Leibniz's Calculus of Real Addition*

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Zusammenfassung

In der vorliegenden Arbeit wird Leibniz' wahrscheinlich detailliertestes und ausgefeiltestes System untersucht: ein Kalkül der Einfügung und eine der Konjunktion ähnliche Operation, die er realis abjectio nennt. Das System soll hinreichend detailliert und mit hinreichender Präzision vorgestellt werden, um zu zeigen, daß es ausgefeilt formal logisch ist und eine Anzahl originärer und wichtiger Züge aufweist. Neben seinem eigenständigen Interesse ist dieses System wichtig wegen seiner Auswirkungen auf andere Aspekte von Leibniz' Logik und Philosophie, und ein weiteres Ziel dieser Arbeit ist, einige dieser Verbindungen aufzuspüren.

In this paper I will examine what is probably Leibniz's most detailed and polished logical system, a calculus of inclusion and a conjunction-like operator that he calls real addition. My first objective is to present this system in sufficient detail, and with sufficient precision, to show that it is a well-developed formal logic with a number of original and important features. But in addition to its intrinsic interest, Leibniz's system is significant because of its bearing on other aspects of his logic and philosophy, and my second objective is to trace some of these connections.

I sketch the motivation for my projection in § 1. In § 2. I consider Leibniz's presentation of his logical calculus in some detail, and in § 3. I present a formal development of it based on this examination. Leibniz's own exposition of his system is surprisingly sophisticated, and its syntax does not require much additional systematization. His semantics is less explicit, but he stresses that his logical calculus is amenable to alternative interpretations, and so he clearly appreciates the distinction between the formal system, on the one hand, and alternative interpretations of it, on the other. Accordingly, in the second half of § 3. I supply Leibniz's calculus with a formal semantics that is suggested by his use of his system, together with his general views about meaning and truth. In § 4., I discuss some of Leibniz's extensions of his logical calculus, examine several modern developments that he anticipated, and consider the bearing of his system on several other aspects of his work in logic and philosophy. In an appendix, I sketch a proof that the logic (as formalized in § 3.) is sound and complete.

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1. Background and Motivation

Although Leibniz is frequently regarded as the founder of symbolic logic, it is often thought that his primary achievement was to envision the general contours of the field, rather than to make any detailed contributions to it[1]. However, in his paper on real addition, Leibniz gives a sophisticated algebraic treatment of logic (discussed below in § 4.4.), presents detailed proofs of over twenty theorems about semilattices and shows their relevance to logic (§ 3. and § 4.4.), provides what is probably the first formal theory of the part-whole relation (§ 4.5.), and discusses alternative interpretations of his formal system (§ 3.2.). All of these are absolutely fundamental contributions to logic, and my first objective is to present Leibniz's system in sufficient detail to show just how this is so. But Leibniz's calculus of real addition also sheds light on other aspects of his logic and philosophy, including his views on the structure of concepts and on infinite analysis, and my second objective is to briefly trace some of these connections (§§ 4.6.– 4.7.).

Leibniz's paper on real addition occupies twelve pages in volume seven of Gerhardt's edition of Leibniz's philosophical writings[2]. Like many of Leibniz's other logical studies, it is untitled and its date is uncertain, but it is clearly a mature work, and there are several reasons why it is a fruitful place to begin a study of his logic[3]. First, although much of Leibniz's logical writing consists of incomplete and exploratory fragments, this paper is a finished and polished piece of work. Second, the paper contains Leibniz's most thoroughly developed logical system, including proofs of twenty-one theorems and several corollaries, three constructions, numerous examples and illustrative diagrams, and discussions of counterexamples that show various arguments formulable in his system to be invalid. Third, the logic developed in this paper is more limited in scope than a number of Leibniz's other calculi, and this makes it a more manageable system with which to begin. But finally, despite its modest aims, Leibniz's calculus of real addition forms the core of a number of his other logics, and so an understanding of it should provide a useful starting point for a study of them.

There have been several formal accounts of Leibniz's logical systems. In an important pioneering work Rescher reconstructs several of Leibniz's calculi, and in two more recent papers Castañeda formalizes one of Leibniz's systems

1 This view is expressed by W. and M. Kneale: *The Development of Logic*, Oxford 1962, pp. 320-321 and G. H. R. Parkinson in his introduction to *Leibniz: Logical Papers*, Oxford 1966, LIX-LXI.

2 See GP VII, 236-247/Leibniz: *Logical Papers* (LLP), transl. by G. H. R. Parkinson, Oxford 1966, pp. 131-144.

3 It was once thought that the paper was written after August 1690 (see LLP LVn5), but more recent research suggests that it may have been written in the period between 1685 and 1687 (see VE N. 419).

from 1686[4]. However, Castañeda focuses almost exclusively on syntax, and although Rescher has a valuable discussion of various interpretations of the systems he examines, he does not include a detailed formal treatment of their semantics. Moreover, although there are several interesting attempts to reconstruct the semantics implicit in Leibniz's work, these have not been geared to the syntax of specific calculi that Leibniz actually devised[5].

My account will differ from such treatments of Leibniz's logic in (i) focusing on his mature paper on real addition, (ii) providing a detailed account of both its syntax and its semantics, and (iii) concentrating mainly on what Leibniz considers the primary interpretations of his calculus, namely those in which its characters denote concepts (rather than sets, as in Lenzen's study[6]). My goal is to provide an account of Leibniz's logical calculus that is at once faithful to his text, yet precise enough to show that it is a satisfactory formal system that is sound and complete (relative to its natural semantics). Because Leibniz wrote long before the advent of contemporary symbolic logic, these two aims are in some tension, but it is surprising just how well they can be jointly accommodated.

2. Leibniz's Calculus of Real Addition

In this section I develop Leibniz's calculus of real addition as a fully formal system; throughout my concern is to stay as close to his text as possible, and so this section also serves as a commentary on the earlier parts of Leibniz's paper.

2.1. Overview of Leibniz's System

The three central notions of Leibniz's logical calculus are *identity*, *inclusion*, and a conjunction-like operator for forming compound names that he calls *real addition* (see LLP 143/GP VII, 246). His paper contains two axioms, two postulates, and six definitions, on the basis of which he constructs detailed proofs of over twenty theorems. However, his conception of an axiom system is more akin to Euclid's than to that current today. Like

4 See N. Rescher: *Leibniz's Interpretation of His Logical Calculi*, in: *The Journal of Symbolic Logic* 19 (1954), pp. 1-13; H.-N. Castañeda: *Leibniz's Syllogistico-Propositional Calculus*, in: *Notre Dame Journal of Formal Logic* 17 (1976), pp. 481-500 and H.-N. Castañeda: *Leibniz's Complete Propositional Logic*, in: *Topoi* 9 (1990), pp. 15-28.

5 Examples of such reconstructions may be found in I. Hacking: *Infinite Analysis*, in: *Studia Leibnitiana* VI/6 (1974), pp. 126-130 and B. Mates: *The Philosophy of Leibniz: Metaphysics and Language*, Oxford 1986.

6 See W. Lenzen: *Arithmetical vs. 'Real' Addition: A Case Study of the Relations Between Logic, Mathematics, and Metaphysics in Leibniz*, in: N. Rescher (ed.): *Leibnizian Inquiries: A Group of Inquiries*, New York 1989, pp. 149-157.

Euclid, he distinguishes axioms from postulates, his propositions include constructions to be performed (in addition to theorems to be proved) and, most importantly, his definitions are presented separately from his axioms, with several central concepts being defined in terms of other notions that are left completely undefined. We should not be surprised by such differences from our conception of an axiomatic system which, after all, derives largely from work in the foundations of geometry by Hilbert and others at the end of the nineteenth century. But these dissimilarities do mean that if we are to view Leibniz's calculus as a satisfactory formal system, we will have to reinterpret at least some of his definitions as something else (e.g., as rules of inference or as informal elucidations of primitive terms).

2.2. Characters and Signification

Leibniz uses the letters 'A', 'B', 'C', etc. as names in what we would now regard as the object language of his formal system. In light of current usage, it would be natural to call these symbols *terms*. However, Leibniz usually reserves the word 'term' (*terminum*) for the thing that such a letter denotes or *signifies* (typically a *concept* or *idea*; (e.g., LLP 39/C 243), rather than for the linguistic expression itself. I will follow Leibniz in his use of *term* and, adopting another of his labels, call the linguistic expression a 'character' (*character*)[7]. Leibniz sometimes urges that natural-language sentences containing nonsignifying characters should be counted as false (largely to ensure bivalence), and he seems to allow such characters in some of his formal systems as well (see LLP 82/C 393). However, he makes no provisions for nonsignifying characters in his paper on real addition, and so I shall assume that when his system is interpreted, all of its characters are to receive significations.

2.3. Definition One: Identity

Leibniz's paper opens with his celebrated definition of *identity*:

> Def 1: terms are the same or coincident which can be substituted for each other wherever we please without loss of truth value (see LLP 131/GP VII, 236).

Leibniz symbolizes the claim that A and B are identical as 'A ∞ B', but I will follow most commentators in writing it as 'A = B'. Definition 1 appears to

7 See e.g., *Monadology and Other Philosophical Essays*, transl. by P. Schrecker and A. M. Schrecker, New York 1965, p. 18/GP VII, 204.

say that the intersubstitutivity of terms is a sufficient condition for their identity, but in all of his uses of this definition Leibniz instead relies on its being a necessary condition. In a fragment containing a system closely related to his calculus of real addition, he makes his intentions clearer, telling us that intersubstitutivity is both sufficient *and* necessary for the identity of terms (see LLP 122/GP VII, 228). However, the principle that intersubstitutivity is a sufficient condition for identity is not easily formulated as an axiom or an inference rule in the language of his system, and in fact it plays no role in his calculus. Hence, I will confine attention to his claim that whenever two terms are identical, either can be substituted for the other *salva veritate*, i.e., without affecting the truth values of sentences that contain them.

