RICHARD MONTAGUE*

PRAGMATICS AND INTENSIONAL LOGIC

The word 'pragmatics' was used in Morris [1] for that branch of philosophy of language which involves, besides linguistic expressions and the objects to which they refer, also the users of the expressions and the possible contexts of use. The other two branches, syntax and semantics, dealing respectively with expressions alone and expressions together with their reference, had already been extensively developed by the time at which Morris wrote, the former by a number of authors and the latter in Tarski [1].

Morris' conception of pragmatics, however, was programmatic and indefinite. A step towards precision was taken by Bar-Hillel, who suggested in Bar-Hillel [1] that pragmatics concern itself with what C. S. Peirce had in the last century called *indexical expressions*.[1] An indexical word or sentence is one of which the reference cannot be determined without knowledge of the context of use; an example is the first person pronoun 'I'. Indexical sentences can be produced in various ways, for instance, by using tenses. Consider 'Caesar will die'. This sentence cannot be considered either true or false independently of the context of use; before a truth value can be determined, the time of utterance, which is one aspect of the context of use, must be specified.

Though Bar-Hillel suggested that pragmatics concern itself with indexical expressions, he was not wholly explicit as to the form this concern should take. It seemed to me desirable that pragmatics should at least initially follow the lead of semantics – or its modern version, model theory[2] – which is primarily concerned with the notions of truth and satisfaction (in a model, or under an interpretation). Pragmatics, then, should employ similar notions, though here we should speak about truth and satisfaction with respect not only to an interpretation but also to a context of use.

These notions I analyzed some years ago in connection with a number of special cases, for instance, those involving personal pronouns, de-

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monstratives, modal operators, tenses, probability operators, contextual ambiguity, and direct self-reference.[3] An important feature of many of these analyses was a treatment of quantifiers due largely to my student Prof. Nino Cocchiarella, and persisting in the general development below.[4]

In each special case, however, truth and satisfaction had to be defined anew; in particular, no unified treatment of operators was seen. Intuitive similarities existed; but full formal unity was not achieved until 1965, and then it came about through joint work of Dr. Charles Howard and myself.

Let me sketch the general treatment. By a *pragmatic language* is understood a language of which the symbols (atomic expressions) are drawn from the following categories:

(1) the logical constants $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\bigwedge$, $\bigvee$, $=$, E (read respectively 'it is not the case that', 'and', 'or', 'if ... then', 'if and only if', 'for all', 'for some', 'is identical with', 'exists'),

(2) parentheses, brackets, and commas,

(3) the individual variables $v_0, \ldots, v_k, \ldots$,

(4) individual constants,

(5) n-place predicate constants, for each natural number (that is, nonnegative integer) n, and

(6) operators.

(The individuals to which such a language refers will be regarded as possible objects; accordingly, the symbol E will occur in such contexts as $\mathsf{E}[x]$, which is read 'x exists' or 'x is actual'. I consider under (6) only what might be called *1-place operators*. These are symbols which, like the negation sign, generate a sentence when placed before another sentence; examples are the modal operators 'necessarily' and 'possibly', as well as the expressions 'it will be the case that', 'usually', and 'it is probable to at least the degree one-half that'. Purely for simplicity I have disallowed operation symbols, descriptive phrases, and many-place operators; but an extension of the present treatment to accommodate such expressions would be completely routine. Indeed, many-place operators can be expressed in both extended pragmatics and intensional logic, which are considered below; and a partial theory of descriptive phrases occurs within intensional logic.)

The *formulas* of a pragmatic language L are built up exactly as one would expect. To be explicit, the set of formulas of L is the smallest set Γ such that (1) Γ contains all expressions

$$\mathsf{E}[\zeta],$$
$$\zeta = \eta,$$
$$P[\zeta_0, \ldots, \zeta_{n-1}],$$

where each of ζ, η, $\zeta_0, \ldots, \zeta_{n-1}$ is an individual constant of L or an individual variable and P is an n-place predicate constant of L, (2) Γ is closed under the application of sentential connectives, (3) $\bigwedge u\phi$ and $\bigvee u\phi$ are in Γ whenever u is an individual variable and ϕ is in Γ, and (4) $N\phi$ is in Γ whenever N is an operator of L and ϕ is in Γ.

To interpret a pragmatic language L we must specify several things. In the first place, we must determine the set of all possible contexts of use – or rather, of all complexes of relevant aspects of possible contexts of use; we may call such complexes *indices*, or to borrow Dana Scott's term, *points of reference*. For example, if the only indexical features of L were the presence of tense operators and the first person pronoun 'I', then a point of reference might be an ordered pair consisting of a person and a real number, understood respectively as the utterer and the moment of utterance.

In the second place, we should have to specify, for each point of reference i, the set A_i of objects present or existing with respect to i. For example, if the points of reference were moments of time, A_i would be understood as the set of objects existing at i.

In the third place, we should have to specify the meaning or *intension* of each predicate and individual constant of L. To do this for a constant c, we should have to determine, for each point of reference i, the denotation or *extension* of c with respect to i. For example, if the points of reference were moments of time and c were the predicate constant 'is green', we should have to specify for each moment i the set of objects to be regarded as green at i. If, on the other hand, c were an individual constant, say 'the Pope', we should have to specify, for each moment i, the person regarded as Pope at i.

The fourth thing we must provide is an interpretation of the operators of L. To do this we associate with each operator of L a relation between points of reference and sets of points of reference. The role played by such relations, as well as the intuitive reasons for regarding them as interpreting operators, can best be discussed later.

In order to be a bit more precise about interpretations, let us introduce a few auxiliary notions. Understand by a $\langle U_0, \dots, U_{n-1}\rangle$-*relation* a subset of $U_0 \times \cdots \times U_{n-1}$ (by which we intend the Cartesian product $\prod_{i<n} U_i$ of the sets $U_0, \dots, U_{n-1}$), and by an $\langle I, U_0, \dots, U_{n-1}\rangle$-*predicate* a function from the set I into the set of all $\langle U_0, \dots, U_{n-1}\rangle$-relations. (I use the word 'relation' for a possible candidate for the extension of a predicate constant, while 'predicate' is reserved for the intension of such a constant. Consider the special case in which $n=1$. Then the $\langle U_0\rangle$-relations will coincide with the sets of elements of U_0, the $\langle I, U_0\rangle$-predicates are what we might regard as *properties* (indexed by I) of elements of U_0, and both will correspond to 1-place predicate constants. In case $n=0$, we should speak of Λ-relations (where Λ is the empty sequence, that is, the empty set); and these are the subsets of the empty Cartesian product, which is of course $\{\Lambda\}$. Thus the only Λ-relations will be the empty set Λ and its unit set $\{\Lambda\}$; let us think of these two objects as the truth-values F and T respectively. The corresponding predicates are $\langle I\rangle$-predicates; and they will be functions from the set I to truth-values, that is, what we might regard as *propositions*[5] indexed by I.)

By a *k-place relation among members* of a set U and by a *k-place I-predicate of members of U* are understood a $\langle U_0, \dots, U_{k-1}\rangle$-relation and an $\langle I, U_0, \dots, U_{k-1}\rangle$-predicate respectively, where each U_p (for $p<k$) is U.

DEFINITION I. A *possible interpretation for a pragmatic language* L is a triple $\langle A, F, R\rangle$ such that (1) A is a function, (2) for each i in the domain of A, A_i is a set (I use the notations 'A_i' and '$A(i)$' indiscriminately for function value), (3) F is a function whose domain is the set of predicate and individual constants of L, (4) whenever c is an individual constant of L, F_c is a function whose domain is the domain of A and such that, for all j in the domain of A, $F_c(j)$ is a member of the union of the sets A_i for i in the domain of A, (5) whenever P is an n-place predicate constant of L, F_P is an n-place $\mathbf{D}A$-predicate of members of the union of the sets A_i (for $i \in \mathbf{D}A$), where $\mathbf{D}A$ is the domain of A, (6) R is a function whose domain is the set of operators of L, and (7) whenever N is in the domain of R, R_N is a $\langle \mathbf{D}A, \mathbf{SD}A\rangle$-relation, where $\mathbf{SD}A$ is the power set (set of all subsets) of $\mathbf{D}A$.

A few remarks are perhaps in order in connection with this definition.

