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Leibniz on Intension and Extension

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Leibniz is well-known for his *intensional* interpretation of logic, according to which a subject-predicate sentence is true just in case the concept signified by the predicate is included in the concept signified by the subject. But he also discusses, and sometimes even employs, an *extensional* approach, according to which a subject-predicate sentence is true just in case the set of things in the extension of the subject is included in the set of things in the extension of the predicate. Leibniz has various brief, but interesting, discussions of the relationships between these two approaches, and my aim here is to examine his views on intension, extension, and the connections between them. Among other things, I shall argue that Leibnizian intensions and extensions share a common structure that explains the relationships among the various interpretations he proposes for his logics, that because of this common structure extensions express intensions in Leibniz's important, technical sense of expression, and that Leibniz's views on intension and extension (in conjunction with his views about truth) require that concepts be extensional.

In §1 I sketch Leibniz's intensional and extensional accounts of the truth conditions of subject-predicate sentences. Leibniz most frequently discusses the relationships between these two accounts in his writings on logic, and in §2 I briefly review one of his central logical systems in order to locate his discussion in the formal context in which he typically places it. In §3 I examine Leibniz's version of the principle of the inverse variation of intension and extension and consider its bearing on the extensionality of Leibnizian concepts. In §4 I examine Leibniz's claim that his logical systems are amenable to both intensional and extensional interpretations and in §5 investigate the relationships between these two kinds of interpretations. In §6 I show how the discussion of earlier sections carries over to several of Leibniz's richer logical systems.

1 Leibniz's Intensional and Extensional Accounts of Truth

On Leibniz's standard, *intensional,* account a subject-predicate sentence is true just in case the concept signified or denoted by the predicate is included in the concept signified or denoted by the subject. Leibniz calls this his 'great principle', and a typical statement of it reads

> ... in every true affirmative proposition, necessary or contingent, universal or particular, the concept of the predicate is in a sense included in that of the subject; the predicate is present in the subject. *Praedicatum inest subjecto;* otherwise I do not know what truth is [G.*ii,* 56 = LA, 63; cf. G.*ii,* 43–46 = LA, 46–50; C, 16–17 = PW, 96; C, 518–19 = PW, 87–8; NE, 486].[1]

For example, the universal sentence 'Rectangles are parallelograms' is true just in case the concept of a parallelogram is included in the concept of a rectangle, and the singular sentence 'Adam is human' is true just in case the concept of a human is included in the (individual) concept of Adam [G.*vii,* 240 = LLP, 136; G.*iv,* 436–37 = PPL, 310–11].[2] This doctrine, often called Leibniz's *predicate-in-subject* account of truth, only applies directly to sentences of the forms ⌜*S* is *P*⌝, ⌜*S*s are *P*s⌝, and their variants like ⌜Every *S* is *P*⌝. But the doctrine takes on much wider significance in Leibniz's own work, since he holds that a wide variety of natural-language sentences can be reduced to, or at least paraphrased by, subject-predicate sentences [e.g., C, 244–5 = LLP, 12–13; C, 395 = LLP, 84].

Leibniz believes that Aristotle's logic is intensional [G.*vii,* 215 = LLP, 120; NE, 486; C, 519 = PW, 87; C, 388 = LLP, 77], whereas the logic of the Scholastics is extensional [C, 53 = LLP, 20]. He clearly thinks that the intensional approach is superior, but he acknowledges that the extensional approach can also be useful, and he even adopts it in some of his own work [e.g., C, 193ff = LLP, 95ff; C, 410ff = LLP, 105ff; cf. C, 82 = LLP, 29–30]. On the extensional approach, sentences of the form ⌜*S* is *P*⌝ and their variants are true just in case the extension of the subject is included in the extension of the predicate. Thus, 'Every *B* is *C*' is true just in case 'the individuals belonging to *B* [*individua ipsius B*] are contained in the individuals belonging to *C*', that is, just in case 'all the individuals belonging to *B* are comprehended in the individuals belonging to *C*' [C, 411 = LLP, 105–06; cf. C, 53 = LLP, 20–21; G.*vii,* 216 = LLP, 120; NE, 486]. For today's reader, this is likely to suggest that the set of *B*s is a subset of the set of *C*s, and I will employ a few notions (like that of a subset) from basic set theory in order to streamline the discussion that follows. However, these notions are eliminable in favor of more complex locutions, so this will not anachronistically saddle Leibniz with any substantive views about sets.[3]

Extensional inclusion cannot be the subset relation in the case of singular sentences, however, since the extension of an individual concept is typically an individual (like Adam), rather than a set. We could accommodate this by saying that α is *extensionally included* in the set β just in case α is an individual that is a member of β or α is a set that is a subset of β. But it is simpler to identify the extension of an individual concept with the set whose sole member is the individual falling under that concept (if there is such an individual, and with the empty set otherwise). This means that extensional inclusion can be modeled by the subset relation in all cases (as Leibniz almost suggests at NE, 485). And since this approach can always be translated into the more complex account suggested earlier in this paragraph, I will adopt it here to avoid unnecessary complexity.[4]

Both Leibniz's intensional and extensional accounts of truth are based on inclusion relations. Extensional inclusion is basically our subset relation, but intensional inclusion is less familiar, and in the next section I will begin an examination of its formal properties. Among other things, we will find that intensional and extensional inclusion have a number of structural features in common, which leads Leibniz to treat them as species of a single, generic inclusion relation [G.*vii,* 244–5 = LLP, 141].

2 Leibniz's Logic of Real Addition

Leibniz's discussions of the relationship between intension and extension typically occur in his logical writings, and it will be easier to understand his views if we consider them in this context. I will focus on a logical calculus he develops in a paper, probably written about 1686, in which he discusses the possibility of both intensional and extensional interpretations of his logic [G.*vii,* 236–247 = LLP, 131–144]. This system is a useful one to work with, since it is Leibniz's most polished and detailed logic; furthermore, it forms the core of many of his other systems, so we can concentrate on it with little loss of generality.

The central notions of this logic are the two-place relations, ∞ (identity) and *inesse* (inclusion), and a binary, conjunction-like operation, $\oplus$. Leibniz calls this operation *real addition,* and so I will call this system his *calculus of real addition;* for readability, I will abbreviate '*inesse*' as '$\preccurlyeq$', and I will follow most commentators in writing '=' for Leibniz's '∞'.