As has often been lamented, Leibniz's habitual talk of identical *terms* being substitutable for one another involves a use-mention confusion. It is not *terms*, when these are taken to be concepts (or other extra-linguistic items), that can be substituted for each other. Rather, if the sentence 'A = B' is true, then the *characters* 'A' and 'B' signify one and the same thing, and it is these *linguistic* items, rather than the things they signify, that can be substituted for one another. Because seventeenth-century writers were much less attentive to the use-mention distinction than we are today, such lapses are to be expected, and fortunately they rarely lead Leibniz into serious difficulties. Indeed, with the judicious insertion of quotation marks (which are quite foreign to Leibniz's Latin) and attention to the distinction between object-language and meta-language, we can rephrase his account so as to avoid use-mention difficulties without altering his basic ideas. And to help with this, I shall use 'α', 'β', etc. as metalinguistic variables ranging over object-language characters (like 'A' and 'B') and 'φ'and 'ψ' as metalinguistic variables ranging over object-language sentences (like 'A = B').

In addition to blurring use and mention, Leibniz's characterization of identity employs undefined substantive notions like *substitution* and *truth-value*, and so by contemporary standards, it is defective as a definition in a formal system. However, Leibniz *uses* this principle to license *inferences* from a pair of sentences of the form φ(α) and ⌜α = β⌝ to the sentence $\varphi({}^{\alpha}_{\beta})$ (where φ(α) is any object language sentence with at least one occurrence of the character α, $\varphi({}^{\alpha}_{\beta})$ is a sentence that results from φ(α) by replacing one or more occurrences of α by the character β. Hence, I will treat Leibniz's "definition" of identity as a rule of inference which, because of its similarity to the natural-deduction rule of identity elimination, I will call '=E'[8].

8 We will see in § 3.1.5. that all instances of ⌜α = α⌝ are theorems of Leibniz's system, and in the presence of = E, this means that his characterization of identity is the same as that of contemporary first-order logic with identity. It should be noted that in *Leibniz's Syllogistico-Propositional Calculus*, and *Leibniz's Complete Propositional Logic*, Castañeda

2.4. Definition Two: Diversity

Leibniz's second definition is of *diversity*:

> Def 2: terms are diverse which are not the same, or, in which substitution sometimes does not hold (see LLP 131/GP VII, 236).

Leibniz symbolizes the assertion that α and β are diverse as $\ulcorner\alpha$ non ∞ $\beta\urcorner$. The claim that terms are not identical amounts to the negation of the claim that they are. Hence Leibniz's system includes a sentential connective for negation, and I will symbolize the negation of $\ulcorner\alpha = \beta\urcorner$ as $\ulcorner\neg\alpha = \beta\urcorner$ (or, more concisely, as $\ulcorner\alpha \neq \beta\urcorner$).

2.5. Indirect Proof and Bivalence

Leibniz proceeds to use Definition 1 to prove that identity is symmetrical (Proposition 1), transitive (Prop 3 and its corollary), and that '$A \neq C$' follows from the premises '$A = B$' and '$B \neq C$' (Prop 4). However, his derivation of '$B \neq A$' from '$A \neq B$' in Prop 2 is an indirect proof in which he assumes the opposite of the conclusion he wants to establish and derives a contradictory of a sentence that occurs earlier in the demonstration. Although his present paper does not contain anything that explicitly sanctions this mode of reasoning, Leibniz endorses it in many other places, telling us, for example, that it is obviously true that "to reduce a proposition to absurdity is to demonstrate its contradictory"[9]. Furthermore, he explicitly incorporates a rule of indirect proof in some of his other logics[10].

Leibniz's use of indirect proof is closely related to his view that language, including the formal language of his calculus, is *bivalent*, i.e., that (when it is interpreted) each of its sentences is either true or false[11]. In fact, bivalence justifies indirect proofs. Every sentence is either true or false, and a sentence and its contradictories have opposite truth values[12]. Thus, the derivation of a

argues that Leibniz's relation of coincidence is not identity, but a weaker congruence relation on concepts. He adduces little direct evidence for this claim, however, and it is difficult to reconcile with Leibniz's insistence on the centrality of identities in logic and his frequent characterizations of coincidence as *identity* or *sameness*. Additional arguments against Castañeda's position may be found in H. Ishiguro: *Leibniz's Philosophy of Logic and Language*, Cambridge ^{2}1990, Ch. 2 and p. 206n7.

9 *Nouveaux Essais* (*NE*) IV, VIII § 2/A VI, 6, 428/transl. by P. Remnant and J. Bennett, Cambridge 1981 (the pagination follows that of the Academy edition).

10 E.g., LLP 107/C 412; LLP 112/GP VII, 208; LLP 115-116/GP VII, 211-212.

11 See *NE* IV, II § 1/A VI, 6, 362; LLP 61/C 371.

12 See Leibniz: *Philosophical Papers and Letters* (*PPL*), transl. by L. E. Loemker, Dordrecht ^{2}1970, p. 225/GP VII, 299; LLP 112/GP VII, 208.

contradictory pair of sentences from a sentence φ means that φ is false and, hence, that any contradictory of φ is true.

2.6. Real Addition

Immediately after proving his first four theorems, Leibniz defines the notion of one term's being included in another (Def 3). However, this definition makes essential use of the notion of real addition which he has not yet defined, and so I will reverse the order of Leibniz's presentation and discuss real addition before inclusion.

Although Leibniz stresses that his calculus is amenable to alternative interpretations, he very frequently interprets its characters as signifying concepts and real addition as an operation for conjoining them; for example, the real addition of the concepts *rational* and *animal* is the complex concept *rational animal*. In order to accommodate real addition in the object language, Leibniz employs what we would now regard as a character-forming operator that allows us to join characters like 'A' and 'B' to produce the *composite character* '$A \oplus B$'. In interpretations in which the characters of his calculus denote concepts, '$A \oplus B$' signifies the compound concept that is the *real sum* of the concepts A and B.

The assumption that every character, including composite ones like '$A \oplus B$', has a signification means that real addition is a total function, so that every pair of concepts has a real sum. This is a strong existence claim, but there can be little doubt that Leibniz endorses it. For example, his second postulate permits the real addition of any two terms (see § 2.9.), and he stresses that "any term can be compounded with any term", even one that is "incompatible" with it (LLP 139/GP VII, 243). The use of '$\oplus$' as an operation symbol also requires that this sum be unique, and Leibniz always treats it as such.

After Definition 3 and three less central definitions (discussed in § 2.8. below), Leibniz presents two *axioms* for real addition. Unlike his definitions and postulates, these are sentences in the object language of his calculus, namely '$B \oplus N = N \oplus B$' and '$A \oplus A = A$' (see LLP 132/GP VII, 237). His use of these axioms clearly indicates that he means for any sentence having the same form as either of them to count as a theorem (and perhaps as an axiom) as well. Hence, a fully explicit presentation of Leibniz's system will either require a rule of substitution, which allows the substitution of any character for each occurrence of a primitive character in a sentence occurring at an earlier stage in a proof, or else his axioms must be viewed as embryonic axiom schemata that would properly be regarded as metalinguistic abbreviations for each of the infinitely many object-language sentences of the same form as the schema itself.

Axiom schemata were not explicitly introduced until the twentieth century, but earlier writers sometimes employed devices that came to much the same thing. Indeed, passages in some of Leibniz's logical writings can be read in this way[13], although other passages suggest something closer to a rule of substitution (see LLP 42–43/GP VII, 24). It would be equally straightforward to develop the present interpretation of Leibniz's system using either such a rule or schemata, but I will employ schemata here in order to more closely retrace the steps of Leibniz's proofs, which do not make explicit use of a rule of substitution.

Leibniz's discussion often suggests that he views '⊕' as a *binary* character-forming operator i.e., one capable of joining just two characters at a time, although there is also evidence that he regards it as *multigrade*, i.e., as capable of linking any (finite) number of characters at a fell swoop. In § 4.1. I will show how to treat Leibniz's operator as multigrade, but until then, there are several reasons for treating it as binary. First, in some places (including his axioms, where we would expect him to take special care), Leibniz seems to consider '⊕' a binary operation symbol. Second, even if real addition is multigrade, it is not irreducibly so. Repeated applications of binary real addition can always be used to define the real sum of any (finite) number of terms, and so a binary version of real addition and a (finitary) multigrade version come to much the same thing. Third, commentators typically treat this operation as binary, and since Leibniz's text does not settle the issue, I will follow suit so that my account can be more easily compared with theirs[14].

In schematic form, Leibniz's axioms are (A1) $\ulcorner \alpha \oplus \beta = \beta \oplus \alpha \urcorner$ and (A2) $\ulcorner \alpha \oplus \alpha = \alpha \urcorner$. (A1) says that real addition is *commutative*, insensitive to order. (A2) says that it is *idempotent*, so that the real addition of the same term to itself has no effect; $\alpha \oplus \alpha$ reduces to α. Treating real addition as a binary operation also requires a third axiom, (A3) $\ulcorner (\alpha \oplus \beta) \oplus \eta = \alpha \oplus (\beta \oplus \eta) \urcorner$, to ensure that it is *associative*. Although Leibniz does not include this axiom in his calculus, the need for it in some of his other work is noted by Frege, and it is discussed in some detail by Rescher (a chief reason for adding (A3) is that several of Leibniz's proofs fail without it; we will see an example of this in § 3.1.6.)[15]. Taken together, (A1)–(A3) tell us that in real addition order, repetition, and grouping are irrelevant. Once we know the subterms of a real sum, there is nothing more to learn about it[16].

13 See e.g., LLP 40/GP VII, 221; LLP 57/C 367.

14 Real addition is treated as a binary operation by Rescher (see note 4), p. 2, Castañeda in *Leibniz's Syllogistico-Propositional Calculus* (see note 4), p. 491, and Lenzen (see note 6), passim; a notable exception to this approach is Castañeda's *Leibniz's Complete Propositional Logic* (see note 4), § 3. Infinitary versions of real addition are also possible, and I will discuss one in § 4.7.

15 See G. Frege: *The Foundations of Arithmetic*, transl. by J. L. Austin, Evanston 1968, § 6 (first published as: *Die Grundlagen der Arithmetik*, Breslau 1884); Rescher (see note 4), p.11.