Let $\mathfrak{A}$ be a possible interpretation for a pragmatic language L, and let $\mathfrak{A}$ have the form $\langle A, F, R\rangle$. We understand the domain of the function A to be the set of all points of reference according to $\mathfrak{A}$. If i is a point of reference, A_i is understood as the set of objects existing with respect to i (according to $\mathfrak{A}$). The union of the sets A_i for i in $\mathbf{D}A$ is thus what we might regard as the set of all possible individuals (according to $\mathfrak{A}$). By the definition above, an individual constant denotes a *possible* individual, and a 1-place predicate constant a set of *possible* individuals, with respect to a given point of reference. To see that it would be overly restrictive to demand that the respective denotations be an individual that exists with respect to the given point of reference or a set of such individuals, suppose that the points of reference are instants of time, and consider the individual constant 'the previous Pope' and the predicate constant 'is remembered by someone'. A similar point can be made in connection with predicate constants of more than one place. Consider, for instance, the 2-place predicate constant 'thinks of' (as in 'Jones thinks of Jove'). Under a standard interpretation of which the points of reference are possible worlds, the extension of this constant with respect to a given world would be a relation between individuals existing in that world and possible individuals (that is, objects existing in some world).[6]

The notions central to pragmatics, those of *truth* and *satisfaction*, are expressed by the phrases 'the sentence (that is, formula without free variables) ϕ is true with respect to the point of reference i under the interpretation $\mathfrak{A}$' and 'the possible individual x satisfies the formula ϕ with respect to the point of reference i under the interpretation $\mathfrak{A}$', which we may abbreviate by 'ϕ is $\text{true}_{i,\mathfrak{A}}$' and '$x\ \text{sat}_{i,\mathfrak{A}}\,\phi$' respectively. The following clauses do not constitute definitions of truth and satisfaction, but are rather to be regarded as true assertions exhibiting the salient features of those notions; the full definitions will be given later.

CRITERIA OF PRAGMATIC TRUTH AND SATISFACTION. Let $\mathfrak{A}$ be a possible interpretation, having the form $\langle A, F, R\rangle$, for a pragmatic language L; let $i \in \mathbf{D}A$; let x be a member of the union of the sets A_j (for $j \in \mathbf{D}A$); let P be a 2-place predicate constant of L; and let u be an individual variable. Then:

(1) $P[c, d]$ is $\text{true}_{i,\mathfrak{A}}$ if and only if $\langle F_c(i), F_d(i)\rangle \in F_P(i)$;

(2) $x\ \text{sat}_{i,\mathfrak{A}}\ P[c, u]$ if and only if $\langle F_c(i), x\rangle \in F_P(i)$;

(3) $x\ \text{sat}_{i,\mathfrak{A}}\ c = u$ if and only if $F_c(i)$ is identical with x;

(4) x sat$_{i,\mathfrak{A}}$ $\mathsf{E}[u]$ if and only if $x \in A_i$;

(5) if ϕ is a sentence of L, then $\neg\phi$ is true$_{i,\mathfrak{A}}$ if and only if ϕ is not true$_{i,\mathfrak{A}}$;

(6) if ϕ, ψ are sentences of L, then $(\phi \wedge \psi)$ is true$_{i,\mathfrak{A}}$ if and only if both ϕ and ψ are true$_{i,\mathfrak{A}}$;

(7) if ϕ is a formula of L of which the only free variable is u, then $\bigvee u\phi$ is true$_{i,\mathfrak{A}}$ if and only if there is an object y in the union of the sets A_j (for $j \in \mathbf{D}A$) such that y sat$_{i,\mathfrak{A}}$ ϕ;

(8) if ϕ is a sentence of L and N an operator of L, then $N\phi$ is true$_{i,\mathfrak{A}}$ if and only if $\langle i, \{j : j \in \mathbf{D}A$ and ϕ is true$_{j,\mathfrak{A}}\}\rangle \in R_N$.

According to (8), $N\phi$ is true at i (under $\mathfrak{A}$) if and only if i bears the relation R_N to the set of points of reference at which ϕ is true (under $\mathfrak{A}$). To see that (8) comprehends the proper treatment of, for example, the past tense operator, consider an interpretation $\mathfrak{A}$ in which $\mathbf{D}A$ is the set of real numbers (that is, instants of time) and R_N is the set of pairs $\langle i, J\rangle$ such that $i \in \mathbf{D}A$, $J \subseteq \mathbf{D}A$, and there exists $j \in J$ such that $j < i$. Then, by (8), $N\phi$ will be true at i (under $\mathfrak{A}$) if and only if there exists $j < i$ such that ϕ is true at j (under $\mathfrak{A}$); and therefore N will correctly express 'it has been the case that'. It is clear that the future tense, as well as the modal operators (interpreted by relevance relations) of Kripke [1], can be similarly accommodated. These examples, however, could all be treated within a simpler framework, in which R_N is always a relation between two points of reference (rather than having as its second relatum a *set* of points of reference). To see the necessity of the more general approach, we could consider probability operators, conditional necessity, or, to invoke an especially perspicuous example of Dana Scott, the present progressive tense. To elaborate on the last, let the interpretation $\mathfrak{A}$ again have the real numbers as its points of reference; and let R_N be the set of pairs $\langle i, J\rangle$ such that $i \in \mathbf{D}A$, $J \subseteq \mathbf{D}A$, and J is a neighborhood of i (that is, J includes an open interval of which i is a member). Then, by (8), $N\phi$ will be true at i (under $\mathfrak{A}$) if and only if there is an open interval containing i throughout which ϕ is true (under $\mathfrak{A}$). Thus N might receive the awkward reading 'it is being the case that', in the sense in which 'it is being the case that Jones leaves' is synonymous with 'Jones is leaving'.

According to (7), quantification is over *possible* (and not merely *actual*) individuals. The desirability of this can be seen by considering, within the special case of tense logic, the sentence 'there was a man whom no one remembers'. One can of course express quantification over actual individuals by combining quantifiers with the symbol E of existence.

To be quite precise, the desiderata (1)–(8) can be achieved by the following sequence of definitions; that is to say, (1)–(8) are simple consequences of Definitions II–V below. We assume for these that $\mathfrak{A}$ is a possible interpretation for a pragmatic language L, $\mathfrak{A}=\langle A, F, R\rangle$, U is the union of the sets A_j for $j \in \mathbf{D}A$, U^ω is the set of all infinite sequences (of type ω) of members of U, $i \in \mathbf{D}A$, and n is a natural number.

DEFINITION II. If ζ is an individual variable or individual constant of L, then by $\mathrm{Ext}_{i,\mathfrak{A}}(\zeta)$, or the *extension* of ζ at i (with respect to $\mathfrak{A}$) is understood that function H with domain U^ω which is determined as follows:

(1) if ζ is the variable v_n and $x \in U^\omega$, then $H(x)=x_n$;

(2) if ζ is an individual constant and $x \in U^\omega$, then $H(x)=F_\zeta(i)$.

The *extension* of a *formula* of L at a point of reference (and with respect to $\mathfrak{A}$) is introduced by the following recursive definition.

DEFINITION III. (1) If ζ is an individual constant of L or an individual variable, then $\mathrm{Ext}_{i,\mathfrak{A}}(\mathsf{E}[\zeta])$ is $\{x : x \in U^\omega$ and $(\mathrm{Ext}_{i,\mathfrak{A}}(\zeta))(x) \in A_i\}$.

(2) If each of ζ, η is either an individual constant of L or an individual variable, then $\mathrm{Ext}_{i,\mathfrak{A}}(\zeta=\eta)$ is $\{x : x \in U^\omega$ and $(\mathrm{Ext}_{i,\mathfrak{A}}(\zeta))(x)$ is identical with $(\mathrm{Ext}_{i,\mathfrak{A}}(\eta))(x)\}$.

(3) If P is an n-place predicate constant of L and each of $\zeta_0, \ldots, \zeta_{n-1}$ is an individual constant of L or an individual variable, then $\mathrm{Ext}_{i,\mathfrak{A}}(P[\zeta_0, \ldots, \zeta_{n-1}])$ is $\{x : x \in U^\omega$ and $\langle(\mathrm{Ext}_{i,\mathfrak{A}}(\zeta_0))(x), \ldots, (\mathrm{Ext}_{i,\mathfrak{A}}(\zeta_{n-1}))(x)\rangle \in F_P(i)\}$.

(4) If ϕ, ψ are formulas of L, then $\mathrm{Ext}_{i,\mathfrak{A}}(\neg\phi)$ is $U^\omega - \mathrm{Ext}_{i,\mathfrak{A}}(\phi)$, $\mathrm{Ext}_{i,\mathfrak{A}}((\phi \wedge \psi))$ is $\mathrm{Ext}_{i,\mathfrak{A}}(\phi) \cap \mathrm{Ext}_{i,\mathfrak{A}}(\psi)$, and similarly for the other sentential connectives.