Leibniz provides two axioms to ensure that real addition is commutative ($\alpha \oplus \beta = \beta \oplus \alpha$) and idempotent ($\alpha \oplus \alpha = \alpha$). And if his proofs are to work, we also need a third axiom (which he doesn't supply) to ensure that it's associative ($(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$; Frege [1968, §6]; Rescher [1954, 11]). Identity works much as it does in logic nowadays, and α is *included* in β just in case it is a conjunct of β, i.e., just in case there is some χ such that $\alpha \oplus \chi = \beta$ [G.*vii,* 237 = LLP, 132]. Leibniz goes on to prove two theorems (*Propositions* 13 and 14 [G.*vii,* 239 = LLP, 135]) that together yield what I will call

Leibniz's Equivalence: $\alpha \preccurlyeq \beta$ if and only if $\alpha \oplus \beta = \beta$,

and this relationship between inclusion and real addition will be important in what follows.[5]

In Leibniz's intensional interpretations of his system, Roman capital letters signify concepts or, as he also calls them, ideas or terms [C, 243 = LLP, 39; G.*iii,* 224]. By way of example, let 'R' signify the concept *rational,* 'A' the concept *animal,* and 'H' the concept *human*. Then the composite character '$R \oplus A$' signifies the complex concept, *rational animal,* which is the *real sum* of the concepts *rational* and *animal.* The sentence '$R \oplus A = H$' says that the concept *rational animal* is identical with the concept human, and '$A \preccurlyeq H$' says that the concept *animal* is included in the concept *human*.

Leibniz is mindful of the distinction between syntax and semantics [NE, 287], and he stresses that his calculus of real addition is amenable to alternative interpretations, including both intensional and extensional ones [G, *vii,* 245 = LLP, 142; cf. G.*vii,* 240 = LLP, 136; G.*vii,* 223 = LLP, 42; C, 531 = LS, 74–5]. For today's reader, it is natural to construe such interpretations as *models* of Leibniz's formal system, and in order to bring out the abstract, structural features that these interpretations have in common, it will be useful to view Leibniz's axioms for his system as characterizing what I will call *Leibnizian Relational Structures* (as with our use of sets, this machinery is eliminable; see note 9). Leibnizian Relational Structures are ordered triples of the form $\langle \mathscr{C}, \oplus, \preccurlyeq \rangle$, where $\mathscr{C}$ is a nonempty set, $\oplus$ is a binary operation on $\mathscr{C}$ that is commutative, idempotent and associative, and $\preccurlyeq$ is a binary relation on $\mathscr{C}$ such that for all α and β in $\mathscr{C}$, $\alpha \preccurlyeq \beta$ just in case there is some χ in $\mathscr{C}$ such that $\alpha \oplus \chi = \beta$ (or, equivalently, just in case $\alpha \oplus \beta = \beta$).

In Leibniz's extensional interpretations of his logical system, $\mathscr{C}$ is a nonempty family of sets, $\oplus$ is the set-theoretic operation of union (not intersection; see §4 below), and $\preccurlyeq$ is the standard subset relation. But in keeping with his preference for the intensional approach, the *primary* interpretations that Leibniz has in mind for his logical calculus are those in which its characters signify *concepts,* and here $\mathscr{C}$ will be a set of concepts, $\oplus$ a logical operation that forms *real sums* (conjunctions) of concepts, and $\preccurlyeq$ the relation of concept inclusion.

3 Leibniz on the Inverse Variation of Intension and Extension

3.1 Intension and Extension Leibniz would probably have traced the distinction between intension and extension back to Aristotle.[6] However, the influence of this distinction in modern logic stems largely from Arnauld and Nicole's treatment of it the Port Royal Logic (*The Art of Thinking*) of 1662. Not surprisingly, philosophers disagreed about the exact meanings of intension and extension, but when the two notions were construed as features of concepts or ideas (as they were by Leibniz), the extension of a concept was typically taken to consist of the individuals to which the concept applied; e.g., the extension of the concept *human* was held to consist of all individual human beings (which for convenience we may think of as the set of humans). There was less accord about the nature of intension, but it was often agreed that the intension of a concept consists of the subconcepts or attributes or qualities that compose it; e.g., the intension of the concept *human* was frequently said to consist of the concepts *rational* and *animal.* Thus, Arnauld and Nicole held that the extension (*extension*) of an idea comprised the things falling under it and that the intension or, as they called it, the *comprehension* (*compréhension*) of an idea consisted of 'the attributes that it contains' (PRL, Pt. II, ch. 17; cf. PRL, Pt. I, ch. 6].

Leibniz only rarely mentions intension and extension by name (as he does at NE, 486), but he often employs what clearly amount to these notions. On his view, the extension of a concept consists of the individuals that fall under it [C, 411 = LLP, 105–06], and its intention consists of its constituent concepts (in the case of a simple concept, this is just the concept itself), which means those subconcepts that are conjoined by the operation of real addition to produce the concept in question [NE, 486; C, 53 = LLP, 20; C, 82 = LLP, 29–30; G.*vii,* 240 = LLP, 136]. So, fittingly enough, Leibniz's intentional account of truth is based on intensions, while his extensional account is based on extensions.

3.2 Relationships between Intensions and Extensions It gradually became a commonplace among logicians from the late seventeenth to the late nineteenth century that intension and extension were inversely related: the greater the intension of a concept, the smaller its extension, and conversely. This slogan was often formulated rather carelessly, but one reasonably clear version of it boils down to a conditional and its converse. First, if the *intension* of the concept α is included in the *intension* of the concept β, then the *extension* of β will be included in the *extension* of α. Second, if the extension of β is included in the extension of α, then the concept α will be included in the concept β. Although these two theses were sometimes treated as a single principle, clarity will be gained by separating them. To this end, I will call the first, which sanctions the move from a claim about intensions to a claim about extensions, the *IE principle,* and its converse, which sanctions a move from a claim about extensions back to a claim about intensions, the *EI principle.*

The *IE* principle is less controversial than its converse. It is nicely illustrated by an example of Leibniz's: the concept *animal* is included in the concept *man,* and so the set of men is included in the set of animals (G.*vii,* 240 = LLP, 136]. The intuitive idea here is that the *intension* of a compound concept like *rational animal* is "larger" than the intension of the concept *animal* in the sense that it contains more subconcepts. By contrast, the *extension* of the concept *rational animal* is smaller than that of the concept *animal,* since something will belong to the former only if it is already in the extension of the original concept *animal* and also has the *additional* property of being rational. In other words, when we add the concept *rational* to the concept *animal,* the extension of the resulting concept will be smaller than the extension of *animal* alone, since all *non*-rational animals will be subtracted from it.