16 Although Leibniz's treatment of real addition in his present paper is more detailed than in most of his other work, both it and (A1) and (A2) occur in numerous other studies. He often represents real addition by the concatenation of characters like 'AB' or 'ab' (e.g., LLP 56 ff/C 366 ff), although he sometimes uses '+' (e.g., LLP 122-129/GP VII, 228-235) or (as here) '⊕' (e.g., C 256). He employs an axiom of commutivity in a number of other writings, (e.g., LLP 40/GP VII, 222; LLP 90/C 235; LLP 93/C 412, and C 421; see also *NE* III, III § 10/A VI, 6, 292). The idempotence axiom also occurs in many other works, (e.g., LLP 40/GP VII, 222; C 260; C 262; LLP 56/C 366; LLP 85/C 396; LLP 90/C 235; LLP 93/C 412; C 421, and LLP 124/GP VII, 230). Leibniz's present treatments of identity and inclusion are also similar to those in most of these other studies.

2.7. Inclusion

With the axioms for real addition in hand, we can return to Leibniz's account of the *inclusion* of one term in another:

> Def 3: A is in (*inesse*) L or L contains (*continere*) A is the same as that L is assumed to coincide with a plurality of terms, among which is A (see LLP 132/ GP VII, 237).

Leibniz's use of this definition clearly shows that by a *plurality of terms* he means a real sum of terms, one of which is A. Intuitively, α is included in β just in case α is one of the conjuncts or summands of β. For example, the concept *rational* is included in the (conjunctive) concept *rational animal*, i.e., in *rational* ⊕ *animal* (Leibniz's inclusion relation subsumes *improper* inclusion, so *rational* is also included in *rational*; Prop 7). For brevity, I will write the claim that α is included in β ($\ulcorner \alpha$ *inesse* $\beta \urcorner$) as $\ulcorner \alpha \leq \beta \urcorner$.

Definition 3 amounts to the claim that α is included in β just in case there is some term X such that $\ulcorner \alpha \oplus X = \beta \urcorner$. In many contemporary logics, this would be a perfectly respectable definition, but it requires an existential quantification, and these cannot be expressed in Leibniz's system. However, Leibniz uses his definition in three ways, two of which treat it rather like a pair of inference rules that allow the introduction and the elimination of the symbol '≤', just as an explicit definition would do. And because it will do much less violence to Leibniz's system to treat Def 3 as including such a pair of rules, rather than as a definition that requires an existential quantifier, I will do so here.

More specifically, Leibniz uses his definition of inclusion in the following three ways:

1. It is sometimes used to move from an identity to an "inclusion". For example, it is used to move from 'A ⊕ B = L' to 'A ≤ L'(e.g., the scholium to Prop 19).

2. It is sometimes used to move from an inclusion to an identity. For example, it is used to move from 'B ≤ L' to 'L = B ⊕ A' (e.g., Prop 14).

3. It is sometimes used, rather like an axiom, to enter a sentence of the form $\ulcorner\alpha \leq \alpha \oplus \beta\urcorner$ in a derivation. For example, it is used to justify the direct entry of sentences like 'A ≤ A ⊕ B' and 'A ⊕ B ≤ (A ⊕ B) ⊕ N' in proofs (see Prop 12).

In the first use, Def 3 functions as a rule taking us from an identity like 'A ⊕ B = C' to the inclusion 'A ≤ C', and given the intended meaning of inclusion, this step preserves truth.

However, arguments that run back the other direction, *from* 'A ≤ C' *to* 'A ⊕ B = C', are not valid. If A is in C, then A together with *some* other term coincides with C, but that other term needn't be B. The idea behind Leibniz's second kind of use of Def 3 is more subtle than this, however, as can be illustrated by the following line of reasoning. Suppose that A is in C. Then A together with *some* other term composes C. Call this other term 'N', selecting a *new* character for the purpose to ensure that it isn't already being used to name something that isn't in C (see LLP 57/C 367). The idea is reminiscent of what Greek geometers and logicians called *ekthesis*[17]. It is also similar to the Gentzen-style natural-deduction rule of existential elimination, which tells us that if we can deduce a sentence φ from the provisional assumption of an instance of an existential quantification (using an instantial constant that doesn't appear earlier in the proof or in φ), then we can deduce φ directly from the existential quantification itself.

Gentzen's rule may seem to embody an inordinately elaborate pattern to attribute to a seventeenth-century thinker, but in fact Leibniz uses something very like it; in particular,

1. He is committed to what amounts to the use of provisional assumptions that are discharged later in demonstrations (because of his use of indirect proof; e.g., Prop 2).

2. He always introduces a *new* character ν each time he moves from a sentence of the form $\ulcorner\alpha \leq \beta\urcorner$ to one of the form $\ulcorner\alpha \oplus \nu = \beta\urcorner$.

3. He uses these new characters in the course of proofs, but they do not appear in his conclusions.

In short, Leibniz's first two uses of his definition of inclusion involve something very like the following pair of inference rules. The first rule, which I will call '≤ E', says that if, in the presence of $\ulcorner\alpha \leq \beta\urcorner$, we can use a provisional assumption of $\ulcorner\alpha \oplus \nu = \beta\urcorner$ to derive a sentence φ (where ν is a non-complex character that doesn't occur earlier in the proof, in $\ulcorner\alpha \leq \beta\urcorner$, or in φ), then φ follows from just $\ulcorner\alpha \leq \beta\urcorner$ (and any undischarged background premises). The second rule, ≤ I, allows us to move from an identity $\ulcorner\alpha \oplus \beta = \eta\urcorner$ to an inclusion $\ulcorner\alpha \leq \eta\urcorner$. These rules will be formulated precisely in § 3.1.5., where we will also see that Leibniz's third use of Def 3, to enter sentences of the form $\ulcorner\alpha \leq \alpha \oplus \beta\urcorner$ directly in derivations, can be justified in terms of the other rules of his system[18].

17 See Aristotle's discussion in the *Prior Analytics* 25a16.

18 Other reconstructions of Leibniz's second use of Def 3 are possible. In various other writings, he distinguishes between *definite* characters, like 'A' and 'B', which are used to signify specific things, and *indefinite* characters, like 'X' and 'Y', which are variable-like symbols used to signify 'indefinitely'. And in at least one fragment he suggests that 'A ≤ C' means the same thing as 'A ⊕ Y = C' (see C 265). This suggests a rule allowing us to move from $\ulcorner\alpha \leq \beta\urcorner$ to $\ulcorner a \oplus \mu = \beta\urcorner$, where μ is a (new) *indefinite* character. We will see in § 3.1.7. that it is also possible to define inclusion in terms of real addition in a way that avoids the need for any special rules or axioms for inclusion and that the resulting system is equivalent to the present one. I will not adopt either of these approaches to Def 3, however, because each is too different from Leibniz's presentation in his present paper, and my goal is not merely to formulate a system that permits the derivation of Leibniz's theorems, but to be able to reproduce his derivations step by step.

2.8. Further Definitions

Leibniz's first three definitions introduce fundamental principles governing identity, diversity (and with it sentential negation), and inclusion. Immediately after Def 3, he presents three further definitions, but these merely introduce additional terminology. The fourth definition tells us that the constituent terms of a real sum are its *components* (although as the paper proceeds, Leibniz more frequently calls them *inexistents*) and, correlatively, that a complex term is *composed of*, or *constituted by*, its components. The fifth definition tells us that terms are *subalternants* if either includes the other, and the sixth that terms are *disparate* when they are not subalternants.

2.9. Postulates

Leibniz rounds out his system with two postulates. Unlike his axioms, they are not identities and, indeed, are not even presented in the object language of his calculus. I will begin with Leibniz's less problematic second postulate, which plays the dual role of ensuring that real addition is a total function and of sanctioning the use of definitions:

> Postulate Two "Any plurality of terms, such as A and B, can be taken together to compose one term, A ⊕ B, or, L" (LLP 132/GP VII, 237).

In part, this postulate licenses joining any pair of terms to form their real sum, but Leibniz also uses it to introduce defined characters (e.g., LLP 138–139/GP VII, 242–243). A similar postulate in a paper containing a system closely related to the present one is slightly more detailed:

> "Several terms, whatever they may be, can be taken together to constitute one; thus, if there are A and B there can be formed from these A ⊕ B, which can be called L" (LLP 124/GP VII, 230)[19].

19 See also LLP 42/GP VII, 224; LLP 38-39/C 242; LLP 57/C 367.

In short, Leibniz is not only providing for the formation of complex characters, but is also establishing a convention for introducing definitional abbreviations for them. As he puts it in his discussion of a related logical system:

"If we have assumed some simple term as equivalent to some composite term, i.e. as expressing the same thing, then the simple term will be the 'defined term' and the composite term will be the 'definition'. This defined term, expressed by a symbol, we shall call henceforth the 'name' of the thing" (LLP 44/GP VII, 226–227)[20].

And such definitions are important, since they let us substitute abbreviations for complex characters that would be difficult to remember or manipulate[21].

If a definition is to serve as a mere abbreviation, however, it must be eliminable (in favor of the defining expressions) and noncreative (not allowing us to prove anything in the original language – before the defined character was added – that we couldn't prove without it). I will make this more precise in § 3.1.4., but the basic idea is that $\ulcorner \nu = \alpha \oplus \beta \urcorner$ is a legitimate definition of ν just in case α and β contain only primitive (or previously defined) characters and ν is a primitive character that is new to the language (and so not in α or in β). When Leibniz introduces defined characters with Postulate 2, they are in fact always new to the problem he is working on. Hence, if we consider the sublanguages of his system that contain just the primitive vocabulary employed in particular problems, his uses of Postulate 2 to introduce defined characters do satisfy these requirements.

In Leibniz's logical work postulates also serve to tell us what sorts of hypotheses are legitimate[22], and this is the task of his first postulate:

> Postulate One "Given any term, some term can be assumed which is different from it, and, if one pleases, disparate, i.e. such that the one is not in the other" (LLP 132/GP VII, 237).

This is a strong existence assumption, telling us that for each term we can assume the existence of a second term that neither includes, nor is included in, the first. Hence, Postulate 1 is quite different in spirit from anything else in Leibniz's calculus, and since he only invokes it in performing two constructions, I will set it aside until § 4.2., where we will see how to incorporate it into his system.

20 Here and in the other quoted passages on definitions, Leibniz uses 'term' to mean 'linguistic expression'.
21 See *PPL* 292/GP IV, 423; *Selections*, transl. by P. P. Wiener, New York 1951, p. 28/C 326.
22 See e.g., LLP 42/GP VII, 224; LLP 90/C 235; LLP 124/GP VII, 230.

