> defined relative to <u>F</u>.

<u>Proof</u>. It is shown that <u>H</u>=<u>W</u>.

Part I, that <u>H</u>⊆<u>W</u>. Let vε<u>H</u>. It is shown that v(A)=<c(A),ν(A)> for some cεC*, some C'ε<u>F</u>, and some ν projected from C' and c. Since vε<u>H</u>, v is established by some C'ε<u>F</u> and cεC'. Now define ν as follows: ν(A)=x iff there is a unique y, such that v(A)=<y,x>. It is claimed that ν is the desired valuation projected from C' and c. That it meets the condition for atomic sentences by the case in which C' confirms p. Then v(p)=<1,1> and ν(p)=1. Other cases are similar. That it meets the defining conditions for molecular sentences is shown by considering representative subcases for conjunction. (1) If v(A)=T and v(B)=t, then C' confirms A and neutralizes B, c(B)=1, c(A&B)=1, C' neutralizes A&B, and v(A&B)=t. (2) If v(A)=t and v(B)=f, then C' neutralizes A and B, c(A)=1, c(B)=0, c(A&B)=0, C' does not confirm A&B. If {A,B} is C'-satisfiable, then C' does not falsify A&B, C' neutralizes A&B, and v(A&B)=f. Otherwise, C' falsifies A&B and v(A&B)=F. In all other cases for conjunction and the other operators the reasoning is similar, showing that v conforms to the $K\circ \not Z^4$ projection. Part II, that <u>W</u>⊆<u>H</u>. Let wε<u>W</u>. Then there is a family <u>F</u> of subsets of C*⊊C, a C'ε <u>F</u>, a cεC', and a ν projected from C' and c. It is claimed that w is established by C' and c. Argument is by induction on length of A. The atomic case is straight forward. The argument for all operators will by illustrated by conjunction the subcases for which will be presented in a tabular form. Let 0 mean is falsified by C', 1 mean is confirmed, 2 mean is neutralized and assigned 1 by c, 3 mean is neutralized and assigned 0 by c.

Possible Cases			By Hypo.		By Matrix
A	B	A&B	w(A)	w(B)	w(A&B)
0	0	0	F	F	F
0	1	0	F	T	F
0	2	0	F	t	F
0	3	0	F	f	F
1	0	0	T	F	F
1	1	1	T	T	T
1	2	2	T	t	t
1	3	3	T	f	f

Possible Cases			By Hypo.		By Matrix
A	B	A&B	w(A)	w(B)	w(A&B)
2	0	0	t	F	F
2	1	2	t	T	t
2	2	2	t	t	t
2	3	3,0	t	f	f,F
3	0	0	f	F	F
3	1	0	f	T	f
3	2	3,0	f	t	f,F
3	3	3,0	f	f	f,F

Clearly, if C' confirms A&B, w(A&B)=T; if C' falsifies A&B, w(A&B)=F; if C' neutralizes A&B and c(A&B)=1, w(A&B)=t; and if C' neutralizes A&B and c(A&B)=0, then w(A&B)=f. QED

In conclusion, I would like to state how, like supervaluations, $K\circ \text{Ł}^4$ and H languages force T to reflect logical theory.

Theorem If A classically entails B, then $A \models_W B$.

Proof. Let A classically entail B and let w(A)=T. Then C' confirms A and c(A)=1, for w defined relative to C' and c. Consider now the supervaluation s such that C_s=C'. Clearly, s=cw. Also since s(A)=T and supervaluation entailment includes classical entailment, s(B)=T. Hence C_s and C' confirm B and c(B)=1. Hence w(B)=T. QED

Finally, I would like to close with some remarks applying the concept of classical completion to a 4-valued language <C*,W> and its collapse into a superlanguage <C*,S> both generated from a family F of subsets of C*. A classical completion of s, as we know, records one of the acceptable resolutions of indeterminate values in s. But as is clear from the definition of H (in an H-expansion), a 4-valued w amounts to a recording of both the information of s plus that from one of its classical completions. Let $W' \subseteq W$ be the set of all w established by C' and some $c' \in C'$ and let s be that supervaluation established by C'. Then trivially, for $w \in W'$, s=cw and w and s share the same classical values and are non-bivalent together. Further, by the distinction between t and f, w records how c' resolves the indeterminateness of s. Thus, we might reasonably say that the classical completion of w has already been carried out in its construction, and that we may define the classical completion of w as that unique c' by which w was established. By hypothesis, both s and w are established by C'. Thus, the establishing class of s and the set of all classical completions of elements of W' are the same. More importantly, if <C*,S> numbers among the superlanguages that are perhaps the most intuitive in that all members of F are maximal establishing classes (all superlanguages defined in terms of partial models are such), then the set of classical completions of s coincided exactly with the set of all classical completions of all elements of W'. Thus, if the set of classically possible worlds is viewed as the "primary" sense of possible world, both Herzberger's languages and the superlanguages they collapse into determine the same concept of possible world in the primary sense. Hence, though it is true as Herzberger remarks that many 4-valued valuations collapse into one supervaluation, there is a sense in which the two theories describe the same set of possible worlds.

References

[1] Hans G. Herzberger, "The Algebra of Supervaluations," typescript, 1974.

[2] ________, "Canonical Superlanguages," Journal of Philosophical Logic, 4(1975), 45-65.

[3] ________, "Dimensions of Truth," Journal of Philosophical Logic, 2(1973), 535-556

[4] ________, "Supervaluations in Two Dimensions" in Proceedings of the 1975 International Symposium on Multiple-Valued Logic, George Epstein et. al., eds. (Bloomington, Indiana: IEEE, 1975), 429-435.

[5] ________, "Truth and Modality in Semantically Closed Languages" in The Paradox of the Liar, R.L. Martin, ed. (New Haven: Yale U.P., 1970), 25-46.

[6] Stephen Cole Kleene, Introduction to Metamathematics, (Amsterdam: North-Holland, 1959).

[7] Jan Łukasiewicz, "On Determinism," in Polish Logic 1920-1939, Storrs McCall, ed. (Oxford: Clarendon Press, 1967), 19-39.

[8] John Martin, Sortal Presupposition, unpublished Ph.D. dissertation (University of Toronto, 1973).

[9] ________, "A Many-valued Semantics for Category Mistakes," Synthese 31 (1975), 63-83.

[10] Richmond H. Thomason, "A Semantic Theory of Sortal Incorrectness," Journal of Philosophical Logic, 1(1972), 209-258.

[11] Bas C. van Fraassen, Formal Semantics and Logic, (New York: Macmillan, 1971).

[12] ________, "Singular Terms, Truth-value Gaps, and Free Logic," Journal of Philosophy, 63(1966), 481-495.