A qualification is needed, since adding one concept to another doesn't always produce a concept with a smaller extension than the original one. For example, no human beings weigh more than a ton, so adding the concept *weighing less* *than a ton* to the concept *rational animal* won't shrink its extension. Still, whenever the concept α is included in the concept β, β's extension cannot be any larger than α's. Hence, a more circumspect statement of the *IE* principle says that if the concept α is included in the concept β, then the extension of β is included in, or identical with, the extension of α.[7]

The *IE* principle is quite independent of the predicate-in-subject theory of truth. Indeed, all that is really needed to motivate it is a view of concepts (or related intensional entities, like attributes or meanings) as things that are structured in such a way that some of them can be included as conjuncts of others. And many philosophers have endorsed such a picture while denying that the concept of the predicate of a true sentence is always (or, in the case of contingent sentences, ever) included in that of the subject. However, the *IE* principle assumes much greater significance in the context of Leibniz's predicate-in-subject account of truth, since on this account, the concept of the predicate is included in the concept of the subject in *all* true subject-predicate sentences, so that the *IE* principle comes into play for *every* sentence of this form.

3.3 Leibniz on the Relationship between Intensions and Extensions Leibniz holds that intension and extension vary inversely. In a paper of April, 1679 he says:

> ... the concept of gold and the concept of metal differ as part and whole; for in the concept of gold there is contained the concept of metal and something else—e.g., the concept of the heaviest among metals. Consequently, the concept of gold is greater than the concept of metal ... . The Scholastics [who employ the extensional approach] speak differently; for they consider, not concepts, but instances which are brought under universal concepts. So they say that metal is wider than gold, since it contains more species than gold, and if we wish to enumerate the individuals made of gold on the one hand and those made of metal on the other, the latter will be more than the former, which will therefore be contained in the latter as a part in the whole. By the use of this observation, and with suitable symbols, we could prove all the rules of logic by a calculus somewhat different from the present [intensional] one—that is, simply by a kind of inversion of it [C, 53 = LLP, 20].

In another paper written in the same month he says that in his own work in logic:

> ... I do not consider a genus as something greater than the species, i.e., as a whole composed of species, as is commonly done (and not done wrongly, since the individuals of the genus are related to the individuals of the species as whole to part). I consider the genus as a part of the species, since the concept of the species is produced from the concept of the genus and of the differentia [C, 81–82 = LLP, 29–30].

A quarter of a century later he writes in the *New Essays* that

> ... when I say 'every man is an animal' I mean that all the men are included amongst all the animals; but at the same time I mean that the idea [i.e., concept] of animal is

> included in the idea of man. Animal comprises more individuals than man does, but man comprises more ideas or more attributes; ... one has the greater extension, the other the greater intension [*l'un a plus d'extension, l'autre plus d'intension*] [NE, 486; cf. NE, 275].

And he concludes a similar discussion in a fragment of 1690 with the pithy remark that an increase in the concepts or conditions (*conditiones*) means a decrease in the number (of instances) [C, 235].

Leibniz expresses much the same view in his paper on real addition, where he says

> ... all men are contained in all animals, and all animals in all corporeal substances; therefore all men are contained in corporeal substances. On the other hand, the concept of corporeal substance is in the concept of animal and the concept of animal is in the concept of man; for being a man contains being an animal [G.*vii*, 240 = LLP, 136].

And, more generally, the intensional approach

> ... can be inverted, if instead of concepts considered in themselves we consider the individuals comprehended under a concept [*ibid;* cf. G.*vii*, 244 = LLP, 141; G.*vii*, 223 = LLP, 42; C, 384–5 = LLP, 74].

These passages span a period of twenty-five years, so it appears that Leibniz's views on the relationship between intension and extension remained relatively constant during his mature career. Throughout, he maintains that if the concept α is included in the concept β, then the extension of β will be included in (or identical with) the extension of α; moreover, although he says less about the converse of this principle, we will see in a moment that his views also commit him to it.[8]

A bit of notation will help bring out the criss-crossing structural relationships between intensions and extensions. Let ext be a function that assigns a (possibly empty) set of individuals as an *extension* to each concept. Thus, for each concept α, $\mathit{ext}(\alpha)$ is the set of individuals that fall under it (e.g., if '*R*' denotes the concept *red,* $\mathit{ext}(R)$ is the set of red things). There is nothing in the *IE* principle itself that prevents distinct concepts from having the same extension, so we will not require that the function ext be one-one (i.e., we will not require that it assign distinct extensions to distinct concepts).

Using this notation, the *IE* principle says that for all concepts α and β,

IE: If $\alpha \preccurlyeq \beta$, then $\mathit{ext}(\beta) \subseteq \mathit{ext}(\alpha)$.

Now something is in the extension of the conjunctive concept *rational animal* (i.e., of *rational* $\oplus$ *animal*) just in case it is rational *and* also an animal, that is, just in case it is a member of the *intersection* of the set of rational beings and the set of animals [cf. C, 52–53 = LLP, 20–21]. More generally, the extension of a conjunctive concept is the intersection of the extensions of its conjuncts, i.e., it is determined in accordance with the principle that for all concepts α and β,

Ext: $\mathit{ext}(\alpha \oplus \beta) = \mathit{ext}(\alpha) \cap \mathit{ext}(\beta)$.

It is worth nothing that in Leibniz's framework, the *IE* principle follows from just this simple characterization of the extension of a conjunctive concept. To see why, assume that $\alpha \preccurlyeq \beta$. Then by Leibniz's Equivalence (from §2), $\alpha \oplus \beta = \beta$. The sentence $\mathit{ext}(\alpha \oplus \beta) = \mathit{ext}(\alpha \oplus \beta)$ is a logical truth, and so by the substitutivity of identity, $\mathit{ext}(\alpha \oplus \beta) = \mathit{ext}(\beta)$ [G.*vii*, 236 = LLP, 131]. By the principle *Ext,* this is equivalent to $\mathit{ext}(\alpha) \cap \mathit{ext}(\beta) = \mathit{ext}(\beta)$, which by (virtual) set theory yields $\mathit{ext}(\beta) \subseteq \mathit{ext}(\alpha)$, as desired.[9]

Although the IE principle has had many champions, its converse,

EI: If $\mathit{ext}(\beta) \subseteq \mathit{ext}(\alpha)$, then $\alpha \preccurlyeq \beta$.

has inspired less devotion. This is scarcely surprising, since it is typically thought to be a contingent, rather than a logical, matter which individuals are in the extension of a given concept. For example, it might just happen that all cyclists are mathematicians, so that the extension of the concept *being a cyclist* is a subset of the extension of the concept *being a mathematician.* But few philosophers would conclude that the concept *being a mathematician* is in any sense included in the concept *being a cyclist.*

Leibniz agrees that it is often a contingent matter that a given individual is in the extension of a concept α. But for him this means that it would require an infinite analysis to show that α is included in the concept of that individual, and because of his predicate-in-subject theory of truth, he is committed to *EI*. To see why, suppose that $\mathit{ext}(\beta) \subseteq \mathit{ext}(\alpha)$. Given the meaning of extensional inclusion, it follows that every β is an α [C, 411 = LLP, 105–6]. Hence, it is *true* that all βs are αs, and so by the predicate-in-subject account of truth $\alpha \preccurlyeq \beta$.