(5) If ϕ is a formula of L, then $\mathrm{Ext}_{i,\mathfrak{A}}(\bigvee v_n \phi)$ is $\{x : x \in U^\omega$ and, for some $y \in U$, the sequence $\langle x_0, \ldots, x_{n-1}, y, x_{n+1}, \ldots\rangle \in \mathrm{Ext}_{i,\mathfrak{A}}(\phi)\}$, and similarly for $\bigwedge v_n \phi$.

(6) If ϕ is a formula of L and N an operator of L, then $\mathrm{Ext}_{i,\mathfrak{A}}(N\phi)$ is $\{x : x \in U^\omega$ and $\langle i, \{j : j \in \mathbf{D}A$ and $x \in \mathrm{Ext}_{j,\mathfrak{A}}(\phi)\}\rangle \in R_N\}$.

DEFINITION IV. If ϕ is a sentence of L, then ϕ is $\mathrm{true}_{i,\mathfrak{A}}$ if and only if $\mathrm{Ext}_{i,\mathfrak{A}}(\phi)=U^\omega$.

DEFINITION V. If ϕ is a formula of L of which the only free variable is v_n, then $y\ \mathrm{sat}_{i,\mathfrak{A}}\phi$ if and only if there exists $x \in \mathrm{Ext}_{i,\mathfrak{A}}(\phi)$ such that $x_n=y$.

It is seen from the definitions above that the extension of a formula (at a point of reference) is a set of sequences (indeed, the set of sequences 'satisfying' that formula at that point of reference, in the sense in which sequences, rather than individuals satisfy) and that the extension of an individual constant or variable (again, at a given point of reference) is a function assigning a possible individual to each sequence in U^{ω}. How does this construction accord with the fundamental discussion in Frege [1]? It should be remembered that Frege considered explicitly the extensions only of expressions without free variables – thus, as far as our present language is concerned, only of sentences and individual constants. For Frege the extension (or *ordinary extension*) of a sentence was a truth value; but it is easily seen that according to Definition III the extension of a sentence of L will always be either U^{ω} or the empty set, which in this context can be appropriately identified with truth and falsehood respectively. For Frege the extension (or *ordinary extension*) of an individual constant was the object it denotes, while for us the extension is the constant function with that object as value (and with U^{ω} as domain). Apart from set-theoretic manipulations, then, Frege's extensions agree with ours in all common cases.

I introduce for the sake of later discussion the *intensions* of certain expressions with respect to $\mathfrak{A}$, as well as the notions of *logical consequence*, *logical truth*, and *logical equivalence* appropriate to pragmatics.

DEFINITION VI. If ϕ is an individual constant of L, a formula of L, or an individual variable, then $\text{Int}_{\mathfrak{A}}(\phi)$ is that function H with domain $\mathbf{D}A$ such that, for each $i \in \mathbf{D}A$, $H(i) = \text{Ext}_{i,\mathfrak{A}}(\phi)$.

DEFINITION VII. A sentence ϕ is a *logical consequence* (in the sense of pragmatics) of a set Γ of sentences if and only if for every pragmatic language L and all $\mathfrak{A}$, A, F, R, i, if $\mathfrak{A} = \langle A, F, R\rangle$, $\mathfrak{A}$ is a possible interpretation for L, $i \in \mathbf{D}A$, $\Gamma \cup \{\phi\}$ is a set of sentences of L, and for every $\psi \in \Gamma$, ψ is $\text{true}_{i,\mathfrak{A}}$, then ϕ is $\text{true}_{i,\mathfrak{A}}$. A sentence is *logically true* if and only if it is a logical consequence of the empty set. A sentence ϕ is *logically equivalent* to a sentence ψ if and only if the sentence $(\phi \leftrightarrow \psi)$ is logically true.[7]

If we understand the extension of a predicate constant P (at i and with respect to $\mathfrak{A}$) to be $F_P(i)$, then inspection of Definition III will show that Frege's functionality principle applies fully to our notion of extension: the extension of a formula is a function of the extensions (ordinary extensions) of those of its parts not standing within indirect contexts (that is, for the present language, not standing within the scope of an operator), together with the intensions (what Frege also called *indirect extensions*) of those parts that do stand within indirect contexts. It is clause (6) of Definition III which creates the dependence of certain extensions on intensions, and which consequently makes it impossible to regard Definition III as a simple recursion on the length of formulas. Instead, the recursion is on a well-founded relation S between ordered pairs, characterized as follows: $\langle\langle j, \psi\rangle, \langle i, \phi\rangle\rangle \in S$ if and only if $i, j \in \mathbf{D}A$, ϕ, ψ are formulas of L, and ψ is a proper part of ϕ.[8]

On the other hand, we could have adopted another order, introducing intensions first and defining extensions explicitly in terms of them. In that case, as is easily seen, we could have introduced intensions by a simple recursion on the length of formulas; in other words, the intension of a complex expression is a function purely of the intensions of its components. (We thus answer negatively, for pragmatic languages at least, a question raised by Frege, whether we need to consider *indirect intensions* as well as ordinary extensions and ordinary intensions. The answer remains negative even for the richer languages considered below.)

The general treatment of operators, embodied in clause (6) of Definition III and due to Charles Howard and me, has the advantage of comprehending all known special cases but the drawback of a seemingly *ad hoc* and unintuitive character. This semblance can be removed, and at the same time a theoretical reduction accomplished, by the consideration of *intensional logic*. Attempts to construct intensional languages suitable for handling belief contexts and the like have been made previously, but without complete success; I report now my own efforts in this direction.

By an *intensional language* is understood a language of which the symbols are drawn from the following categories:

(1) the logical constants of pragmatic languages,

(2) parentheses, brackets, and commas,

(3) the individual variables $v_0, \ldots, v_n, \ldots$,

(4) individual constants,

(5) the n-place predicate variables $G_{0,n}, \ldots, G_{k,n}, \ldots$, for each natural number n,

(6) predicate constants of type s, for each finite sequence s of integers $\geqslant -1$,

(7) the operator $\Box$ (read 'necessarily'),

(8) the descriptive symbol T (read 'the unique ... such that' and regarded, along with the symbols under (1) and (7), as a logical constant).

Under (6) we admit predicate constants taking predicate variables, as well as individual symbols, as arguments. The type of such a constant indicates the grammatical categories of a suitable sequence of arguments, -1 indicating an individual symbol and a nonnegative integer n indicating an n-place predicate variable. Thus our previous n-place predicate constants are comprehended, and can be identified with predicate constants of type $\langle s_0, \ldots, s_{n-1} \rangle$, where each s_i (for $i<n$) is -1. The descriptive symbol will be applied only to predicate variables; this is because it will be needed only in such contexts and because its use in connection with individual variables would require some small but extraneous attention to the choice of a 'null entity'.[9] The descriptive phrases we admit will be completely eliminable, and are introduced solely to facilitate certain later examples.

The set of *formulas* of an *intensional* language L is the smallest set Γ such that (1) Γ contains the expressions

$$\mathrm{E}[\zeta],$$
$$\zeta = \eta,$$
$$G[\zeta_0, \ldots, \zeta_{n-1}],$$

where each of ζ, η, $\zeta_0, \ldots, \zeta_{n-1}$ is an individual constant of L or an individual variable and G is an n-place predicate variable of L, as well as all expressions

$$P[\zeta_0, \ldots, \zeta_{n-1}],$$

where P is a predicate constant of L having type $\langle s_0, \ldots, s_{n-1} \rangle$ and, for each $i<n$, either $s_i \geqslant 0$ and ζ_i is an s_i-place predicate variable, or $s_i = -1$ and ζ_i is an individual constant of L or an individual variable, (2) Γ is closed under the application of sentential connectives, (3) $\bigwedge u\phi$ and $\bigvee u\phi$ are in Γ whenever ϕ is in Γ and u is either an individual variable or a predicate variable, (4) $\Box\phi$ is in Γ whenever ϕ is in Γ, and (5) whenever ϕ, ψ are in Γ, and G is a predicate variable, then Γ also contains the result of replacing in ϕ all occurrences of G which do not immediately follow $\bigwedge$, $\bigvee$, or T by T$G\psi$.

By a *term* of L is understood either an individual constant of L, a variable, or an expression $\mathsf{T}G\phi$, where G is a predicate variable and ϕ a formula of L.