2.10. Regimentation of Natural Language

Few claims are more central to Leibniz's mature philosophy than his doctrine that a subject-predicate sentence of a natural language is true just in case the concept signified by the predicate is included in the concept signified by the subject. This claim is especially important for him, since he also holds that virtually all natural-language sentences can be reduced to those of subject-predicate form[23]. But the syntax of his calculus of real addition does not allow the direct expression of subject-predicate sentences like 'Gold is a metal' or 'Adam sinned'.

It would be possible to extend Leibniz's system in various ways to allow a more direct representation of natural-language sentences in it. However, he holds that ordinary language has numerous defects that render it incapable of functioning as a calculus[24], and he typically adopts the logically conservative policy of regimenting natural language in the canonical idiom of his formal systems. Since Frege's day, it has been part of our translation lore to represent sentences of the form $\ulcorner$All αs are βs$\urcorner$ as first-order sentences of the form $\ulcorner \forall \mu(\alpha\mu \supset \beta\mu) \urcorner$. Similarly, I believe, Leibniz has an incipient doctrine of logical form, according to which a sentence $\ulcorner \sigma$ is $\pi \urcorner$ is represented by a sentence in his formal system of the form $\ulcorner \pi \leq \sigma \urcorner$[25]. And the role of his calculus is to then provide a precise and detailed treatment of these formal counterparts of natural-language sentences.

3. $\mathcal{L}^{\oplus}$: The Formalization of Leibniz's Calculus of Real Addition

In § 3.1. I collect the principles discussed in the previous section together into a formal system, $\mathcal{L}^{\oplus}$, and in § 3.2. provide $\mathcal{L}^{\oplus}$ with a formal semantics that is suggested by Leibniz's use of his calculus, together with his general views about meaning and truth.

3.1. Syntax of $\mathcal{L}^{\oplus}$

One of the chief purposes of Leibniz's logics is to serve as instruments for human reasoning. Hence, because our cognitive abilities are finite – "one cannot go to infinity in proofs" (PPL 225/GP VII, 299)[26] – $\mathcal{L}^{\oplus}$ should not

23 See e.g., LLP 12-13/C 244-245; LLP 84/C 395.
24 See Leibniz (see note 7), p. 18/GP VII, 205; *NE* III, I § 5/A VI, 6, 276; see also LLP 12-13/C 243-245; LLP 13-16/C 286-290.
25 See LLP 67/C 378; *Philosophical Writings* (*PW*), transl. by M. Morris and G. H. R. Parkinson, London 1973, p. 87/ C 518-519.
26 See also *PW* 97/C 17-18; *PW* 75/GP VI, 309.

contain infinitely long characters, sentences, or proofs. It will be convenient to allow denumerably many primitive characters, but this will not make the system infinitary in any objectionable sense, since particular applications of it need only involve sublanguages with a finite number of primitives.

3.1.1. Primitive Vocabulary

We begin with the *vocabulary* of the language of $\mathcal{L}^{\oplus}$:

Primitive (Simple) Characters: Denumerably many Roman capital letters, 'A', ... , 'T', with or without positive integer subscripts.
Binary Character-Forming Operator: $\oplus$
Two-place Predicates: $=, \leq$
Sentential Connective: $\neg$
Punctuation marks: (,)

3.1.2. Formation Rules

We next give a recursive definition of the set of *characters of* $\mathcal{L}^{\oplus}$:

(FC1) If α is a primitive character, then α is a character.
(FC2) If α and β are characters, then $\ulcorner(\alpha \oplus \beta)\urcorner$ is a (composite) character (I will call it the *real conjunction* of α and β, and will say that α and β are its *immediate subcharacters*)[27].

We then define the set of *sentences* of $\mathcal{L}^{\oplus}$; if α and β are characters and φ is a sentence, then

(FS1) $\ulcorner\alpha = \beta\urcorner$ is a(n atomic) sentence (and is called an *identity*).
(FS2) $\ulcorner\alpha \leq \beta\urcorner$ is a(n atomic) sentence (and is called an *inclusion*).
(FS3) $\ulcorner \neg\varphi\urcorner$ is a sentence (and is called the *negation of* φ).

Only expressions that can be generated by a finite number of applications of these rules are characters and sentences of $\mathcal{L}^{\oplus}$. To enhance readability, I will drop unnecessary parentheses, and when issues of use and mention aren't explicitly at stake, I will follow the convention of autonymous use, letting each simple expression stand for itself and each juxtaposition of expressions stand for their concatenation.

27 It is often natural to call characters like 'A $\oplus$ B' *real sums*, and it is usually harmless to do so. However, in § 2.6. we adopted this label for the compound terms that such composite characters signify.

3.1.3. Axioms

All instances of the following schemata are *axioms of* $\mathcal{L}^{\oplus}$:

(A1) $\alpha \oplus \beta = \beta \oplus \alpha$,
(A2) $\alpha \oplus \alpha = \alpha$
(A3) $(\alpha \oplus \beta) \oplus \eta = \alpha \oplus (\beta \oplus \eta)$

3.1.4. Postulates

In § 4.2. we will see how to incorporate Postulate 1 in $\mathcal{L}^{\oplus}$ as a rule of inference. However, this postulate raises several special problems, and because Leibniz rarely uses it, I will defer discussion of it to that subsection.

One task of Postulate 2 is to enable us to form the real conjunction of any two characters, and the formation rule (FC2) already takes care of this. A second task is to allow the introduction of defined characters in the manner discussed in § 2.9. For this purpose, we suppose that we have available a supply of auxiliary characters, $A'_1, A'_2, \ldots$. Then $\nu = \alpha \oplus \beta$ is a correct definition of ν just in case (i) each sub-character of α and of β is either a (non-auxiliary) primitive character or else a previously defined character and (ii) ν is a primitive character that is new to the language (or to the sublanguage of $\mathcal{L}^{\oplus}$ with which we are working).

3.1.5. Rules of Inference

If χ is an atomic sentence, I will say that its *simple contradictory* is $\neg\chi$; if χ is a negation, $\neg\theta$, it has two simple contradictories, θ and $\neg\neg\theta$. Let φ^* be a simple contradictory of φ, $\varphi(\alpha)$ a sentence with at least one occurrence of the character α, and $\varphi(\genfrac{}{}{0pt}{}{\beta}{\alpha})$ a sentence obtained from $\varphi(\alpha)$ by replacing one or more occurrences of α by the character β. With these conventions, the *inference rules* for $\mathcal{L}^{\oplus}$ have the following schematic form:

$$\text{IP:}\quad \frac{\begin{array}{c}\underline{\varphi}\\ \vdots\\ \psi^* \quad \psi\end{array}}{\varphi^*}
\qquad\qquad
\leq\text{E:}\quad \frac{\begin{array}{c}\underline{\alpha \oplus \nu = \beta}\\ \vdots\\ \alpha \leq \beta \quad \psi\end{array}}{\psi}$$

Where ν is a simple character and neither it, nor any character ultimately defined in terms of it, occurs in $\alpha \leq \beta$, ψ or any undischarged premises.

$$= \text{E}: \ \frac{\varphi(\alpha) \quad \alpha = \beta}{\varphi(^{\beta}_{\alpha})} \qquad = \text{E}: \ \frac{\varphi(\beta) \quad \alpha = \beta}{\varphi(^{\alpha}_{\beta})} \qquad \leq \text{I}: \ \frac{\alpha \oplus \beta = \eta}{\alpha \leq \eta}$$

The notation works as in standard presentations of modern natural deduction systems. Thus, the rules = E and ≤ I sanction inferences of sentences of the form below the line from sentences of the form above it. The remaining rules, IP and ≤ E, involve provisional assumptions (φ, in IP; $\alpha \oplus \nu = \beta$, in ≤ E). This means that if we deduce a sentence (perhaps satisfying certain conditions) from the provisional assumption, then we can discharge that assumption and transfer dependence of our conclusion to the premise of the sentence immediately above the line (ψ^*, in IP; $\alpha \leq \beta$, in ≤ E), together with any other undischarged premises. For readability, I have not stated all of the variations on order that can be obtained using the symmetry of identity and the commutivity of real addition, but I will also count such modifications as instances of these rules (for example, in ≤ I the premise $\alpha \oplus \beta = \eta$ would license the inference to $\beta \leq \eta$ as well as to $\alpha \leq \eta$).

The *premise set* for a rule is the set of premises on which a sentence obtained by an application of the rule depends. Premises depend on themselves, and the premise set of an axiom is empty. The premise sets for = E and ≤ I are the premise sets of their premises. The premise set for IP is the union of the premise sets of ψ and ψ^*, minus φ, and that for ≤ E is the union of the premise sets of $\alpha \leq \beta$ and the first occurrence of ψ, minus $\alpha \oplus \nu = \beta$. A *derivation* is defined in the now standard way, and when ψ is derivable from the sentences $\varphi_1, \dots, \varphi_n$, I will write $\varphi_1, \dots, \varphi_n \vdash \psi$.

There are two important schemata that Leibniz treats rather like axioms, but which can in fact be derived in $\mathcal{L}^{\oplus}$. First, although he frequently declares that we need not (indeed cannot) demonstrate "formal and explicit identities" like A = A (PPL 226/GP VII, 300), such sentences can be derived in his calculus by entering $\alpha = \alpha \oplus \alpha$ (by A2), then using the substitutivity of identity (= E) to obtain $\alpha = \alpha$. Hence, we can allow the direct entry of such identities in proofs using a derived rule I will call = I. Second, we saw in § 2.7. that in addition to the uses of the definition of inclusion (Def 3) captured by ≤ I and ≤ E, Leibniz sometimes uses this principle to justify the direct entry of instances of $\alpha \leq \alpha \oplus \beta$ in proofs. We can justify this by using = I to derive $\alpha \oplus \beta = \alpha \oplus \beta$, then using ≤ I to obtain $\alpha \leq \alpha \oplus \beta$. Leibniz often calls Def 3 the definition of *inexistence*, and so I will say that such instances of $\alpha \leq \alpha \oplus \beta$ are obtained by the (derived) rule *Inexistence*.