It follows from the *EI* principle that Leibnizian concepts are extensional, i.e., that concepts with the same extension are identical (so that the extension assignment turns out to be one-one after all).[10] Leibniz held that an individual concept, like the concept of Adam, is complete in the sense that for each concept α, it contains either α or else α's negation. Thus *individual* concepts with the same extension must contain exactly the same subconcepts, since otherwise some subconcept would apply to the individual falling under it (since that subconcept is included in one of the individual concepts) and also fail to apply to it (since it is not included in the other). And when two concepts contain precisely the same subconcepts, they should indeed be identical. But Leibniz doesn't note (and perhaps doesn't recognize) that the identity of *all* coextensive concepts follows from two of his basic principles, namely his predicate-in-subject account of truth (which yields *EI*) and the uncontroversial principle *Ext* (which he tacitly uses quite often). So it is not surprising that the doctrine that coextensional concepts are identical does not play a major role in his philosophy.[11]

4 Intensional and Extensional Interpretations of Leibniz's Logic

One of Leibniz's deepest insights is that logics can be viewed as abstract formal systems that are amenable to alternative interpretations. He is particularly explicit about this in his paper on real addition, where he tells us that his system's inclusion relation can be construed in different ways.

> We say that the concept of the genus is in the concept of the species, the individuals of the species in the individuals of the genus; a part in the whole, and the indivisible in the continuum [G.*vii,* 244 = LLP, 141].

A page later he says of his axioms for real addition that 'whenever these laws are observed, the present calculus can be applied' [G.*vii,* 245 = LLP, 142]. In short, it is the purely formal features of $\preccurlyeq$ and $\oplus$ that are the key to Leibniz's logical calculus.

In Leibniz's intensional interpretations of his system, $\oplus$ is a *conjunction*-like operation on concepts, but in his extensional interpretations, it becomes a *disjunction*-like operation on extensions (in effect, it becomes set-theoretic union). Speaking of a related system that includes the axioms of his calculus of real addition, Leibniz tells us that its theorems are

> ... easily proved from the one assumption that the subject is as it were a container, and the predicate a simultaneous or conjunctive content [this is the intensional interpretation]; or conversely, that the subject is as it were a content, and the predicate an alternative or disjunctive container [*praedicatum ut continens alternativum seu disjunctivum;* this is the extensional interpretation] [G.*vii,* 223 = LLP, 42].

And in discussing the possibility of extensional interpretations of his calculus of real addition, he says

> ... our proofs hold even of those terms which compose something distributively, as all species together compose the genus [G.*vii,* 244 = LLP, 141].

A closer look at the algebraic structure of Leibniz's logical system will clarify the relationships between its various interpretations and explain why this shift to a disjunctive reading of '$\oplus$' is required.

Leibniz proves that the inclusion relation of his calculus of real addition is reflexive ($\alpha \preccurlyeq \alpha$), anti-symmetric (if $\alpha \preccurlyeq \beta$ and $\beta \preccurlyeq \alpha$, then $\alpha = \beta$), and transitive (if $\alpha \preccurlyeq \beta$ and $\beta \preccurlyeq \gamma$, then $\alpha \preccurlyeq \gamma$) [*Propositions* 7, 17, and 15, respectively], and so $\preccurlyeq$ is what is now called a *partial ordering*. Furthermore, a relational structure of the form $\langle \mathscr{C}, \oplus, \leqslant \rangle$ in which $\leqslant$ is a partial ordering and $\oplus$ is an idempotent, commutative and associative operation is what is now known as a *semilattice*. Indeed, given Leibniz's treatment of the relationship between inclusion and real addition, a Leibnizian Relational Structure is a *join* semilattice, with $\oplus$ its *join*.[12]

In modern work on semilattices, Leibniz's Equivalence—'$\alpha \leqslant \beta$ and only if $\alpha \oplus \beta = \beta$'—is often used to *define* the inclusion relation in terms of the join, but we can also define $\oplus$ in terms of inclusion. An element α of a partially ordered set $\langle \mathscr{C}, \leqslant \rangle$ is a *least upper bound* of a set $\mathscr{B} \subseteq \mathscr{C}$ just in case α is an upper bound of $\mathscr{B}$ (i.e., for each β in $\mathscr{B}$, $\beta \leqslant \alpha$) and for each upper bound γ of $\mathscr{B}$, $\alpha \leqslant \gamma$. It is an elementary theorem about semilattices that if $\leqslant$ is a partial ordering and each pair set of elements α and β in $\mathscr{C}$ has a least upper bound, then $\langle \mathscr{C}, \leqslant \rangle$ is a join semilattice with the least upper bound of α and β being their join. We thus require that each set $\{\alpha, \beta\}$ have a least upper bound, and this turns out to be the unique element $\alpha \oplus \beta$. Hence, a Leibnizian Relational Structure is a partially-ordered set in which each pair of elements has a least upper bound, namely their real sum.

In Leibniz's *extensional* interpretation of his logic, inclusion is basically the subset relation, so we can replace '$\leqslant$' by '$\subseteq$' and rewrite Leibniz's Equivalence more perspicuously as '$\alpha \subseteq \beta$ if and only if $\alpha \oplus \beta = \beta$'. And (virtual) set theory assures us that for $\alpha \oplus \beta = \beta$ to hold in the realm of sets, $\oplus$ must be the *disjunction*-like operation of set-theoretic union, rather than the conjunction-like operation of intersection. After all if $\alpha \subseteq \beta$, then $\alpha \cap \beta$ is α, rather than β; however, $\alpha \cup \beta$ is β. Hence, Leibniz's extensional interpretations can be modeled by relational structures of the form $\langle \mathscr{C}, \cup, \subseteq \rangle$, where $\mathscr{C}$ is a nonempty family of sets, $\cup$ is union, and $\subseteq$ the subset relation. By way of example, on such interpretations Leibniz's theorem that α is included in the real sum of α and β means that $\alpha \subseteq \alpha \cup \beta$. It is readily verified that $\subseteq$ is a partial ordering and that the union of any pair of sets in $\mathscr{C}$ forms their least upper bound. Thus, such relational structures are join semilattices, i.e., Leibnizian Relational Structures, and so they obey all the laws of Leibniz's logical system. Hence, letting $\leqslant$ stand for the subset relation and $\oplus$ for union provides an interpretation of Leibniz's calculus of real addition in which $\oplus$ functions as a kind of disjunction, just as he says [G.*vii,* 223 = LLP, 42; G.*vii,* 244 = LLP, 141].[13]