DEFINITION VIII. A *possible interpretation for an intensional language* L is a pair $\langle A, F\rangle$ such that clauses (1)–(4) of Definition I hold, and in addition (5′) whenever P is a predicate constant of L having type $\langle s_0, \ldots, s_{n-1}\rangle$, F_P is a $\langle \mathbf{D}A, U_0, \ldots, U_{n-1}\rangle$-predicate, where, for each $i<n$, either $s_i=-1$ and U_i is the union of the sets A_i for $i \in \mathbf{D}A$, or $s_i \geqslant 0$ and U_i is the set of all s_i-place $\mathbf{D}A$-predicates of members of the union of the sets A_i for $i \in \mathbf{D}A$.

Clause (5) of Definition I is a special case of the present (5′), taking $s_0 = \ldots = s_{n-1} = -1$.

Again we shall be primarily interested in notions of truth and satisfaction, expressed by the phrases 'the sentence ϕ is true with respect to the point of reference i under the interpretation $\mathfrak{A}$', and 'x satisfies the formula ϕ with respect to the point of reference i under the interpretation $\mathfrak{A}$'. Since, however, our formulas may now contain free predicate variables as well as free individual variables, we must understand 'x' to refer either to a possible individual or to a predicate of individuals. The intuitions underlying the present development will become clear upon consideration of the following criteria.

CRITERIA OF INTENSIONAL TRUTH AND SATISFACTION. Let $\mathfrak{A}$ be a possible interpretation, having the form $\langle A, F\rangle$, for an intensional language L; let $i \in \mathbf{D}A$; let U be the union of the sets A_j (for $j \in \mathbf{D}A$); let $x \in U$; let P be a predicate constant of L of type $\langle -1, -1\rangle$; let c, d be individual constants of L; and let u be an individual variable. Then:

(1)–(7) of the criteria of pragmatic truth and satisfaction.

(8′) If ϕ is a formula of L of which the only free variable is the n-place predicate variable G, then $\bigvee G\phi$ is $\text{true}_{i,\mathfrak{A}}$ if and only if there is an n-place $\mathbf{D}A$-predicate X of members of U such that $X \, \text{sat}_{i,\mathfrak{A}}\phi$.

(9′) If G is an n-place predicate variable, $\mathscr{P}$ a predicate constant of L of type $\langle n\rangle$, and X an n-place $\mathbf{D}A$-predicate of members of U, then X $\text{sat}_{i,\mathfrak{A}}\, \mathscr{P}[G]$ if and only if $\langle X\rangle \in F_{\mathscr{P}}(i)$.

(10′) If ϕ is a sentence of L, then $\Box\phi$ is $\text{true}_{i,\mathfrak{A}}$ if and only if ϕ is $\text{true}_{j,\mathfrak{A}}$ for all $j \in \mathbf{D}A$.

(11′) If G is an n-place predicate variable, $\mathscr{P}$ a predicate constant of L of type $\langle n\rangle$, and ϕ a formula of L of which the only free variable is G, then $\mathscr{P}[\mathsf{T}G\phi]$ is $\text{true}_{i,\mathfrak{A}}$ if and only if either there is exactly one n-place $\mathbf{D}A$-predicate X of members of U such that $X\ \text{sat}_{i,\mathfrak{A}}\ \phi$, and that predicate is in $F_{\mathscr{P}}(i)$; or it is not the case that there is exactly one such predicate, and the empty predicate (that is, $\mathbf{D}A \times \{\Lambda\}$) is in $F_{\mathscr{P}}(i)$.

(12′) If G is a 0-place predicate variable and X a $\langle \mathbf{D}A \rangle$-predicate, then $X\ \text{sat}_{i,\mathfrak{A}}\ G[\ ]$ if and only if the empty sequence is a member of $X(i)$ (hence, if and only if $X(i)=\{\Lambda\}$).

In view of (8′), predicate variables range over predicates of possible individuals. In view of (10′), $\Box$ should be regarded as the *standard* necessity operator. In view of (8′) and earlier remarks, 0-place predicate variables range over propositions; accordingly, we may, by (12′), read $G[\ ]$ as 'the proposition G is true'.

Quantification over individual concepts and over relations (in the extensional sense) is lacking, but its effect can nevertheless be achieved. Let $\langle A, F \rangle$ be a possible interpretation for an intensional language, and let U be the union of the sets A_i for $i \in \mathbf{D}A$. By an *individual concept* of $\langle A, F \rangle$ is understood a function from $\mathbf{D}A$ into U. But individual concepts of $\langle A, F \rangle$ can be identified with $\langle \mathbf{D}A, U \rangle$-predicates satisfying the formula

$$\Box \bigvee u \bigwedge v(G[v] \leftrightarrow v=u).$$

Further, as J. A. W. Kamp has observed, $\langle U, U \rangle$-relations can be identified with $\langle \mathbf{D}A, U, U \rangle$-predicates satisfying the formula

$$\bigwedge u \bigwedge v(\Box G[u, v] \vee \Box \neg G[u, v]);$$

and a similar identification can be performed for relations of more or fewer places.

Let us now introduce precise definitions having Criteria (1)–(12′) as consequences. We assume that $\mathfrak{A}$ is a possible interpretation for an intensional language L, $\mathfrak{A}=\langle A, F, \rangle$, U is the union of the sets A_j for $j \in \mathbf{D}A$, and $i \in \mathbf{D}A$. We can no longer regard simple infinite sequences as assigning values to variables; the presence of variables of various sorts requires the consideration of *double* sequences in which one of the indices determines the sort of variable in question. In particular, let us understand by a *system* associated with $\mathfrak{A}$ a function x having as its domain the set of pairs $\langle n, k \rangle$ for which n is a natural number and k an integer

$\geqslant -1$, and such that whenever $\langle n, k\rangle$ is such a pair, either $k=-1$ and $x(\langle n, k\rangle) \in U$, or $k \geqslant 0$ and $x(\langle n, k\rangle)$ is a k-place **D**A-predicate of members of U. We assume that S is the set of all systems associated with $\mathfrak{A}$; as is customary, we shall understand by $x_{n,k}$ the function value $x(\langle n, k\rangle)$. In addition, we assume that n, k are natural numbers; and if x is a function, we understand by x^a_b the function obtained from x by substituting b for the original value of x for the argument a, that is, the function $(x-\{\langle a, x(a)\rangle\}) \cup \{\langle a, b\rangle\}$.

The *extension* of a *term* or *formula* is introduced by a single recursion.

DEFINITION IX. (1) If c is an individual constant of L, then $\mathrm{Ext}_{i,\mathfrak{A}}$ (c) is that function H with domain S such that, for all $x \in S$, $H(x)=F_c(i)$.

(2) $\mathrm{Ext}_{i,\mathfrak{A}}(v_n)$ is that function H with domain S such that, for all $x \in S$, $H(x)=x_{n,-1}$.

(3) $\mathrm{Ext}_{i,\mathfrak{A}}(G_{n,k})$ is that function H with domain S such that, for all $x \in S$, $H(x)=x_{n,k}$.

(4) If ϕ is a formula of L, then $\mathrm{Ext}_{i,\mathfrak{A}}(\mathsf{T}G_{n,k}\phi)$ is that function H with domain S such that, for all $x \in S$, either $\{H(x)\}=\{Y: x^{\langle n,k\rangle}_Y \in \mathrm{Ext}_{i,\mathfrak{A}}(\phi)\}$, or there is no Z for which $\{Z\}=\{Y: x^{\langle n,k\rangle}_Y \in \mathrm{Ext}_{i,\mathfrak{A}}(\phi)\}$, and $H(x)$ is $\mathbf{D}A \times \{\Lambda\}$.

(5) If ζ is an individual constant of L or an individual variable, then $\mathrm{Ext}_{i,\mathfrak{A}}(\mathsf{E}[\zeta])$ is $\{x: x \in S$ and $(\mathrm{Ext}_{i,\mathfrak{A}}(\zeta))(x) \in A_i\}$.

(6) If each of ζ, η is either an individual constant of L or an individual variable, then $\mathrm{Ext}_{i,\mathfrak{A}}(\zeta=\eta)$ is $\{x: x \in S$ and $(\mathrm{Ext}_{i,\mathfrak{A}}(\zeta))(x)$ is identical with $(\mathrm{Ext}_{i,\mathfrak{A}}(\eta))(x)\}$.