3.1.6. A Sample Derivation

I will illustrate the workings of the inference rules of $\mathcal{L}^{\oplus}$ by reconstructing a representative derivation from Leibniz's paper. Because $\mathcal{L}^{\oplus}$ does not contain a conditional, the resulting theorem must be viewed as a metalinguistic claim about derivability, rather than as a sentence in the object language itself.

> Proposition 12: If B is in L, then A ⊕ B will be in A ⊕ L... For let L = B ⊕ N (by the definition of inexistent); A ⊕ B is also in B ⊕ N ⊕ A (by the same definition), that is [A ⊕ B is] in A ⊕ L (see LLP 134/GP VII, 239).

In short, $B \leq L \vdash A \oplus B \leq A \oplus L$; in $\mathcal{L}^{\oplus}$ the proof runs as follows:

{1}	(1) $B \leq L$	Premise
{2}	(2) $L = B \oplus N$	Provisional Assumption (for ≤ E)
Ø	(3) $A \oplus B \leq (A \oplus B) \oplus N$	Inexistence
Ø	(4) $A \oplus B \leq A \oplus (B \oplus N)$	3 A3
Ø	(5) $A \oplus B \leq (B \oplus N) \oplus A$	4 A1
{2}	(6) $A \oplus B \leq L \oplus A$	5, 2 = E
{1}	(7) $A \oplus B \leq L \oplus A$	1, 2, 6 ≤ E
{1}	(8) $A \oplus B \leq A \oplus L$	7 A1

Note that the instantial character, 'N', is new to the proof and doesn't occur in the conclusion. Leibniz's remaining theorems can be reproduced in similar fashion (as in this example, many of them require the associativity axiom), and the arguments he shows to be intuitively invalid can be given countermodels using the semantics developed in § 3.2.

3.1.7. A (non-Leibnizian) Definition of Inclusion

In modern algebra, inclusion relations like ≤ are often defined by a biconditional like $\alpha \leq \beta$ iff $\alpha \oplus \beta = \beta$ (see § 4.4.). Leibniz proves that the two halves of this biconditional are equivalent[28], and it is routine to construct derivations in $\mathcal{L}^{\oplus}$ that retrace his arguments. This would allow us to use this definition of ≤ to replace the rules ≤ I and ≤ E. We could dispense with ≤ I, since once we have this definition, a brief *reductio* shows that $\alpha \oplus \beta = \eta \vdash \alpha \leq \eta$. To see how to dispense with ≤ E, suppose that we have $\alpha \leq \beta$. Using our new definition of inclusion, we rewrite this as $\alpha \oplus \beta = \beta$. To introduce the new character ν of ≤ E, we invoke Postulate 2 to obtain $\nu = \beta \oplus \beta$, then use (A2) to reduce this to $\nu = \beta$. By = E we infer $\alpha \oplus \nu = \beta$, and we then proceed to derive φ as before. The modern definition of ≤ thus provides an alternative formalization of $\mathcal{L}^{\oplus}$ that allows us to dispense with ≤ as a primitive of the system. But whatever its attractions, it is not the formalization that Leibniz provides.

28 In Propositions 13 and 14 (see LLP 135/GP VII, 239).

3.2. Semantics of $\mathcal{L}^{\oplus}$

Formal semantics as we now think of it did not exist until Tarski's work in the 1930s. However, Leibniz is mindful of the distinction between syntax and semantics (e.g., *NE* III, II § 5 = A VI, 6, 287), and his views about the nature of real addition can be developed into a formal semantics for $\mathcal{L}^{\oplus}$ in a quite natural way. Doing so will enable us to justify the syntactic principles of his logic by showing that its axioms are necessarily true and its inference rules necessarily truth preserving. It will also highlight the abstract formal structure of Leibniz's system in a way that is less sensitive to the nuances of particular presentations (e.g., to which of its semantically equivalent axiomatizations is selected for study) than syntactic treatments are.

Leibniz stresses that his calculus of real addition can be interpreted extensionally or intensionally; indeed, "whenever [...] [its axioms] are observed, the present calculus can be applied" (LLP 142/GP VII, 245)[29]. Thus he clearly appreciates the distinction between the syntax of the system, on the one hand, and various meanings that can be assigned to its characters, on the other[30]. Furthermore, Leibniz's views on meaning and truth involve something very like what is now known as *referential* semantics, according to which the meaning of a character is (at least in part) the extra-linguistic thing that it signifies. This comes out, for example, in his criticisms of Hobbes's view that truth is a matter of convention, which, Leibniz contends, stems from a failure to appreciate the distinction between language (which *is* conventional) and the extra-linguistic reality that linguistic expressions *signify* (which is independent of conventions)[31]. Leibniz believes that various sorts of interpretations of his calculus can be useful, so it would be misleading to speak of an *intended* interpretation of $\mathcal{L}^{\oplus}$. But the *primary* interpretations he has in mind for his system are those in which its characters signify *concepts*[32].

29 See also Leibniz: *Selections* (see note 21), p. 74/C 531.

30 Leibniz presents most of his formal calculus without invoking any semantic notions like signification or truth, and this makes it reasonably straightforward to separate the syntax and semantics of $\mathcal{L}$. However, his definition of identity does rely on the notion of truth. I have urged that this definition is best construed as a semantic characterization of identity that justifies the syntactic rule = E of § 2.3., but this is not to deny that Leibniz's grip on the use-mention distinction often weakens when identity is involved.

31 See LLP 33/GP VII, 219; *PPL* 182-185/GP VII, 190-193; *NE* IV, V § 11/A VI, 6, 397-398.

32 In many of the examples in Leibniz's present paper, characters signify concepts, and in numerous other works they do so as well (e.g., LLP 39/C 243). This is motivated, of course, by his view that truth conditions for sentences are specified in terms of the concepts that their constituent characters signify (see § 2.10.).

3.2.1. Leibnizian Relational Structures

The key notions in Leibniz's primary interpretations of his calculus are those of a *concept*, the *real addition* of concepts, and the *signification* of characters. I will systematize these notions using a bit of apparatus from contemporary model theory. The point of this is simply to organize the account, however, and it will not saddle Leibniz with any substantive semantic views that are not supported by the text; for example, the semantic values of characters will still be the things that they signify (rather than some sort of set-theoretic surrogates for these).

We will interpret the language of $\mathcal{L}^{\oplus}$ over what I will call *Leibnizian Relational Structures*. Such structures are ordered pairs of the form, $\mathcal{L} = \langle \mathcal{C}, \underline{\oplus} \rangle$, where $\mathcal{C}$ is a nonempty set and $\underline{\oplus}$ is a total, binary operation on $\mathcal{C}$ that is commutative, idempotent, and associative. In keeping with Leibniz's view that $\mathcal{L}^{\oplus}$ can be interpreted extensionally as well as intensionally, $\mathcal{C}$ could be any nonempty family of sets and $\underline{\oplus}$ the set-theoretic operation of union (see note 47). But in primary interpretations, $\mathcal{C}$ is a set of concepts and $\underline{\oplus}$ is a logical operation that forms *real sums* of the concepts in $\mathcal{C}$. We could add a primitive inclusion relation, $\leqq$, to $\mathcal{L}$, but to keep semantic primitives to a minimum, I will instead define this relation with the equivalence $x \leqq y$ iff $x \underline{\oplus} y = y$ embodied in Leibniz's Propositions 13 and 14. Finally, a *model* of $\mathcal{L}^{\oplus}$ is an ordered pair, $\mathfrak{M} = \langle \mathcal{L}, [\ ]_{\mathfrak{M}} \rangle$, where $\mathcal{L}$ is a Leibnizian relational structure, and $[\ ]_{\mathfrak{M}}$ is an interpretation function that assigns significations to the characters of $\mathcal{L}^{\oplus}$ in the way explained in the following subsection.

3.2.2. Signification and Truth

We define the *signification* of the character α in the model $\mathfrak{M}$ (abbreviated as $[\alpha]_{\mathfrak{M}}$) as follows:

(IC1) If α is a primitive character, then $[\alpha]_{\mathfrak{M}} \in \mathcal{C}$.
(IC2) For each real conjunction $\beta \oplus \eta$, $[\beta \oplus \eta]_{\mathfrak{M}} = [\beta]_{\mathfrak{M}} \underline{\oplus} [\eta]_{\mathfrak{M}}$.

We then define the relation of the sentence φ's being *true in the model* $\mathfrak{M}$ (abbreviated $\mathfrak{M} \models \varphi$):

(IS1) $\mathfrak{M} \models \alpha = \beta$ iff $[\alpha]_{\mathfrak{M}} = [\beta]_{\mathfrak{M}}$.
(IS2) $\mathfrak{M} \models \alpha \leq \beta$ iff $[\alpha]_{\mathfrak{M}} \leqq [\beta]_{\mathfrak{M}}$ (i. e., iff $[\alpha]_{\mathfrak{M}} \underline{\oplus} [\beta]_{\mathfrak{M}} = [\beta]_{\mathfrak{M}}$).
(IS3) $\mathfrak{M} \models \neg\varphi$ iff it is not the case that $\mathfrak{M} \models \varphi$.

Leibniz endorses bivalence, so the sentence φ is *false* in $\mathfrak{M}$ just in case it is not true in $\mathfrak{M}$. As usual, a set of sentences Γ is *satisfiable* just in case it has a model, and Γ *entails* the sentence φ ($\Gamma \models \varphi$) just in case every model of Γ is also a model of φ. Proofs that $\mathcal{L}^{\oplus}$ is sound and complete (relative to this semantics) are sketched in the appendix.

The syntactic structure of Leibniz's calculus (embodied in the five formation rules of § 3.1.2.) mirrors its semantic structure (embodied in the five semantic evaluation rules of the previous paragraph). This is as it should be, for Leibniz is convinced that a logically perspicuous language

"[...] would represent our thoughts truly and distinctly, and [...] when a thought [e.g., a concept, which is the semantic value of a character in Leibniz's primary interpretations of his logic] is composed of other simpler ones, its character would also be similarly composed"[33].