It is also possible to construct extensional interpretations in which $\oplus$ represents the *conjunction*-like operation of set intersection, but this requires a compensatory adjustment in the interpretation of $\leqslant$. Writing $\cap$ for $\oplus$, Leibniz's Equivalence now becomes $\alpha \leqslant \beta$ if and only if $\alpha \cap \beta = \beta$. When $\alpha \cap \beta = \beta$, α will not in general be a subset of β (indeed, it can never be a proper subset of it); however, β *will* be a subset of α. Hence, for $\alpha \leqslant \beta$ to hold exactly when $\alpha \cap \beta = \beta$ does, $\leqslant$ must be the superset relation, $\supseteq$, and so we are dealing with relational structures of the form $\langle \mathscr{C}, \cap, \supseteq \rangle$. The superset relation is the converse of the subset relation, and since $\subseteq$ is a partial ordering, $\supseteq$ is a partial ordering too (this follows from what is known as the principle of duality for partially-ordered sets; cf. e.g., Birkhoff [1967, Ch. 1]). Moreover, basic set theory assures us that the intersection of any two sets in the family of sets $\mathscr{C}$ will be their least upper bound, and so $\langle \mathscr{C}, \cap, \supseteq \rangle$ is a Leibnizian Relational Structure and obeys the laws of Leibniz's logical calculus. Hence, although Leibniz doesn't consider such interpretations, letting $\leqslant$ stand for the superset relation and $\oplus$ for set intersection provides a second kind of extensional interpretation of his calculus of real addition.

Just as duality gives rise to two kinds of extensional interpretations of Leibniz's system, it also yields two kinds of intensional interpretations. Indeed, Leibniz sometimes comes close to noting this in his frequent discussions of the relation of *concept containment.* Concept containment is the converse of concept inclusion, so the concept α contains the concept β (which I will abbreviate $\alpha \geqslant \beta$) just in case $\beta \leqslant \alpha$ [G.*vii,* 237 = LLP, 132]. By the IE principle, the concept β is included in the concept α only if the extension of α is a subset of the extension of β. Hence, if α contains β, then anything falling under α will also fall under β, and so $\geqslant$ is a sort of entailment relation among concepts.[14]

When we treat concept containment ($\geqslant$) rather than inclusion ($\leqslant$) as the ordering relation on concepts, Leibniz's Equivalence takes the form $\alpha \geqslant \beta$ just in case $\alpha \oplus \beta = \beta$. But since we are now working with the *converse* of the inclusion relation, $\oplus$ can no longer be the *conjunctive* operation of real addition; after all, if α contains β, the conjunction of α and β will be α, rather than β. Given the familiar duality between conjunction and disjunction, we might expect $\oplus$ to be a *disjunctive* operation on concepts, so that the extension of $\alpha \oplus \beta$ is now the set of things that are either α or β (or both), and one way to see that this is so is to take $\geqslant$ seriously as a kind of entailment.

In classical propositional logic, the sentence *P* entails the sentence *Q* just in case the disjunction *P or Q* is logically equivalent to the disjunct *Q,* i.e., just in case *P or Q* and *Q* entail each other, and when we turn to an entailment relation on concepts, Leibniz's Equivalence embodies a similar idea. This is so, because a sentence of the form $A \oplus B = B$ is equivalent to the conjunction of $A \oplus B \geqslant B$ and $B \geqslant A \oplus B$ [by Prop 8, G.*vii,* 238 = LLP, 133 and Prop 17, G.*vii,* 240 = LLP, 136]. This means that the concept *A* entails the concept *B* just in case the concepts $A \oplus B$ and *B* entail one another. So on the present, disjunctive, interpretation of $\oplus$, Leibniz's Equivalence says that the concept *A* entails the concept *B* just in case *B* and the disjunctive concept *being either A or B* entail one another. This is precisely how disjunction should behave, and hence the reading of $\geqslant$ as concept entailment yields a reading of $\oplus$ as concept disjunction. Leibniz pays little attention to disjunctions of concepts, and so it isn't surprising that he doesn't discuss such intentional interpretations of his system. Nevertheless, on the disjunctive construal of $\oplus$, it remains an idempotent, commutative, and associative operation, and the present interpretation of Leibniz's logical calculus is a completely legitimate one.[15]

5 Intensions, Extensions, and Leibnizian Expression

In this section I will briefly examine an important structural relationship between intensions and extensions. To this end, let us define an *intensional* Leibnizian Relational Structure as an ordered triple, $\mathscr{L}_I = \langle \mathscr{C}, \oplus, \preccurlyeq \rangle$, where as before $\mathscr{C}$ is a set of concepts, $\oplus$ is the operation of real addition of concepts, and $\preccurlyeq$ is the relation of concept inclusion. And to highlight the structural similarity between the realm of intensions and that of extensions, we will consider *extensional* Leibnizian Relational Structures of the form $\mathscr{L}_E = \langle \mathscr{S}, \cap, \supseteq \rangle$, where $\mathscr{S}$ is a nonempty family of sets, $\cap$ is set intersection, and $\supseteq$ is the superset relation. (The relations $\preccurlyeq$ and $\supseteq$ are definable in terms of $\oplus$ and $\cap$ respectively, but since they play a central role in Leibniz's discussions of logic and truth, I will include them explicitly.)

The notion of a *structure preserving mapping* is a central concept of modern algebra, and in the present context it turns out that an assignment of extensions to concepts is just such a mapping. To see what this means, note that an extension assignment $\mathscr{ext}$ is a function from a set of concepts $\mathscr{C}$ to an appropriately structured family of sets $\mathscr{S}$, which serve as their extensions. Moreover, this function *preserves* the join and the ordering of an intensional structure $\mathscr{L}_I$, i.e., for all concepts α and β in $\mathscr{C}$

(i) $\mathscr{ext}(\alpha \oplus \beta) = \mathscr{ext}(\alpha) \cap \mathscr{ext}(\beta)$, and
(ii) if $\alpha \preccurlyeq \beta$, then $\mathscr{ext}(\alpha) \supseteq \mathscr{ext}(\beta)$.

Clause (i) is just the condition *Ext* that was seen to govern the extensions of conjunctive concepts in §3.3, and clause (ii) is equivalent to the *IE* principle. This means that an assignment of extensions to concepts is the sort of structure-preserving mapping that is now known as a *(join) homomorphism,* and when there is such a mapping from $\mathscr{L}_I$ to $\mathscr{L}_E$, the two structures are said to be *(join) homomorphic.*

When such a mapping exists, the defined operations and relations in $\mathscr{L}_I$ will also be preserved under *ext*. Indeed, *IE* falls out of *Ext* precisely because $\preccurlyeq$ and $\supseteq$ are defined in parallel ways, which means that $\preccurlyeq$ is preserved by $\supseteq$. Similarly the defined relation of containment ($\succcurlyeq$) in $\mathscr{L}_I$ is preserved by the subset relation of $\mathscr{L}_E$, and the operation of intensional disjunction of §4 is preserved by union.