(7) If η is an n-place predicate variable or a term $\mathsf{T}G\phi$ (with G an n-place predicate variable), and each of $\zeta_0, \ldots, \zeta_{n-1}$ is an individual constant of L or an individual variable, then $\mathrm{Ext}_{i,\mathfrak{A}}(\eta[\zeta_0, \ldots, \zeta_{n-1}])$ is $\{x: x \in S$ and $\langle(\mathrm{Ext}_{i,\mathfrak{A}}(\zeta_0))(x), \ldots, (\mathrm{Ext}_{i,\mathfrak{A}}(\zeta_{n-1}))(x)\rangle \in (\mathrm{Ext}_{i,\mathfrak{A}}(\eta))(x)(i)\}$.

(8) If P is a predicate constant of L of type $\langle s_0, \ldots, s_{n-1}\rangle$ and, for each $i<n$, either $s_i \geqslant 0$ and ζ_i is either an s_i-place predicate variable or a term $\mathsf{T}G\phi$ in which G is an s_i-place predicate variable and ϕ a formula of L, or $s_i=-1$ and ζ_i is an individual constant of L or an individual variable, then $\mathrm{Ext}_{i,\mathfrak{A}}(P[\zeta_0, \ldots, \zeta_{n-1}])$ is $\{x: x \in S$ and $\langle(\mathrm{Ext}_{i,\mathfrak{A}}(\zeta_0))(x), \ldots, (\mathrm{Ext}_{i,\mathfrak{A}}(\zeta_{n-1}))(x)\rangle \in F_P(i)\}$.

(9) If ϕ, ψ are formulas of L, then $\mathrm{Ext}_{i,\mathfrak{A}}(\neg\phi)$ is $S-\mathrm{Ext}_{i,\mathfrak{A}}(\phi)$, and similarly for the other sentential connectives.

(10) If ϕ is a formula of L, then $\text{Ext}_{i,\mathfrak{A}}(\bigvee v_n\phi)$ is $\{x : x \in S$ and, for some $y \in U$, the system $x^{\langle n, -1\rangle}_{y} \in \text{Ext}_{i,\mathfrak{A}}(\phi)\}$, and similarly for $\bigwedge v_n\phi$.

(11) If ϕ is a formula of L, then $\text{Ext}_{i,\mathfrak{A}}(\bigvee G_{n,k}\phi)$ is $\{x : x \in S$ and, for some k-place $\mathbf{D}A$-predicate Y of members of U, the system $x^{\langle n,k\rangle}_{Y} \in \text{Ext}_{i,\mathfrak{A}}(\phi)\}$, and similarly for $\bigwedge G_{n,k}\phi$.

(12) If ϕ is a formula of L, then $\text{Ext}_{i,\mathfrak{A}}(\Box\phi)$ is $\{x : x \in S$ and, for all $j \in \mathbf{D}A$, $x \in \text{Ext}_{j,\mathfrak{A}}(\phi)\}$.

DEFINITION X. If ϕ is a sentence of L, then ϕ is $\text{true}_{i,\mathfrak{A}}$ if and only if $\text{Ext}_{i,\mathfrak{A}}(\phi) = S$.

DEFINITION XI. If ϕ is a formula of L with exactly one free variable, then $y\ \text{sat}_{i,\mathfrak{A}}\phi$ if and only if either there is a natural number n such that the free variable of ϕ is v_n and there exists $x \in \text{Ext}_{i,\mathfrak{A}}(\phi)$ such that $x_{n,-1} = y$, or there are natural numbers n, k such that the free variable of ϕ is $G_{n,k}$ and there exists $x \in \text{Ext}_{i,\mathfrak{A}}(\phi)$ such that $x_{n,k} = y$.

DEFINITION XII. If ϕ is a term or formula of L, then $\text{Int}_{\mathfrak{A}}(\phi)$, or the *intension* of ϕ with respect to $\mathfrak{A}$, is that function H with domain $\mathbf{D}A$ such that, for each $i \in \mathbf{D}A$, $H(i) = \text{Ext}_{i,\mathfrak{A}}(\phi)$.

DEFINITION XIII. A sentence ϕ is a *logical consequence* (in the sense of intensional logic) of a set Γ of sentences if and only if for every intensional language L and all $\mathfrak{A}$, A, F, i, if $\mathfrak{A} = \langle A, F\rangle$, $\mathfrak{A}$ is a possible interpretation for L, $i \in \mathbf{D}A$, $\Gamma \cup \{\phi\}$ is a set of sentences of L, and for every $\psi \in \Gamma$, ψ is $\text{true}_{i,\mathfrak{A}}$, then ϕ is $\text{true}_{i,\mathfrak{A}}$. A sentence is *logically true* if and only if it is a logical consequence of the empty set. A sentence ϕ is *logically equivalent* to a sentence ψ if and only if the sentence $(\phi \leftrightarrow \psi)$ is logically true.

The remarks about extensions and intensions made in connection with pragmatic languages continue to apply here, with infinite sequences everywhere replaced by systems. Further, Criteria (1)–(12′) are immediate consequences of Definitions IX–XI.

It was said earlier that descriptive phrases of the sort we admit, that is, descriptive phrases involving predicate variables, are eliminable. We can now make a more precise statement: if ϕ is any sentence of an intensional language L, then there is a sentence of L without descriptive phrases that is logically equivalent to ϕ. For instance, if ϕ is

$$\mathscr{P}[\urcorner G\mathscr{Q}[G]],$$

where G is a 1-place predicate variable and $\mathscr{P}$, $\mathscr{Q}$ are predicate constants of type $\langle 1 \rangle$, then ϕ is logically equivalent to

$$\bigvee G(\mathscr{Q}[G] \wedge \bigwedge H(\mathscr{Q}[H] \rightarrow \Box \bigwedge x(H[x] \leftrightarrow G[x])) \wedge \mathscr{P}[G]) \vee (\neg \bigvee G(\mathscr{Q}[G] \wedge \bigwedge H(\mathscr{Q}[H] \rightarrow \Box \bigwedge x(H[x] \leftrightarrow G[x]))) \wedge \bigvee G(\Box \bigwedge x \neg G[x] \wedge \mathscr{P}[G])).$$

The convenience of descriptive phrases is found in the construction of names of specific predicates. For instance, we can distinguish as follows expressions designating properties or 2-place predicates expressed by particular formulas (with respect to places marked by particular individual variables): if ϕ is a formula and u, v are distinct individual variables, understand by $\hat{u}\phi$ (which may be read 'the property of u such that ϕ') the term $\mathsf{T}G \bigwedge u \Box (G[u] \leftrightarrow \phi)$, and by $\hat{u}\hat{v}\phi$ (read 'the predicate of u and v such that ϕ') the term $\mathsf{T}H \bigwedge u \bigwedge v \Box (H[u, v] \leftrightarrow \phi)$, where G, H are respectively the first 1-place and the first 2-place predicate variables not occurring in ϕ. We can of course proceed upward to three variables or more; but – and this is more interesting – we can proceed downward to the empty sequence of variables. In particular, if ϕ is any formula, understand by $^\wedge\phi$ the term $\mathsf{T}G \Box (G[\ ] \leftrightarrow \phi)$; this term designates the proposition expressed by the formula ϕ, may be read 'the proposition that ϕ' or simply 'that ϕ', and serves the purposes for which the term '$\bar{\phi}$' of Kaplan [1] was constructed.

It is clear from Definition IX that sentences of intensional languages, unlike those of pragmatic languages, may contain indirect components – that is, components of which the *intension* must be taken into account in determining the *extension* of the compound – of only one sort; and these are components standing within the scope of the particular operator $\Box$. An equivalent construction would have taken the indirect context $^\wedge\phi$ rather than $\Box\phi$ as basic, together with the notion of identity of propositions; we could then have defined $\Box\phi$ as $^\wedge\phi = {}^\wedge \bigwedge v_0 v_0 = v_0$.