Such passages contain perhaps the earliest suggestions of a compositional semantic theory in the history of logic. They also reflect Leibniz's preoccupation with combinatorial rules and their use as an effective procedure – 'a mechanical thread' – to draw inferences based on the forms of sentences[34]. Indeed, (IC2) and (IS1) – (IS3) simply are combinatorial rules that provide an algorithm for determining the semantic value for each form of complex linguistic expression, given the semantic values of its simpler parts[35].

4. Extensions, Anticipations, and Applications

In this section I examine several additional features of Leibniz's calculus and discuss some of its implications for other aspects of his logic and philosophy.

4.1. Real Addition as a Multigrade Operation

Commentators typically treat '⊕' as a symbol for a *binary* operation. Some passages in Leibniz certainly suggest such a reading, but there is also evidence that he views it as a symbol for a *multigrade* operation, one capable of joining any finite number of characters in a single stroke. For one thing, he sometimes treats relations as multigrate; for example, in work on the geometry of situation (*analysis situs*) of 1679, his central primitive notion is a generalized congruence relation that can relate any (even) number of points (see *PPL* 249–253/GM II, 20–27). Furthermore, Postulate 2 of his present paper says that "any plurality of terms [...] can be taken together to compose one term" (LLP 132/GP VII, 237)[36], and virtually none of his logical writings contain grouping devices like parentheses. If '⊕' is a binary operation symbol, this means that Leibniz often fails to discuss associativity when he should; for example, he fails to include an axiom to accommodate it, despite his insistence that features as basic as commutivity be explicitly addressed. If real addition is a multigrade operation, however, we needn't ascribe such carelessness to him.

We can treat real addition as a multigrade operation by making the following modifications in $\mathcal{L}^{\oplus}$. The vocabulary remains the same, but we rewrite the formation rule for composite characters to say that if $\varphi_1, \dots, \varphi_k$ ($k \geq 2$) are characters, then $\ulcorner\oplus(\varphi_1, \dots, \varphi_k)\urcorner$ is also a character (which for readability can be written in infix notation as $\ulcorner\varphi_1 \oplus \dots \oplus \varphi_k\urcorner$). The rules of inference and the first two axioms stay the same, but the axiom for associativity is no longer required. Derivations remain much like those in § 3.1.6., although with the demise of the associativity axiom, they parallel Leibniz's own proofs even more closely. On the semantic side, we require that every subset of terms $x_1, \dots, x_k$ ($k \geq 2$) in C has a real sum $\underline{\oplus}(x_1, \dots, x_k)$, also in C. As with its linguistic counterpart, we retain the commutivity and idempotence axioms for $\underline{\oplus}$, but discard associativity. Finally, we generalize (IC2) so that $[\oplus(\varphi_1, \dots, \varphi_n)]_{\mathfrak{M}} = \underline{\oplus}([\varphi_1]_{\mathfrak{M}}, \dots, [\varphi_n]_{\mathfrak{M}})$. For simplicity, I shall employ the binary version of real addition in what follows, but it would be relatively straightforward to recast subsequent discussion in terms of the multigrade version introduced here.

4.2. Postulate 1

In § 2. I deferred discussion of Postulate 1 to this subsection. This assumption, which Leibniz only employs in constructions (rather than in proofs of theorems), reads:

"Given any term, some term can be assumed which is different from it, and, if one pleases, disparate, i. e. such that the one is not in the other" (LLP 132/GP VII, 237).

Leibniz's presentation at the beginning of his paper is reminiscent of Euclid's presentation at the beginning of the *Elements*. Euclid begins with five postulates and five axioms ('common notions'), the axioms supposedly being applicable to any subject matter, the postulates just to geometry. Now Leibniz sometimes equates postulates and axioms (*PPL* 187/A II, 1, 398), and other times he speaks of postulates as "understood not in Euclid's way, but in Aristotle's, namely as assumptions which we are willing to agree on while awaiting an opportunity to prove them" (*NE* IV, VII § 11/A VI, 6, 419). However, a key feature of postulates in his logical writings is to serve as

33 *Philosophical Essays*, transl. by R. Ariew and D. Garber, Indianapolis 1989, p. 240/GP IV, 295-296; see also GM IV, 141; Leibniz: *Selections* (see note 21), p. 10/GP VII, 192; LLP 33/GP VII, 219; LLP 21/C 54; *PPL* 192-194/GM IV, 460-462; Leibniz (see note 7), p. 18/GP VII, 209.

34 See LLP 192-194/GM IV, 460-462; see also *PPL* 670/GM VII, 24; LLP 85-87/C 396-399; GP IV, 27-102.

35 I discuss this aspect of Leibniz's logic and semantics in more detail in *Leibnizian Expression*, forthcoming in the *Journal of the History of Philosophy*.

36 See LLP 124/GP VII, 230; LLP 42/GP VII, 224.

existence assumptions[37]. This is the role of Postulate 1, and in this respect it *is* like Euclid's first three postulates.

Euclid's first three postulates, which authorize the construction of straight lines and circles, in effect serve as existence assumptions about such figures. Unlike most present-day geometry, Euclid's treatise contains many constructions of figures, and in Descartes' *Geometry* of 1637 the emphasis on constructions is even more pronounced. And just as the first twenty-one propositions of Leibniz's paper generalize the idea of proving theorems in geometry[38] to proving theorems in logic about the abstract structure of concepts, his last three propositions generalize the idea of geometrical construction to show that concepts satisfying certain specifications can be constructed. For example, in Prop 22 Leibniz shows how to construct a concept C that is different from both members of any pair of disparate concepts, A and B, yet such that either $A \oplus C$ is in $B \oplus C$, or vice versa. With this generalization, questions about specific methods of geometrical construction are left far behind, but some primitive assumptions about what can be constructed are still necessary for constructions to get off the ground. And much as Euclid's first postulate allows us to draw a straight line connecting any given pair of points, Leibniz's first postulate allows us to introduce the name of a term that is disparate from any given term.

The construction of a concept shows that it is consistent (provided that its constituent concepts are), and this is closely related to Leibniz's frequent claim that a *real definition* of a concept not only catalogues its subconcepts, but also shows that the concept is possible or consistent. Indeed, a paradigm of the real definition of the concept of a geometrical figure is provided by a construction, which shows that the concept is consistent by exhibiting it[39]. Leibniz's constructions also provide a brief formal treatment of one aspect of *problem solving* (namely, finding concepts that satisfy certain conditions), which is noteworthy because of his life-long dream to devise a Universal Language that could be used not only for proving theorems, but also for solving problems.

Although Postulate 1 is quite powerful, Leibniz doesn't pause to defend it. I think the main reason for this is that in primary interpretations of his calculus, its characters signify concepts, and a concept can always be negated; indeed, many of Leibniz's other logical systems explicitly include an operation for negating them. And if, as I shall argue in the next subsection, Leibniz views his calculus of real addition as a minimal system that can be extended to incorporate such operations, this would mean that each term automatically has at least one term disparate from itself, namely its negation.

One way to accomodate Postulate 1 in $\mathcal{L}^{\oplus}$ is to add an inference rule, P1, that licenses the move from $\varphi(\alpha)$ on line 1 of a derivation to $\neg(\nu \leq \alpha)$ on line

37 See e.g., LLP 124/GP VII, 230; LLP 90/C 235.
38 And a few other fields (see *NE* IV, II § 13/A VI, 6, 370-371).
39 See *NE* III, III § 18/A VI, 6, 295; see also *PPL* 230-231/GP VII, 293-295.

$n (n > 1)$ and $\neg(\alpha \leq \nu)$ on $n + 1$ (where ν, which is to serve as the name of a term that is disparate from the term named by α, is a primitive character that does not occur earlier in the proof). On the semantic side, we require that for every concept in the domain of a Leibnizian Relational Structure, there be some other concept that neither includes, nor is included in, it. However, this approach is not in the spirit of the rest of Leibniz's system, and so here I shall treat Postulate 1 as a metalinguistic principle allowing us to assume the existence of disparate terms when performing constructions. It would be straightforward, however, to use P1 to make this postulate an integral part of $\mathcal{L}^{\oplus}$ itself.

4.3. Extensions of $\mathcal{L}^{\oplus}$: Negation

Leibniz's calculus of real addition is limited in scope, but there is good evidence that he regards it as a core system that can be (conservatively) extended in various ways. Thus, near the end of his paper on real addition, he says:

"[...] as various laws of combination [of characters] can be discovered, the result of this is that various methods of computation arise. Here, however, no account is taken of the variation which consists in a change of order alone [...]. Next, no account is taken here of repetition [...]" (LLP 142/GP VII, 245)[40].

A page later he adds that "in due course order also will be considered", although he doesn't do so in the present paper. Moreover, Leibniz's calculus of real addition, or one very like it, forms a subsystem of several of his other logical systems[41]. Indeed, in other papers Leibniz employs several devices that could be used to extend $\mathcal{L}^{\oplus}$, including variable-like symbols ('indefinite' characters), the incorporation of the predicates 'true' and 'false' in the object language, and various treatments of the syllogism[42]. It would take us too far afield to examine these extensions here, but I will briefly indicate how a combinatorial operation of negation can be added to $\mathcal{L}^{\oplus}$.

Leibniz often claims that concepts have negations; for example, the negation of the concept *happy* is the concept *not-happy*, and something falls under the latter just in case it does *not* fall under the former[43]. Moreover, in many of his other logical systems, he includes character-forming *negation* operators, often writing the name for the negation of the concept A as 'not-A'(non-A). His axioms for negation vary from paper to paper, but he almost always includes one of the form ⌜α = not-not-α⌝[44]. In a system of 1690

40 See C 256.
41 See e.g., LLP 40-46/GP VII, 221-227; LLP 90-92/C 235-237; LLP 93-94/C 421-423; see note 16.
42 See e.g., LLP 47-87/C 356-399; LLP 90-92/C 235-237; LLP 93-94/C 421-423.
43 See Leibniz: *Philosophical Essays* (see note 33), p. 11-12/C 86; see also LLP 47/C 356; LLP 53/C 363; LLP 79/C 390.
44 See e.g., LLP 69/C 379; LLP 84-86/C 396-397.

that includes $\mathcal{L}^{\oplus}$, he also employs an axiom of the form ⌜$\alpha \neq (\beta \oplus$ not $\alpha)$⌝ (see LLP 90/C 235), and I will consider these two axiom schemata here, since they enable us to derive many of Leibniz's principles about negation.