The set of empty concepts in $\mathscr{C}$ (those whose extensions are the empty set, $\varnothing$) is closed under the inclusion relation, i.e., if $\alpha \preccurlyeq \beta$ and $\mathscr{ext}(\alpha) = \varnothing$, then $\mathscr{ext}(\beta) = \varnothing$. Similarly, the set of universal concepts is closed under the converse of $\preccurlyeq$. Furthermore, these points carry over to extensions of Leibniz's system where empty and universal concepts are of more interest than they are in his calculus of real addition. For example, if we add the negation operation discussed in §6, we can show that the concept $\alpha \oplus \mathscr{Not}(\alpha)$ has an empty extension [cf. C, 368 = LLP, 58].

Although Leibniz doesn't have the explicit concept of a homomorphism, he does have something very like the notion of a structure preserving mapping or function. Not only did he introduce the terminology of functions into mathematics, but his important concept of *expression* (which figures most prominently in his doctrine that each monad expresses the entire universe) involves a correspondence between things that requires just the sort of preservation of relations found in the structure-preserving mappings familiar in algebra and model theory today. In a paper of 1678 that probably contains his most extended discussion of expression, he characterizes it this way:

> That is said to express a thing in which there are relations which correspond to the relations of the thing expressed [G.*vii*, 263 = PPL, 207; cf. G.*ii*, 112 = PW, 71; C, 15 = PW, 176–177; LH, 80–1; GM.*v*, 141].

This suggests that one thing expresses a second just in case there is a structural similarity between the relations holding among the constituents of the first thing and the relations holding among the constituents of the second, as there is when one can be mapped to another by a structure-preserving mapping like a homomorphism (Swoyer, [forthcoming-b]). It is in this sense that relations like $\cap$ and $\supseteq$ (among extensions) 'correspond to relations' like $\oplus$ and $\preccurlyeq$ (among concepts), with the result that a semilattice of extensions *expresses* its associated semilattice of concepts.

Expression is important because the similarity of structure between an object and its expression allows us to reason about the latter in order to draw conclusions about the former. As Leibniz puts it,

> What is common to all these expressions is that we can pass from a consideration of the relations in the expression to a knowledge of the corresponding properties of the thing expressed [G.*vii*, 263 = L, 207; cf. G.*vii*, 264 = PPL, 208; C, 154–155 = LS, 14].

Leibniz doesn't note the point explicitly, but in the presence of *EI*, distinct concepts have distinct extensions. Hence, the structures of extensions directly mirror the structures of intensions, and so we could always use the extensions of concepts as surrogates in reasoning about the concepts themselves. Indeed, *IE* alone (without its converse) would often allow such reasoning. And this would justify the use of extensional logic while retaining the primacy of intensional accounts of truth and entailment.

6 Intension and Extension in Leibniz's Richer Logics

Leibniz's logical calculus of real addition is limited in scope, but there is good evidence that he regards it as a core system that can be (conservatively) extended in various ways. Indeed, he suggests several extensions in his paper on real addition [G.*vii,* 245–46 = LLP, 142–44; cf. C, 256], and a number of his other logics include this calculus as a subsystem (cf. note 5). I will close by examining the ways in which three such extensions would affect my discussion in previous sections of this paper.

6.1 Additional Operations on Concepts In other works Leibniz often employs an operation for negating concepts, and I will consider it as a case study in how additional logical operations on concepts would affect the relationships between intensions and extensions.[16] Leibniz's axioms for negation vary from paper to paper, but since my concern is with the general way in which additional operations would affect our earlier discussion, I will only consider his most common axiom, $A = \mathcal{N}ot\text{-}\mathcal{N}ot\text{-}A$ [C, 379 = LLP, 69; C, 396–7 = LLP, 84–6; C, 235 = LLP, 90].

Let $\mathcal{C}omp$ be the standard set-theoretic operation of (relative) complementation and let $\mathcal{N}ot$ be the operation of concept negation (which is governed by the requirement that for each concept α, $\alpha = \mathcal{N}ot(\mathcal{N}ot(\alpha))$). The negation of the concept *happy* is the concept *not-happy,* and something falls under the latter just in case it does *not* fall under the former [C, 86 = PE, 11–12; cf. C, 356 = LLP, 47; C, 390 = LLP, 79; NE, 276]. Thus the homomorphism condition for negation

$$ext(\mathcal{N}ot(\alpha)) = \mathcal{C}omp(ext(\alpha)),$$

should govern the extensions of negative concepts.

The addition of the negation operation yields additional logical principles like $\mathcal{N}ot(\alpha) \preccurlyeq \mathcal{N}ot(\beta)$ only if $ext(\alpha) \subseteq ext(\beta)$.[17] However, the presence of negative concepts does not undo the fact that the inclusion relation can still be characterized in terms of $\oplus$ by Leibniz's Equivalence; it is still true that α is in β just in case α and (perhaps) some other concepts completely make up β. Moreover, an individual is still in the extension of a conjunctive concept exactly when it is in the extension of both of its conjuncts, and so we still have *Ext,* and with it the *IE* principle. Finally, Leibniz's predicate-in-subject account of truth still yields the converse of this principle, namely *EI,* by the argument in §3.3. In short, new operations like concept negation yield additional logical principles, but they do not undermine any of the results discussed in previous sections.

6.2 Modality Leibniz is sometimes credited with analyzing necessary truth as truth in all possible worlds. Although textual evidence for this attribution is harder to come by than one might suppose, we can work modality into the picture by thinking of each concept as having an extension at each possible world. To this end, let ext^* be a binary function that assigns extensions to concepts at possible worlds in conformity with the natural model analogue of *Ext,* namely that for each world w, $ext^*(\alpha \oplus \beta, w) = ext^*(\alpha, w) \cap ext^*(\beta, w)$ (we also require that the extensions of a concept at distinct worlds be pairwise disjoint to accommodate Leibniz's view that no individual exists in more than one world). This modal version of *Ext* can then be used to show that the *IE* principle holds within each world, and a straightforward adaptation of the argument of §3.3 yields the intra-world version of *EI*. But although this extension of Leibniz's ideas now seems a natural one, it should be stressed that possible worlds in fact play little role in his own discussions of intension, truth, and modality.

6.3 Propositions as Concepts Perhaps a more interesting extension of Leibniz's treatment of intension and extension is suggested by his claim that propositions can be treated as terms (i.e., as concepts [C, 243 = LLP, 39; G.*iii*, 224]) and that entailment relations among propositions amount to inclusion relations among their associated terms:

> If the proposition *A is B* is treated as a term, ... there arises an abstract term, namely *A's being B,* and if from the proposition *A is B* the proposition *C is D* follows, then from this there is made a new proposition of this kind: *A's being B* is, or contains, *C's being D,* i.e., The *B-ness of A* contains the *D-ness of C* ... [C, 389 = LLP, 78, italics added; cf. C, 382 = LLP, 71].