Now let us see how to accommodate operators within intensional languages. (The observation that this can be done, as well as the present way of doing it, is due jointly to J. A. W. Kamp and me.) Suppose that L is any pragmatic language and $\langle A, F, R \rangle$ any possible interpretation for it. Let the operators N of L be mapped biuniquely onto predicate constants N' of type $\langle 0 \rangle$. Let L′ be an intensional language of which the individual constants are those of L, and the predicate constants are those of L together with the symbols N', for N an operator of L. Let F' be such that $\langle A, F'\rangle$ is a possible interpretation for the intensional language L′, $F \subseteq F'$, and for each operator N of L and each $i \in \mathbf{D}A$, $F'_{N'}(i)$ is $\{\langle U\rangle : U$ is a $\langle \mathbf{D}A\rangle$-predicate and $\langle i, \{j : j \in \mathbf{D}A$ and $U(j) = \{\Lambda\}\}\rangle \in R_N\}$. Then we can easily prove the following: if ϕ is a sentence of L, ϕ' is obtained from ϕ by replacing each subformula of the form $N\psi$, where N is an operator of L and ψ a formula of L, by

$$\bigvee G(\Box(G[\,] \leftrightarrow \psi) \wedge N'[G]),$$

and $i \in \mathbf{D}A$, then ϕ is true with respect to i and the *pragmatic* interpretation $\langle A, F, R\rangle$ if and only if ϕ' is true with respect to i and the *intensional* interpretation $\langle A, F'\rangle$.

We thus have a reduction of pragmatics to intensional logic which amounts, roughly speaking, to treating 1-*place modalities* (that is, relations between points of reference and sets of points of reference) as properties of propositions. Conversely, every property of propositions corresponds to a 1-place modality. Indeed, if $\langle A, F\rangle$ is an interpretation for an intensional language and $\mathscr{X}$ is a property of propositions with respect to $\langle A, F\rangle$ (that is, a $\langle \mathbf{D}A, U\rangle$- predicate, where U is the set of all $\langle \mathbf{D}A\rangle$-predicates), then the corresponding 1-place modality will be the set of pairs $\langle i, J\rangle$ such that $i \in \mathbf{D}A$ and there exists $Y \in \mathscr{X}(i)$ such that $J = \{j : j \in \mathbf{D}A$ and $Y(j) = \{\Lambda\}\}$.

Let us be a little more precise about the sense in which intensional logic can be *partially* reduced to pragmatics. Let L be an intensional language of which the predicate constants are all of type $\langle 0\rangle$ or $\langle s_0, \ldots, s_{n-1}\rangle$, where $s_p = -1$ for all $p < n$, and let $\langle A, F\rangle$ be any interpretation for L. Let the predicate constants $\mathscr{P}$ of L having type $\langle 0\rangle$ be mapped biuniquely onto operators $\mathscr{P}'$, and let N be an operator not among these. Let L′ be a pragmatic language of which the individual constants are those of L, the predicate constants are those of L not having type $\langle 0\rangle$, and the operators consist of N together with the symbols $\mathscr{P}'$ for $\mathscr{P}$ a predicate constant of L of type $\langle 0\rangle$. Let F', R be such that $\langle A, F', R\rangle$ is a possible interpretation for the pragmatic language L′, $F' \subseteq F$, R_N is the set of pairs $\langle i, J\rangle$ such that $i \in \mathbf{D}A$ and $J = \mathbf{D}A$, and for each predicate $\mathscr{P}$ of L of type $\langle 0\rangle$, $R_{\mathscr{P}'}$ is the set of pairs $\langle i, J\rangle$ such that $i \in \mathbf{D}A$ and there exists $Y \in F_{\mathscr{P}}(i)$ such that $J = \{j : j \in \mathbf{D}A$ and $Y(j) = \{\Lambda\}\}$. Then we can easily show that if $i \in \mathbf{D}A$, ϕ is a sentence of L, ϕ' is obtained from ϕ by replacing each subformula $\mathscr{P}[{}^\wedge\psi]$, where $\mathscr{P}$ is a predicate constant of type $\langle 0\rangle$ and ψ

is a formula of L, by $\mathscr{P}'\psi$, and ϕ' is a sentence of the pragmatic language L′ (this imposes certain limitations on the form of ϕ), then ϕ is true with respect to i and the *intensional* interpretation $\langle A, F\rangle$ if and only if ϕ' is true with respect to i and the *pragmatic* interpretation $\langle A, F', R\rangle$.

The fact that 1-place modalities coincide in a sense with properties of propositions is what lends interest to those modalities and provides intuitive sanction for using them to interpret operators. (A completely analogous remark would apply to many-place modalities and many-place operators if these had been included in our system of pragmatics.) The relations among various systems can be roughly expressed as follows. If we understand by *modal logic* that part of intensional logic which concerns formulas containing no predicate variables, then intensional logic can be regarded as *second-order modal logic*, and pragmatics is in a sense contained in it; indeed, pragmatics can be regarded as a first-order reduction of part of intensional logic.

Nothing of course compels us to stop at *second-order* modal logic. We could extend the present construction in a fairly obvious way to obtain various higher-order systems, even of transfinite levels. Only the second-order system, however, is required for the rather direct philosophical applications for which the present paper is intended to provide the groundwork.

For example, belief can be handled in a natural way within intensional logic. Let L be an intensional language containing a predicate constant $\mathscr{B}$ of type $\langle -1, 0\rangle$. If $\langle A, F\rangle$ is a possible interpretation for L, we now regard the domain of A as the set of all possible worlds, A_i as the set of objects existing within the possible world i, and $F_c(i)$ as the extension of the nonlogical constant c within the world i. Then a $\langle \mathbf{D}A\rangle$-predicate can reasonably be regarded as a proposition in the full philosophical sense, not merely the extended sense considered earlier, and the intension of a sentence with respect to $\langle A, F\rangle$ as the proposition expressed by that sentence (under the interpretation $\langle A, F\rangle$). We regard $\mathscr{B}$ as abbreviating 'believes', and accordingly regard $F_{\mathscr{B}}(i)$ as the set of pairs $\langle x, U\rangle$ such that x believes the proposition U in the possible world i. The proposal to regard belief as an empirical relation between individuals and propositions is not new. A number of difficulties connected with that proposal are, however, dispelled by considering it within the present framework; in particular, there remains no problem either of quantifying into belief contexts or of iteration of belief.[10] Consider the assertion 'there exists an object of which Jones believes that Robinson believes that it is perfectly spherical'. This involves both iteration and quantification into indirect contexts, but is represented in L (with respect to $\langle A, F\rangle$) by the simple sentence

$$\bigvee x(\mathsf{E}[x] \wedge \mathscr{B}[J, {}^{\wedge}\mathscr{B}[R, {}^{\wedge}S[x]]]),$$

where J and R are individual constants regarded as designating Jones and Robinson respectively and S is a predicate constant regarded as expressing the property of being perfectly spherical; or, if we prefer to avoid descriptive phrases, by the logically equivalent sentence

$$\bigvee x \bigvee G(\mathsf{E}[x] \wedge \mathscr{B}[J, G] \wedge \Box(G[\,] \leftrightarrow \bigvee H(\mathscr{B}[R, H] \wedge \Box(H[\,] \leftrightarrow S[x])))).$$

Two objections might be raised. In the first place, what empirical sense can be assigned to belief as a relation between persons and propositions? As much, I feel, as is customary with empirical predicates. One can give confirmatory criteria for belief, though probably not a definition, in behavioristic terms. I present two unrefined and incompletely analyzed examples:

(1) If ϕ is any sentence expressing the proposition G, then the assertion that x assents to ϕ confirms (though certainly not conclusively) the assertion that x believes G.

(2) If ϕ is any formula with exactly one free variable that expresses the property H (in the sense that, for all $i \in \mathbf{D}A$, $H(i)$ is the set of possible individuals satisfying ϕ with respect to i and a given interpretation), then the assertion that x assents to ϕ when y is pointed out to x confirms (though again not conclusively) the assertion that x believes the proposition that $H[y]$.

A second objection might concern the fact that if ϕ and ψ are any logically equivalent sentences, then the sentence

$$\mathscr{B}[J, {}^{\wedge}\phi] \rightarrow \mathscr{B}[J, {}^{\wedge}\psi]$$

is logically true, though it might under certain circumstances appear unreasonable. One might reply that the consequence in question seems unavoidable if propositions are indeed to be taken as the objects of belief, that it sheds the appearance of unreasonableness if (1) above is seriously maintained, and that its counterintuitive character can perhaps be traced to the existence of another notion of belief, of which the objects are sentences or, in some cases, complexes consisting in part of open formulas.[11]

As another example, let us consider the verb 'seems', as in

u seems to be perfectly spherical to *v*.