We extend $\mathcal{L}^{\oplus}$ to the calculus $\mathcal{L}^{\oplus\mathcal{N}}$ by adding a unary, character-forming operator, 'not-', to its vocabulary, the formation rule (FC3), which tells us that if α is a character, then ⌜not-α⌝ is also a character, and the axiom schemata (A4), ⌜α = not-not-α⌝ and (A5), ⌜$\alpha \neq (\beta \oplus$ not-$\alpha)$⌝. On the semantic side, we consider ordered triples $<\mathcal{C}, \underline{\oplus}, Neg>$, where $\mathcal{C}$ and $\underline{\oplus}$ are as before and Neg is a unary operation on $\mathcal{C}$ that maps terms to their 'negations' (subject to the restrictions that for each x and y in $\mathcal{C}$, $x = Neg(Neg(x))$ and $x \neq (y \underline{\oplus} Neg(x))$). Finally, we add a clause (IC3) to the definition of an interpretation in § 3.1.2., telling us that $[\text{not-}\alpha]_{\mathfrak{M}} = Neg([\alpha]_{\mathfrak{M}})$.

Many of Leibniz's central principles about negation are provable in $\mathcal{L}^{\oplus\mathcal{N}}$, including $\vdash$ not-α = not-α, $\alpha = \beta \vdash \alpha \neq$ not-β, and $\alpha = \beta \vdash$ not-α = not-β (e.g., LLP 83/C 394–395). It is also possible to provide a justification for Postulate 1 in this system. We begin by noting that we can now prove that α and not-α are disparate, i.e., that all instances of $\neg(\alpha \leq$ not-$\alpha)$ and $\neg($not-$\alpha \leq \alpha)$ are theorems. For example, we prove that $\neg(A \leq$ not-$A)$ is a theorem by assuming the opposite, namely $A \leq$ not-A. By Prop 14, this delivers $A \oplus$ not-A = not-A. By (A1) and (A4) this yields not-A = not-A $\oplus$ not-not-A. We then enter an instance of (A5), not-A $\neq$ (not-A $\oplus$ not-not-A), which is the contradictory of the previous sentence, and use IP to conclude $\neg(A \leq$ not-A). This done, we set not-A equal to a new character ν by Postulate 2 and (A2) to obtain a name, ν, of a term disparate from α, as required[45].

4.4. Leibnizian Algebras

Leibniz's insistence that $\mathcal{L}^{\oplus}$ is amenable to alternative interpretations, together with his modeling of real addition on the standard algebraic operation of numerical addition (with the difference that it is idempotent[46]), inaugurated the algebraic approach to logic that runs from Boole and Pierce in the nineteenth century to work in cylindrical and polyadic algebras in the twentieth. Furthermore, because Leibniz's axioms are quantifier-free equations (see LLP 118/GP VII 214), his formulation of his calculus is also algebraic, and so it is not surprising that a Leibnizian Relational Structure, $\mathcal{L} = <\mathcal{C}, \underline{\oplus}>$, turns out to be an *algebra*. More specifically, because the operation $\underline{\oplus}$ is idempotent,

45 Further axioms for negation, or for additional combinatorial operations like disjunction, can be added on the model employed here. Although Leibniz does not explicitly restrict negation to simple concepts, doing so would fit better with some passages which suggest that concepts can in principle be analyzed into simple concepts and their negations. We could impose this restriction on $\mathcal{L}^{\oplus\mathcal{N}}$ combining it with the system in § 4.6. below.

46 See LLP 143/see also GP VII 246; and LLP 124/GP VII, 230.

commutative, and associative, it is a *semilattice*, and given Leibniz's treatment of the relationship between inclusion and real addition, it is a *join semilattice*, with $\underline{\oplus}$ its join[47].

In a semilattice we can define a binary relation $\leqq$ in terms of the join with the biconditional $x \leqq y$ iff $x \underline{\oplus} y = y$, which is just the equivalence embodied in Leibniz's Propositions 13 and 14. Often the next step in a text on lattices is to prove that $\leqq$ is a *partial ordering*, i.e., that it is reflexive ($x \leqq x$), anti-symmetric (if $x \leqq y$ and $y \leqq x$, then $x = y$), and transitive (if $x \leqq y$ and $y \leqq z$, then $x \leqq z$), and Leibniz gives detailed proofs that $\leqq$ has these three properties in Propositions 7, 17, and 15 respectively. Another rudimentary result in lattice theory is that if $x \leqq z$ and $y \leqq z$, then $x \underline{\oplus} y \leqq z$, which is proved in Proposition 18. Again, it is typically shown that $\underline{\oplus}$ is isotone or order-preserving, i.e., that if $x \leqq y$, then $z \underline{\oplus} x \leqq z \underline{\oplus} y$, and Leibniz proves this in Proposition 12.

Like most of his logics, Leibniz's calculus of real addition does not include a disjunction operation, and so a Leibnizian Relational Structure may well lack a meet and thus not be a full-fledged lattice (much less a Boolean algebra). But it is still a remarkable fact that two centuries before Dedekind launched the modern study of lattices, Leibniz had produced quite thorough and rigorous proofs of over twenty basic theorems about semilattices and shown their relevance to logic.

4.5. Leibnizian Mereology

Leibniz often says that when one concept is included in (but not identical with) a second, the first is a *part* of the second[48]. Elsewhere, including his present paper, he claims that although the formal features of inclusion are a necessary aspect of a part-whole relation, they are not sufficient for it:

47 In many familiar cases where lattices are of logical relevance (e.g., propositional Boolean algebras, algebras of sets), conjunction-like operations (e.g., truth-functional conjunction, set intersection) are *meets* rather than *joins*. This is so, because the conjunction-like entities resulting from the application of such operations are 'included' in the items to which the operation is applied to obtain them (e.g., $x \cap y \subseteq y$). However, the situation is inverted in *intensional* interpretations, where terms are regarded as concepts. This is so, because one concept is included in a second just in case it is a conjunct of it (see LLP 136/GP VII, 240; LLP 20-21/C 53; A VI, 6, 486, 275; LLP 74/C 384-385). Related views about the inverse relationship between intension and extension had been advanced in 1662 by Arnauld and Nicole in the Port Royal Logic (*The Art of Thinking*, Pt. II, ch. 17). I discuss Leibniz's views on the relationships between intensions and extension in more detail in *Leibniz on Intension and Extension*, forthcoming in *Noûs*.

48 See LLP 19-20/C 52; LLP 66-67/C 377; see also *NE* IV, XVII § 8/A VI, 6, 486; LLP 29/C 81-82.

"[...] if the terms which are in something are homogeneous with that in which they are contained, they are called parts and the container is called a whole" (LLP 142/GP VII, 245)[49].

In short, the inclusion relation in the presence of homogeneity is the same thing as the part-whole relation. It would take us too far afield to explore Leibniz's notion of homogeneity, which involves the geometric notions of similarity and dimension[50]. But it is important to note that Leibniz's calculus is probably the earliest formal theory of the part-whole relation (or at least of the relation of inclusion that underlies it).

In recent mereological theories like Leonard and Goodman's calculus of individuals, the part-whole relation is a partial ordering, and every set of things has a mereological sum. In $\mathcal{L}^{\oplus}$, $\leqq$ is a partial ordering, and any finite number of terms have a real sum. Repeated applications of real addition cannot generate infinite sums, however, and so an infinite set of terms will not have a real sum. Recent mereological accounts are sometimes criticized for incorporating such strong existence assumptions, however, and so it is not obvious that Leibniz's account is the worse for this omission[51].

4.6. Simple Concepts

Leibniz holds that all complex concepts are built up from simple concepts by real addition (and probably other operations, at least negation). As the years went by, he came to doubt that human beings were capable of isolating absolutely simple concepts, but he continued to believe that we could discover those concepts that were incapable of further analysis by us[52]. These concepts would comprise an alphabet of human thought, and the primitive characters of a logically perspicuous language would signify them[53]. However, this aspect of Leibniz's views is not reflected in his calculus of real addition; indeed, his system is neutral as to the existence of simple concepts, and in some of its models every concept is infinitely complex, the conjunction of two further concepts, downward without end (see LLP 141-142/GP VII, 245). We can implement Leibniz's view that simple characters should signify simple concepts, though, with a few minor changes in $\mathcal{L}^{\oplus}$.

We select a subset, $\mathcal{S}$, of primitive characters whose member will serve as names of simple concepts. In the semantics, we extend Leibnizian Relational Structures to ordered triples of the form $<\mathcal{C}, \mathcal{C}^s, \underline{\oplus}>$, where $\underline{\oplus}$ is as before, $\mathcal{C}^s$ is a non-empty set, and $\mathcal{C}$ is the closure of $\mathcal{C}^s$ under the operation $\underline{\oplus}$. In primary interpretations, $\mathcal{C}^s$ is a set of simple concepts, and $\mathcal{C}$ contains all of the concepts that can be generated from the members of $\mathcal{C}^s$ by repeated applications of real addition. We then say that a primitive interpretation is a function $[\ ]^s$ that maps each simple character in $\mathcal{S}$ to a member of $\mathcal{C}^s$ (if each simple concept is to have only one name in an ideal language, we should also require that $[\ ]^s$ be one-one). Finally, an interpretation based on $[\ ]^s$ is a function such that for each primitive character α, $[\alpha] = [\alpha]^s$, and for all characters α and β, $[\alpha \oplus \beta] = [\alpha] \underline{\oplus} [\beta]$. The remaining clauses of the definition of an interpretation are as given in § 3.2.2.

4.7. Infinite Analysis

Leibniz maintains that a finite analysis can reduce a necessary truth to an identity, whereas "true contingent propositions require an analysis continued to infinity" (LLP 61/C 371). And near the end of his paper on real addition, he adds that "the whole of synthesis and analysis depends on the principles laid down here" (LLP 142/GP VII, 245). The idea is that a sentence $\ulcorner\sigma$ is $\pi\urcorner$ is contingently true just in case we could in principle begin with the claim that the concept π is included in the concept σ, and by "an analysis of the terms of the proposition and the substitution of the definition of a part of it, for the thing defined" get ever closer to, though never arrive at, transparently true identities of the form $\ulcorner\alpha = \alpha\urcorner$[54]. But if Leibniz's picture of infinite analysis is to make sense, the concepts involved must be infinitely complex, and indeed he holds that individual concepts, like that of Adam, are complete and so infinitely structured[55]. There are many difficulties with Leibniz's account of contingency, but here I want to ask whether his account of the logical structure of concepts, or any natural extension of it, can explain the possibility of infinitely complex concepts.