I will use λ as a term-forming operation that transforms the proposition P into the term $\lambda[P]$, so that, for example, λ[Adam is human] is the concept or term *Adam's being human.* Then Leibniz's point amounts to the claim that P entails Q just in case $\lambda[Q]$ is included in $\lambda[P]$. The latter holds just in case $\lambda[P]$ contains $\lambda[Q]$, and so P entails Q just in case $\lambda[P]$ contains $\lambda[Q]$.

Now P entails Q just in case $\lambda[Q] \preccurlyeq \lambda[P]$, so the *IE* principle together with its converse assure us that P entails Q just in case $\mathit{ext}(\lambda[P]) \subseteq \mathit{ext}(\lambda[Q])$. Leibniz does not say enough about propositions as concepts or terms for us to be certain what (if anything) the extension of a propositional term would be. However, the extension of a standard concept or term is just the set of things to which it applies, and by a natural extrapolation of this idea, the extension of a propositional term would be the set of things, namely situations or worlds, to which it applies. Hence, in the non-modal case the extension of a propositional term would be the set containing the actual world (if its associated proposition is true) or else the empty set (if its associated proposition is false). And in the modal case the extension of a propositional term would be the set of worlds at which its associated proposition is true.

It follows from EI that coextensional concepts are identical, so if $\mathit{ext}(\lambda[P]) = \mathit{ext}(\lambda[Q])$, then $\lambda[P] = \lambda[Q]$. In the non-modal case, this has a number of unhappy consequences, e.g., that all propositions with the same truth value entail each other. Things are somewhat better in the modal case, where we are only forced to identify propositional terms whose associated propositions have the same truth value at all possible worlds, but this still collapses distinctions among propositional terms whose associated propositions are logically equivalent.

In the absence of its converse, however, the IE principle has more promising consequences. In particular, it tells us that if P entails Q, then $\mathit{ext}(\lambda[P]) \subseteq \mathit{ext}(\lambda[Q])$. In other words, P entails Q only if the set of situations in which P is true—its "truth set"—is included in the set of situations in which Q is true. In the non-modal case, the two admissible extensions of a propositional term are the set containing the actual situation and the empty set, and these can be modeled (or, in Leibniz's terminology, *expressed*) by the two-element Boolean algebra whose domain consists of the truth values *true* and *false* that we now use in standard propositional logic. And in the modal case, the extension of a propositional term is the set of worlds at which its associated proposition is true, so that we can represent or express a proposition by the set of worlds at which it is true, as is often done in modal logic today. Of course Leibniz does not actually develop his ideas in this way, but when we combine his views about intension and extension with his treatment of propositions as terms, we are well on the way to such an account.[18]

Notes

[1]I will use the abbreviations in the left margin of the bibliography in citing primary works.

[2]Thus the sense in which Leibniz's logics are intensional is rather different from that in which recent, Kripke-style intensional logics are. The latter give completely *extensional* truth (or satisfaction) conditions for subject-predicate formulae (and, indeed, for all formulae free of intensional operators), and intensionality only enters the picture because the semantic value (at a given possible world) of a formula containing such operators is determined by the semantic values of related formulae at other possible worlds.

[3]x is a *subset* of y ($x \subseteq y$) just in case x and y are sets and all of the members of x are also members of y; x is a *proper* subset of y just in case it is a subset of, but not identical with, y. In note 9, we will see how to express Leibniz's claims about extensions in Quine's theory of virtual classes [1963, pp. 15–21], in which talk of sets is a convenient but dispensable shorthand.

[4]Leibniz sometimes holds that a subject-predicate sentence is false if its subject lacks a denotation [C, 393 = LLP, 82]. One way to work this into his extensional account of truth is to hold that $\ulcorner S$ is $P\urcorner$ is true just in case S has a non-empty extension (or, if S is a proper name, just in case it has a denotation) that is included in the extension of P. But Leibniz's claim that sentences with non-denoting subject terms are false doesn't accord well with his intensional account of truth, since concepts with empty extensions often include other concepts [C, 53 = LLP, 20]. For example, the concept of a parallelogram would be included in the concept of a rectangle even if there were no rectangles, and so 'rectangles are parallelograms' would still be true. But in general Leibniz is not very concerned with non-denoting terms, and since he doesn't provide for them in his discussions on intension and extension, I won't pursue the matter here.

[5]As with many of his other works on logic, Leibniz never published his paper on real addition. Although his treatment of real addition in his present paper is more detailed than in his other works, both the operation and his axioms for it occur in many other writings. For example, the commutivity axiom occurs at G.*vii,* 222 = LLP, 40; C, 235 = LLP, 90; C, 412 = LLP, 93; and C, 421, and the idempotence axiom at G.*vii,* 222 = LLP, 40; C, 260; C, 262; C, 366 = LLP, 56; C, 396 = LLP, 85; C, 235 = LLP, 90; C, 412 = LLP, 93; C, 421; and G.*vii,* 230 = LLP, 124. One or both of these axioms, along with treatments of identity and inclusion similar to those in his present paper, occur in several other works where Leibniz discusses the relationships between intension and extension, including G.*vii,* 222 = LLP, 40; C, 356ff = LLP, 47ff, and NE, 292. A fuller discussion of Leibniz's calculus of real addition and his applications of it may be found in Swoyer [forthcoming-a]. In §4.1 of that paper I note some evidence that Leibniz viewed real addition as a multigrade operation, i.e., one capable of joining any number of concepts in a single stroke, but this possibility won't affect the basic issues discussed here.

[6]Leibniz sometimes says that the intension of the genus is included in the intension of the species,

whereas the extension of the species included in the extension of the genus [G.*vii*, 244 = LLP, 141; C, 82 = LLP, 29–30]. This is reminiscent of Aristotle's remark that when we consider the parts of a definition, the genus is included in the species, but that we can also regard the species as included in the genus (*Metaphysics*, 1023b 22].

[7]This qualification was proposed by J. N. Keynes in the late nineteenth century [1884, p. 37]. However, we will see that it is already present in Leibniz's account, since both his intensional and extensional inclusion relations subsume improper inclusion, i.e., α is included in α (*Proposition* 7 [G.*vii*, 238 = LLP, 133]).