We let L be as above, except that it is now to contain a predicate constant $\mathscr{S}$ of type $\langle -1, 1, -1\rangle$; if $\langle A, F\rangle$ is a possible interpretation for L and $i \in \mathbf{D}A$, $F_{\mathscr{S}}(i)$ is to be regarded as the set of triples $\langle x, U, y\rangle$ such that, in the possible world i, x seems to y to have the property U. The formula displayed above would then be represented in L by the formula

$$\mathscr{S}[u, \hat{w}S[w], v].$$

We have made no attempt to *define* 'believes' or 'seems'. But that need not prevent us from clarifying the logical status of these verbs and the notions of logical truth and logical consequence for discourse involving them; and this would appear to be the main requirement for the evaluation of a number of philosophical arguments. The philosophical utility of intensional logic, however, is not in my opinion thereby exhausted; more important applications can be found in other areas, notably metaphysics and epistemology, and are to some extent discussed in Montague [3].

It is perhaps not inappropriate to sketch here an intermediate system, due to Dana Scott and me, which may be called *extended pragmatics*.[12] The symbols of an *extended pragmatic language* are drawn from the following categories:

(1) the logical constants of pragmatics,
(2) parentheses, brackets, commas,
(3) individual variables,
(4) individual constants,
(5) operators of degree $\langle m, n, p\rangle$, for all natural numbers m, n, p.

The set of *formulas* of such a language L is the smallest set Γ satisfying certain expected conditions, together with the condition that

$$Nu_0 \ldots u_{m-1}[\zeta_0, \ldots, \zeta_{n-1}, \phi_0, \ldots, \phi_{p-1}]$$

is in Γ whenever N is an operator of L having degree $\langle m, n, p \rangle$, $u_0, \ldots, u_{m-1}$ are distinct individual variables, each of $\zeta_0, \ldots, \zeta_{n-1}$ is either an individual constant of L or an individual variable, and $\phi_0, \ldots, \phi_{p-1}$ are in Γ. A *possible interpretation for an extended pragmatic language* L is a pair $\langle A, F \rangle$ satisfying conditions (1), (2), (4) of Definition I, and in addition such that (3′) F is a function whose domain is the set of individual constants and operators of L, and (5′) whenever N is an operator of L of degree $\langle m, n, p \rangle$, F_N is a $\langle \mathbf{D}A, U_0, \ldots, U_{n-1}, V_0, \ldots, V_{p-1} \rangle$-predicate, where each U_i (for $i<n$) is the union of the sets A_j for $j \in \mathbf{D}A$, and each V_i (for $i<p$) is the set of m-place $\mathbf{D}A$-predicates of members of the union of the sets A_j (for $j \in \mathbf{D}A$). The *extension* of an individual variable, an individual constant, or a formula with respect to a possible interpretation $\mathfrak{A}$ having the form $\langle A, F \rangle$ and at a point of reference $i \in \mathbf{D}A$ is characterized as in Definition II, together with a recursion consisting of clauses (1), (2), (4), (5) of Definition III, together with the following clause: if N is an operator of L of degree $\langle m, n, p \rangle$, $k_0, \ldots, k_{m-1}$ are distinct natural numbers, each of $\zeta_0, \ldots, \zeta_{n-1}$ is either an individual constant of L or an individual variable, and $\phi_0, \ldots, \phi_{p-1}$ are formulas of L, then $\mathrm{Ext}_{i,\mathfrak{A}}$ $(N v_{k_0} \ldots v_{k_{m-1}} [\zeta_0, \ldots, \zeta_{n-1}, \phi_0, \ldots, \phi_{p-1}])$ is $\{x : x \in U^\omega$ and $\langle \mathrm{Ext}_{i,\mathfrak{A}}(\zeta_0)$ $(x), \ldots, \mathrm{Ext}_{i,\mathfrak{A}}(\zeta_{n-1})(x), Y_{0,x}, \ldots, Y_{p-1,x} \rangle \in F_N(i)\}$, where, for each $q<p$ and $x \in U^\omega$, $Y_{q,x}$ is $\{\langle j, \{\langle y_0, \ldots, y_{m-1} \rangle : x^{k_0}_{y_0} \ldots {}^{k_{m-1}}_{y_{m-1}} \in \mathrm{Ext}_{j,\mathfrak{A}}(\phi_q)\} \rangle :$ $j \in \mathbf{D}A\}$.

Thus, in particular, if N is an operator of degree $\langle 0, n, 0 \rangle$, then $\mathrm{Ext}_{i,\mathfrak{A}}(N[\zeta_0, \ldots, \zeta_{n-1}])$ is $\{x : x \in U^\omega$ and $\langle \mathrm{Ext}_{i,\mathfrak{A}}(\zeta_0)(x), \ldots, \mathrm{Ext}_{i,\mathfrak{A}}$ $(\zeta_{n-1})(x) \rangle \in F_N(i)\}$, and N will play the role of an n-place predicate constant; and if N has degree $\langle 0, 0, 1 \rangle$, then $\mathrm{Ext}_{i,\mathfrak{A}}(N[\phi])$ is $\{x : x \in U^\omega$ and $\langle \{\langle j, \{\Lambda\} \rangle : j \in \mathbf{D}A$ and $x \in \mathrm{Ext}_{j,\mathfrak{A}}(\phi)\} \cup \{\langle j, \Lambda \rangle : j \in \mathbf{D}A$ and $x \notin \mathrm{Ext}_{j,\mathfrak{A}}(\phi)\} \rangle \in F_N(i)\}$, and N will accordingly serve as a substitute for a (one-place) operator of pragmatics. Further, an operator of extended pragmatics of arbitrary degree $\langle m, n, p \rangle$ can be replaced within intensional logic by a predicate constant of type $\langle s_0, \ldots, s_{n-1}, t_0, \ldots, t_{p-1} \rangle$, where each s_i (for $i<n$) is -1 and each t_i (for $i<p$) is m.

Thus, in a sense, pragmatics is contained in extended pragmatics, which is in turn contained in intensional logic. We can regard extended pragmatics as providing another first-order reduction, more comprehensive than that supplied by ordinary pragmatics, of part of intensional logic. For instance, if $\mathscr{B}$ is, like 'believes', a predicate constant of type $\langle -1, 0 \rangle$

of intensional logic, we could replace $\mathscr{B}$ by an operator $\mathscr{B}'$ of degree $\langle 0, 1, 1\rangle$ (of extended pragmatics) and express the assertion

$$\mathscr{B}[x, {}^{\wedge}\phi]$$

equivalently (under a suitable interpretation) by

$$\mathscr{B}'[x, \phi].$$

Similarly, if $\mathscr{S}$ is, like 'seems', a predicate constant of type $\langle 1, 1, -1\rangle$, we could replace $\mathscr{S}$ by an operator $\mathscr{S}'$ of degree $\langle 1, 2, 1\rangle$ and express the assertion

$$\mathscr{S}[u, \hat{w}\phi, v]$$

by

$$\mathscr{S}'w[u, v, \phi].$$

(It should be clear from this example, as well as from the general definition of extension, that the m variables immediately following an operator of degree $\langle m, n, p\rangle$ are to be regarded as *bound*.) There is of course no contention that all formulas of intensional logic involving $\mathscr{B}$ or $\mathscr{S}$ can be paraphrased within extended pragmatics; for instance, the assertion 'Jones believes something which Robinson does not believe' does not correspond to any formula of extended pragmatics.

We may now consider various technical properties of the three systems introduced in this paper. Notice first that the compactness theorem does not hold for intensional logic. In other words, let us call a set of sentences *satisfiable* if there is a nonempty interpretation $\mathfrak{A}$ and a point of reference i of $\mathfrak{A}$ such that all sentences in the set are true with respect to i and $\mathfrak{A}$; then it is not the case that for every set Γ of sentences of intensional logic,

(3) if every finite subset of Γ is satisfiable, then Γ is satisfiable.

This is obvious in view of the reduction, at which we hinted earlier, of ordinary second-order logic to intensional logic, together with the well-known failure of the compactness theorem for second-order logic. On the other hand, let us call ϕ a *predicative* sentence if ϕ is a sentence of intensional logic not containing the descriptive symbol and such that (1) whenever G is a predicate variable, ψ is a formula, and $\bigwedge G\psi$ is a subformula of ϕ, there are $\mathscr{P}$, $\zeta_0, \ldots, \zeta_n$, χ such that $\mathscr{P}$ is a predicate constant, each ζ_i (for $i \leqslant n$) is either an individual constant, an individual variable, or a predicate variable, χ is a formula, ψ is the formula $(\mathscr{P}[\zeta_0, \ldots, \zeta_n] \rightarrow \chi)$, and G is ζ_i for some $i \leqslant n$, and (2) whenever G is a predicate variable, ψ is a formula, and $\bigvee G\psi$ is a subformula of ϕ, there are $\mathscr{P}$, $\zeta_0, \ldots, \zeta_n$, χ satisfying the same conditions as in (1) except that ψ is now to be $(\mathscr{P}[\zeta_0, \ldots, \zeta_n] \wedge \chi)$. For the *predicative* sentences of intensional logic we do have a compactness theorem; in other words, (3) holds for every set Γ of predicative sentences.[13] From this assertion we can infer full compactness theorems for pragmatics and extended pragmatics, in other words, the assertion that (3) holds for *every* set Γ of sentences of pragmatics and for *every* set Γ of sentences of extended pragmatics; we use reductions of the sort sketched above of those disciplines to intensional logic and notice that the reductions can be performed in such a way as to result exclusively in predicative sentences.