Leibniz typically holds that the stock of simple concepts is finite; indeed, God produces the greatest variety of things with the most economical means possible, so it may well be quite small[56], and the first difficulty is that it is

49 See LLP 122-123/GP VII, 229; *PPL* 668/GM VII, 19.

50 See *PPL* 666-669/GM VII, 18-21; see also LLP 122-123/GP VII, 229; *NE* Preface/A VI, 6, 63-64.

51 In a study probably composed around the same time as the paper on real addition, Leibniz isolates the notions of communicating and uncommunicating terms (see LLP 123/GP VII, 229). These correspond to the relations of overlapping and being discrete from, respectively, of the calculus of individuals (see H. S. Leonard and N. Goodman: *The Calculus of Individuals and its Uses*, in: *Journal of Symbolic Logic* 5 (1940), pp. 45-55).

52 See Leibniz: *Selections* (see note 21), p. 51/C 176; *PW* 10/GP VII, 292.

53 See Leibniz (see note 7), pp. 18-19/GP VII 205; see also LLP 33-34/GP VII, 219.

54 A.-L. Foucher de Careil (ed.): *Nouvelles lettres et opuscules inédits de Leibniz*, Paris 1857, pp. 181-182/*PPL* 264f; see also *PPL* 267-268/C 518-519; *PW* 75/GP VII, 309.

55 See e.g. LLP 66/C 377; *PPL* 332-335/GP II, 48-54; *PW* 77/GP VII, 311.

56 See *PW* 2/C 430; Leibniz (see note 7), p. 18/GP VII, 205.

impossible to construct infinitely complex concepts from a finite set of primitives with just the operation of real addition. This is so because real sums are amorphous; order, repetition, and grouping make no difference, so that once we know the simple constituents of a complex concept, there is nothing more to learn about it. Hence, the set of real sums that can be generated from a finite set of primitive concepts is in one-to-one correspondence with the set of all subsets of that original set. This means that if we begin with n primitive concepts, we can only generate $2^{\mathcal{N}}$ concepts, each with n or fewer (primitive) subconcepts. Consequently, when $\mathcal{N}$ is finite, we can only construct a finite number of concepts, none of which is infinitely complex.

Leibniz does say that "infinite things can be compounded out of the combination of a few" (*PW* 2/C 430), and this may suggest that it was a mistake to regard real addition as a finitary operation. Nowadays infinitary operations are familiar; for example, a *complete* join semilattice is one in which every set of elements, whatever its cardinality, has a join. And perhaps Leibniz vaguely anticipated such operations, thinking that even infinite sets of concepts have real sums. If so, we should extend Leibnizian relational structures to include an infinitary version of $\oplus$. But the most complex concept that can be generated from a set of concepts with such an operation is still just the real sum of all of the concepts in the set, so as long as the set of primitives is finite, even an infinitary version of real addition will only allow the construction of finite concepts.

However, Leibniz sometimes suggests that there might be infinitely many primitive concepts (see *PW* 10/GP VII, 291). If so, a binary version of real addition would be able to generate infinitely many concepts, although since repeated applications of a finitary version of real addition cannot generate sums of infinite sets of concepts, we still could not obtain concepts of infinite complexity. Hence, if we are to generate infinitely complex concepts using anything like the operation of real addition (even when augmented by term negation and disjunction), we need infinitely many primitive concepts *and* an infinitary version of real addition. But aside from a few passing remarks about infinite conjunctions (see LLP 142/GP VII, 245). Leibniz doesn't discuss such operations. And so a gap remains between the infinitely structured concepts that his accounts of individual concepts and contingency require and what his logical apparatus can provide[57].

57 It should be stressed that we are considering an infinitary version of the operation $\oplus$ on concepts, rather than an infinitary version of the character-forming operator $\oplus$ in the object language of $\mathcal{L}^{\oplus}$. As long as the language is finitary, $\oplus$ must be finitary, and so infinitely complex concepts could not be fully described using it. This is what we should expect, however, since as Leibniz stresses, infinitely complex concepts can never be fully grasped by the finite creatures for whom $\mathcal{L}^{\oplus}$ is designed. With an infinitary version of $\oplus$, Leibniz's account of the part-whole relation also moves closer to more recent accounts of it.

5. Conclusion

Leibniz's calculus of real addition, with its concentration on inclusion and conjunction, is undeniably limited in scope, and some of its features, particularly its treatment of definitions, are bound to seem inadequate today. But this should not obscure the fact that his system has a number of striking and important features. Among other things, it sheds light on Leibniz's views about parts and wholes, the structure of concepts, and infinite analysis. It is also remarkable how many features of Leibniz's account, including his algebraic treatment logic, his formal account of something very like the part-whole relation, and his discussion of the possibility of alternative interpretations of his abstract formal system, are detailed anticipations of later developments that are important even today.

Appendix: Metatheory of $\mathcal{L}^{\oplus}$

$\mathcal{L}^{\oplus}$ is strongly sound and complete. Since it is a weak system, these results are not of great intrinsic interest, but they are good evidence that Leibniz's calculus validates all of the theorems it should and that the semantics supplied in § 3. for it is the correct one.

We can prove that $\mathcal{L}^{\oplus}$ is *strongly sound* – i.e., that for any set of sentences Γ and any sentence φ of $\mathcal{L}^{\oplus}$, if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$ – by an induction on the length of derivations, showing that the sentence on each line is entailed by its premise set. The basis clause and the inductive steps for all of the inference rules but $\leq$ E are straightforward, and the proof for this rule is analogous to that for the rule of existential elimination in contemporary natural-deduction systems.

$\mathcal{L}^{\oplus}$ is also *strongly complete*: If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$ A set of sentences is *consistent* just in case there is no sentence such that both it and its negation are derivable from that set, and it is maximally consistent if no further sentences can be added to it without destroying its consistency. So as usual, completeness is equivalent to

(C) If Δ is a consistent set of sentences of $\mathcal{L}^{\oplus}$, then Δ has a model,

which we establish with a Henken proof.

Assume that Δ is a consistent set of sentences. We extend it to a maximal consistent superset, $\mathfrak{M}^*$, by an induction along an enumeration of the sentences of $\mathcal{L}^{\oplus}$ with steps and proofs of the familiar sort. If Δ^* is a maximal consistent set of sentences of $\mathcal{L}^{\oplus}$, then: (a) $\varphi \in \Delta^*$ iff $\sim\varphi \notin \Delta^*$, (b) $\varphi \in \Delta^*$ iff $\Delta^* \vdash \varphi$, (c) if $\varphi(\alpha) \in \Delta^*$ and $\alpha = \beta \in \Delta^*$, then $\varphi\binom{\beta}{\alpha} \in \Delta^*$, (d) $\alpha = \alpha \in \Delta^*$, (e) if φ is an axiom, then $\varphi \in \Delta^*$, and (f) $\alpha \leq \beta \in \Delta^*$ iff $\alpha \oplus \beta = \beta \in \Delta^*$. Clauses (a)–(e) are demonstrated in the usual way. Clause (f) requires the use of the rules $\leq$ I and $\leq$ E, and its demonstration is essentially that given by Leibniz in his proofs of Propositions 13 and 14 (see LLP 135/GP II, 239).

The next step is to construct a model, $\mathfrak{M}^*$, for Δ^*. We begin by factoring out equivalence classes of characters of $\mathcal{L}^{\oplus}$, taking α's equivalence, class $[\alpha]$, as the set $\{\alpha' : \ulcorner\alpha = \alpha'\urcorner \in \Delta^*\}$ (the proof that such sets are equivalence classes relies on properties (c) and (d) of maximal consistent sets). We then take the set of these classes as the domain of $\mathfrak{M}^*$'s underlying Leibnizian relational structure and let each character signify its associated equivalence class, i.e., $[\alpha]_{\mathfrak{M}^*} = [\alpha]$. For example, $[A \oplus B]_{\mathfrak{M}^*}$ includes '$A \oplus B$', '$B \oplus A$', '$(A \oplus A) \oplus B$', and infinitely many other characters as well.

The final step is to show that $\mathfrak{M}^*$ is a model of Δ^*, i.e., that for each sentence φ of $\mathcal{L}^{\oplus}$,

(Mod) $\varphi \in \Delta^*$ if and only if $\mathfrak{M}^* \models \varphi$.

The proof proceeds by induction on the number of negations in φ. The basis clause divides into a case for identities and one for inclusions. $\mathfrak{M}^*$ is contrived to ensure that (Mod) holds for identities, but the situation for inclusions is more interesting. Interpretations in familiar systems like first-order logic assign extensions to predicates like '$\leq$', but this is not how things work in Leibniz's system. Hence, a bit of effort is needed to prove that $\alpha \leq \beta \in \Delta^*$ iff $\mathfrak{M}^* \models \alpha \leq \beta$. Going left to right, assume that $\alpha \leq \beta \in \Delta^*$. Then $\alpha \oplus \beta = \beta \in \Delta^*$ (by properties (b) and (f)). Since (Mod) holds for identities, $\mathfrak{M}^* \models \alpha \oplus \beta = \beta$. And so, by the definition of $\leqq$, $\mathfrak{M}^* \models \alpha \leq \beta$. For the converse, assume that $\mathfrak{M}^* \models \alpha \leq \beta$. Then, by the definition of $\leqq$, $\mathfrak{M}^* \models \alpha \oplus \beta = \beta$. Since (Mod) holds for identities, $\alpha \oplus \beta = \beta \in \Delta^*$. Hence, $\alpha \leq \beta \in \Delta^*$ (again by (b) and (f)). The inductive step proceeds in the usual way for negation. Finally, the restriction of $\mathfrak{M}^*$ to Δ is a model of Δ, which concludes the proof of (C), and so of the completeness of $\mathcal{L}^{\oplus}$. The compactness and downward Löwenheim-Skolem theorems follow as immediate corollaries in the usual way.

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