[8]The labels 'intension' and 'extension' are often credited to Sir William Hamilton, but Leibniz uses their French counterparts over a century earlier to mark essentially the same distinction that later writers did [NE, 486]. Intension and extension are closely related to Mill's *connotation* and *denotation,* DeMorgan's *force* and *scope,* Peirce's *depth* and *breadth* (a pair of terms that really was first suggested by Hamilton), Kant's *content* (*Inhalt*) and *extension* (*Umfang*), and Frege's *concept* ((*Begriff*), whose subparts are *Merkmale*) and *extension* (*Werthverlauf*). Although the principle of inverse variation was espoused by such luminaries as Kant [1974, p. 101], Mill [1843, Bk 1, ch 7, §5], and Peirce, [1984, pp. 76ff], interest in it waned in the twentieth century as the traditional notion of intension faded out of logic.

[9]We can avoid sets by rewriting *IE* as the claim that if the concept α is included in the concept β, then all βs are αs and rewriting *Ext* as the claim that something is in the extension of the conjunctive concept $\alpha \oplus \beta$ just in case it is in the extension of α and also in the extension of β. All other uses of set-theoretic notions in this paper can be eliminated in similar fashion, and we can avoid the use of relational structures by more intricate paraphrases. For example, in §2 we interpreted Leibniz's calculus of real addition over a Leibnizian Relational Structure, $\langle \mathscr{C}, \oplus, \preccurlyeq \rangle$, but we could instead say that on Leibniz's intentional interpretations, names denote concepts, '$\oplus$' stands for the operation of real addition, and '$\preccurlyeq$' stands for the relation of concept inclusion. However, the use of relational structures will greatly facilitate discussion later, when we consider the structural relationships between intensions and extensions.

[10]In the context of *Ext* (the straightforward characterization of the extension of conjunctive concepts) it is routine to prove that *EI* is true if and only if $\mathscr{ext}$ is one-one, though only the entailment from left to right is relevant here. To see why it holds, assume that *EI* is true, i.e., assume that for all concepts γ and δ, $\mathscr{ext}(\gamma) \subseteq \mathscr{ext}(\delta)$ only if $\delta \preccurlyeq \gamma$. Now suppose that for some arbitrary pair of concepts, α and β, $\mathscr{ext}(\alpha) = \mathscr{ext}(\beta)$. To prove the extension assignment is one-one, we must show that $\alpha = \beta$. By hypothesis, $\mathscr{ext}(\alpha) = \mathscr{ext}(\beta)$, and so by (virtual) set theory $\mathscr{ext}(\alpha) \subseteq \text{ext}(\beta)$. We then invoke *EI* to conclude that $\beta \preccurlyeq \alpha$. Similarly, $\mathscr{ext}(\alpha) = \mathscr{ext}(\beta)$ entails $\mathscr{ext}(\beta) \subseteq \mathscr{ext}(\alpha)$, and a second application of *EI* yields $\alpha \preccurlyeq \beta$. But $\beta \preccurlyeq \alpha$ and $\alpha \preccurlyeq \beta$ together entail $\alpha = \beta$ (as Leibniz proves in *Prop* 17 [G.*vii*, 240 = LLP, 136]). Hence, if *EI* holds, then $\mathscr{ext}$ is one-one and concepts that in fact have the same extension are identical. This foreshadows Frege's view two centuries later that concepts with the same extension stand in that relation between concepts which corresponds to the relation of identity among objects [1984, p. 200].

[11]Most commentators take Leibniz's relation of coincidence (∞) to be identity, but in two important papers Castañeda [1976], [1990] argues that it is a weaker congruence relation on concepts. I think Ishiguro [1990, Ch. 2] is right that Castañeda's reading is difficult to reconcile with Leibniz's insistence on the centrality of identities in logic and his frequent characterizations of coincidence as *identity* or *sameness*. But perhaps the fact that any pair of coextensive concepts stand in the ∞-relation provides some support for Castañeda's reading, since it may seem more plausible to hold that coextensive concepts are always congruent than to hold that they are always identical.

[12]We could express this without relational structures by saying simply that Leibniz's concepts have the structure of a join semilattice. Like most of his logics, Leibniz's calculus of real addition has only one binary operation, and so a Leibnizian Relational Structure may lack a meet and thus not be a full-fledged lattice (much less a Boolean algebra). In Leibniz's primary interpretations of his calculus, its binary operation works like a conjunction. In familiar cases where lattices are now of logical relevance (e.g., algebras of sets, propositional Boolean algebras), conjunction-like operations (e.g., set intersection, truth-functional conjunction) yield *meets* rather than *joins*. This is so because conjunction-like entities resulting from the application of such operations are "included" in the items to which the operation is applied to obtain them (e.g., $\gamma \cap \delta \subseteq \gamma$). However, intensional approaches invert this picture, so that conjuncts are included in conjunctions ($\alpha \preccurlyeq \alpha \oplus \beta$).

[13]Just as the intersection of the sets α and β ($\alpha \cap \beta$) is the set whose members are precisely the

things that are in α *and* in β, the union of α and β ($\alpha \cup \beta$) is the set whose members are precisely the things that are in α *or* in β, which is why unions behave much like disjunctions. To avoid excessive use of quotation marks, I will henceforth follow the convention of autonymous use, letting each simple expression stand for itself and each juxtaposition of expressions stand for their concatenation.

[14]Such a relation among concepts or properties has been of interest to a number of philosophers, arguably beginning with Plato. It involves what Vlastos [1974] calls *Pauline Predication*, and he argues that when Plato says that fire is hot, he means that the Forms Fire and Heat are necessarily related in such a way that anything exemplifying the former must also exemplify the latter.

[15]In his pioneering study of Leibniz's logics, C. I. Lewis [1918, p. 17] notes the possibility of both sorts of extensional interpretations discussed in this section, and Kneale & Kneale [1962, pp. 343–44] note the two sorts of intensional interpretations. However, none of these writers discuss these interpretations in any detail or examine the algebraic relationships among them.

[16]Leibniz also occasionally mentions disjunction and a type of conjunction in which order is relevant [C, 532 = LS, 75; G.*vii*, 246 = LLP, 143–44; C, 556–57; cf. G.*vii*, 206 = M, 20; C, 256].

[17]The relative complement (on a domain $\mathscr{D}$) of the set α is the set of things in $\mathscr{D}$ that are not in α. To see why $\mathscr{N}eg(\alpha) \preccurlyeq \mathscr{N}eg(\beta)$ only if $\mathscr{ext}(\alpha) \subseteq \mathscr{ext}(\beta)$, suppose that $\mathscr{N}eg(\alpha) \preccurlyeq \mathscr{N}eg(\beta)$. By *IE* this yields, $\mathscr{ext}(\mathscr{N}eg(\beta)) \subseteq \mathscr{ext}(\mathscr{N}eg(\alpha))$. By the homomorphism condition for negation, this is equivalent to $\mathscr{C}omp(\mathscr{ext}(\beta)) \subseteq \mathscr{C}omp(\mathscr{ext}(\alpha))$, which by basic set theory entails $\mathscr{ext}(\alpha) \subseteq \mathscr{ext}(\beta)$.

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