Similar remarks apply to the recursive enumerability of the logical truths of the three systems we have considered. We must, however, say a word about the meaning of recursive enumerability in this context. We have not required that the symbols from which our languages are constructed form a countable set; it would thus be inappropriate to speak of a Gödel numbering of all expressions. We may, however, suppose that a Gödel numbering satisfying the usual conditions has been given for a certain denumerable *subset* S of the set of all expressions; we may further suppose that all logical constants, the parentheses and brackets, the comma, all individual variables, all predicate variables, infinitely many n-place predicate constants (for each n), infinitely many predicate constants of each type, infinitely many 1-place operators, and infinitely many operators of each degree are in S, and that S is closed under the concatenation of two expressions. When we say that a set of expressions is recursive or recursively enumerable we shall understand that it is a subset of S which is recursive or recursively enumerable under our fixed Gödel numbering.

Let us identify a language with the set of symbols it contains; we may accordingly speak of recursive languages. It is then easily shown, by the same methods as those sketched in connection with compactness, that (1) there are recursive intensional languages of which the sets of logical truths are not recursively enumerable; (2) if L is any recursive intensional language, then the set of predicative sentences of L which are logically true is recursively enumerable; (3) if L is any recursive pragmatic language, then the set of all logical truths of L is recursively enumerable; (4) if L is any recursive extended pragmatic language, then the set of all logical truths of L is recursively enumerable.

On the basis of (2)–(4), together with a theorem of Craig [1], we can of course show for each of the three sets mentioned in (2)–(4) the existence of a recursive subset which axiomatizes the set in question under the rule of detachment. It would be desirable, however, to find natural and simple recursive axiomatizations of these sets. Of the three problems that thus arise one has been definitely solved: David Kaplan has recently axiomatized the set of logical truths of (ordinary) pragmatics. He has also axiomatized the set of logical truths of a system closely resembling extended pragmatics; and it is likely that when his axiomatization becomes available, it will be capable of adaptation to extended pragmatics. The problem, however, of axiomatizing predicative intensional logic remains open.

In connection with problems of axiomatizability it is perhaps not inappropriate to mention that all three of our systems are purely referential in one sense, specifically, in the sense that

$$(4) \qquad \bigwedge u \bigwedge v(u = v \rightarrow (\phi \leftrightarrow \phi'))$$

is logically true whenever u, v are individual variables, ϕ is a formula of the language in question, and ϕ' is obtained from ϕ by replacing a free occurrence of u by a free occurrence of v, but *not* purely referential in another sense: it is not generally true that whenever c, d are individual constants, ϕ is a formula of one of the languages under consideration, and ϕ' is obtained from ϕ by replacing an occurrence of c by d, the formula

$$(5) \qquad c = d \rightarrow (\phi \leftrightarrow \phi')$$

is logically true. It follows, of course, that the principle of universal instantiation does not always hold; it holds when one instantiates to variables but not in general when one instantiates to individual constants.

There is rather general (though not universal) agreement that (5) ought not to be regarded as logically true when modal and belief contexts are present; for consider the following familiar example of (5):

> If the Morning Star = the Evening Star, then Jones believes that the Morning Star appears in the morning if and only if Jones believes that the Evening Star appears in the morning.

This viewpoint has led some philosophers, however, to reject also the logical truth of (4). The desirability of maintaining (4) as a logical truth but not (5) was, to my knowledge, first explicitly argued in the 1955 talk reported in Montague [2], but has more recently been advanced in Føllesdal [1] and Cocchiarella [2], and in addresses of Professors Richmond Thomason and Dagfinn Føllesdal.

Let me conclude with a few historical remarks concerning intensional logic. The first serious and detailed attempt to construct such a logic appears to be that of Church [1]. Carnap had independently proposed in conversation that intensional objects be identified with functions from possible worlds to extensions of appropriate sorts, but that, in distinction from the later proposal of Kripke adopted in the present paper, possible worlds be identified with models. David Kaplan, in his dissertation Kaplan [1], pointed out certain deficiencies of Church's system, presented a modified version designed to correct these, and supplied a model theory for the revised system based on Carnap's proposal. Kaplan's system, however, suffered from the drawback indicated above involving the iteration of empirical properties of propositions; the difficulty stemmed largely from Carnap's suggestion that possible worlds be identified with models. More recent attempts by Charles Howard, David Kaplan, and Dana Scott (some preceding and some following the talk reported by the main body of the present paper) have avoided this difficulty but have shared with Kaplan [1] the drawback of not allowing unrestricted quantification over ordinary individuals. Without such quantification, however, I do not believe that one can treat ordinary language in a natural way or meet adequately Quine's objections to quantification into indirect contexts.

University of California, Los Angeles

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1 Other terms for these expressions include 'egocentric particulars' (Russell), 'token-reflexive expressions' (Reichenbach), 'indicator words' (Goodman), and 'noneternal sentences' (Quine, for sentences that are indexical).

2 For an account of the fundamental concepts of model theory see Tarski [2].

3 This work was reported in a talk I delivered before the U.C.L.A. Philosophy Colloquium on December 18, 1964. The treatment of special cases within the general framework of the present paper will be discussed in another publication.

4 Cocchiarella considered quantification only in connection with tense logic; his treatment may be found in the abstract Cocchiarella [1] and the unpublished doctoral dissertation Cocchiarella [2].

5 The idea of construing propositions, properties, and relations-in-intension as functions of the sorts above occurs first, I believe, in Kripke [1].

6 This simple and obvious approach is not the only possible treatment of 'thinks of', a phrase that has been discussed in the philosophical literature, for instance, in Anscombe [1], with incomplete success; but it is, I think, *one* possible treatment of *one* sense – the referential – of that phrase. For a treatment of the nonreferential sense see Montague [3].

7 Let us call an interpretation $\langle A, F, R\rangle$ *empty* if the union of the sets A_i for $i \in \mathbf{D}A$ is the empty set. We have not excluded empty interpretations from consideration, and it might be feared that minor difficulties might consequently arise in connection with the notions introduced in Definition VII. Such fears would be unjustified; it can easily be shown that the definition given above of logical consequence is equivalent to the result of adding to it the restriction that $\mathfrak{A}$ be a nonempty interpretation. On the other hand, some of the criteria given above of truth and satisfaction would fail for empty interpretations; but the case of empty interpretations is excluded by the assumption 'x is a member of the union of the sets A_j'.

8 Recursion on well-founded relations was first explicitly introduced in Montague [1]; for a discussion of it see Montague, Scott, Tarski [1].

9 The present system could, however, be extended so as to contain a full theory of definite descriptions in any of the well-known ways, for instance, that of Montague and Kalish [1]. It is partly in order to avoid irrelevant controversy over the best treatment of descriptions that I introduce them so sparingly here.

10 Problems of the first sort have been pointed out many times by Quine, for instance, in Quine [1]; and problems of the second sort arose in connection with Kaplan [1], the system of which appeared incapable of being extended in such a way as adequately to accommodate iteration of belief.

[11] A partial treatment of such a notion may be found in Montague and Kalish [2]. The discussion there is, however, incomplete in that it fails to provide for such cases as those for which the confirmatory criterion (2) was designed – cases in which beliefs may concern objects for which the believer has no name.

[12] The outline of extended pragmatics did not occur in the original version of this paper, but was added after I had seen a treatment of modal logic developed by Scott in June, 1967, and had discussed it with him and David Kaplan. The principal difference between Scott's system and extended pragmatics is that in the former no allowance is made for quantification over individuals, but only over individual concepts.

[13] This assertion, the formulation of which is partly due to J. A. W. Kamp, can be shown rather easily on the basis of the completeness theorem for ω-order logic of Henkin [1], and is not peculiar to *second*-order modal logic: indeed, the compactness theorem would hold for the predicative sentences of a higher-order modal logic containing variables of all finite levels.