WOLFGANG LENZEN

LEIBNIZ'S LOGIC

1 INTRODUCTION

The meaning of the word 'logic' has changed quite a lot during the development of logic from ancient to present times. Therefore any attempt to describe "the logic" of a historical author (or school) faces the problem of deciding whether one wants to concentrate on what the author himself understood by 'logic' or what is considered as a genuinely logical issue from our contemporary point of view. E.g., if someone is going to write about Aristotle's logic, does he have to take the entire *Organon* into account, or only the First (and possibly the Second) Analytics? This problem also afflicts the logic of Gottfried Wilhelm Leibniz (1646–1716).

In the late 17th century, logic both as an academic discipline and as a formal science basically coincided with Aristotelian syllogistics. Leibniz's logical work, too, was to a large extent related to the theory of the syllogism, but at the same time it aimed at the construction of a much more powerful "universal calculus". This calculus would primarily serve as a general tool for determining which formal inferences (not only of syllogistic form) are *logically valid*. Moreover, Leibniz was looking for a "universal characteristic" by means of which he hoped to become able to apply the logical calculus to arbitrary (scientific) propositions so that their *factual truth* could be "calculated" in a purely mechanical way. This overoptimistic idea was expressed in the famous passage:

> If this is done, whenever controversies arise, there will be no more need for arguing among two philosophers than among two mathematicians. For it will suffice to take the pens into the hand and to sit down by the abacus, saying to each other (and if they wish also to a friend called for help): *Let us calculate.*[1]

Louis Couturat's well-known monograph *La logique de Leibniz*, published in 1901, contains, besides a series of five appendices, nine different chapters on "La Syllogistique, La Combinatoire, La Langue Universelle, La Caractéristique Universelle, L'Encyclopédie, La Science Générale, La Mathématique Universelle, Le Calcul Logique, Le Calcul Géométrique ". This very broad range of topics may perhaps properly reflect Leibniz's own

---

[1]Cf. **GP** 7, 200: "Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo (accito si placet amico) dicere: *Calculemus*". The abbreviations for the editions of Leibniz's works are explained at the beginning of the bibliography.

understanding of 'logic', and it certainly does justice to the close interconnections between Leibniz's ideas on logic, mathematics, and metaphysics as expressed in often quoted statements such as "My Metaphysics is entirely Mathematics"[2] or "I have come to see that the true Metaphysics is hardly different from the true Logic"[3]. In contrast to Couturat's approach (and in contrast to similar approaches in Knecht [1981] and Burkhardt [1980]), I will here confine myself to an extensive reconstruction of the *formal core of Leibniz's logic* (sections 4–7) and show how the theory of the syllogism becomes provable within logical calculus (section 8). In addition, it will be sketched in section 9 how a part of Leibniz's "true Metaphysics" may be reconstructed in terms of his own "true logic" which had been prophetically announced in a letter to Gabriel Wagner as follows:

> It is certainly not a small thing that *Aristotle* brought these forms into unfailing laws, and thus was the first who wrote mathematically outside *Mathematics*. [...] This work of *Aristotle*, however, is only the beginning and quasi the *ABC*, since there are more composed and more difficult forms as for example *Euclid*'s forms of inference which can be used only after they have been verified by means of the first and easy forms [...] The same holds for *algebra* and many other formal proofs which are naked, though, and yet perfect. It is namely not necessary that all inferences are formulated as: *omnis, atqui, ergo*. In all unfailing sciences, if they are proven exactly, quasi higher logical forms are incorporated which partly flow from Aristotle's [forms] and partly resort to something else.
>
> [...] I hold for certain that the art of reasoning can be further developed in uncomparable ways, and I also believe to see it, to have some anticipation of it, which I would not have obtained without *Mathematicks*. And though I already discovered some foundation when I was not even in the mathematical *novitiate* [...], I eventually felt how entangled the paths are and how difficult it would have been to find a way out without the help of an inner mathematicks. Now what, in my opinion, might be achieved in this field is of such great an idea that, I am afraid, no one will believe before presenting real examples.[4]

The systematic reconstruction of Leibniz's logic to be developed in this chapter reveals five different calculi which can be arranged as follows:

---

[2]Cf. **GM** 2, 258: "Ma Metaphysique est toute mathematique".

[3]Cf. **GP** 4, 292: "j'ay reconnu que la vraye Metaphysique n'est guères differente de la vraye Logique".

[4]Cf. Leibniz's old-fashioned German in **GP** 7, 522.

$$L0.4$$
$$\downarrow$$
$$L0.8$$
$$\downarrow$$
$$L1 \longrightarrow PL1$$
$$\downarrow$$
$$L2$$
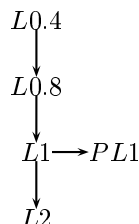
Four of these calculi form a chain of increasingly stronger logics $L0.4$, $L0.8$, $L1$, and $L2$, where the decimals are meant to indicate the respective logical strength of the system. All these systems are *concept logics* or term-logics, to use the familiar name from the historiography of logic. Only the fifth calculus, *PL1*, is a system of *propositional logic* which can be obtained from $L1$ by mapping the concepts and conceptual operators into the set of propositions and propositional operators.

The most important calculus is $L1$, the full *algebra of concepts* which Leibniz developed mainly in the *General Inquiries* (**GI**) of 1686 and which will be described in some detail in section 4 below. As was shown in Lenzen [1984b], $L1$ is deductively equivalent or isomorphic to the ordinary algebra of sets. Since Leibniz happened to provide a complete set of axioms for $L1$, he "discovered" the Boolean algebra 160 years before Boole.

Also of great interest is the subsystem $L0.8$. Instead of the conceptual operator of negation, it contains subtraction (and some other auxiliary operators). Since, furthermore, the conjunction of concepts is symbolized there by the addition sign, it is usually referred to as *Plus-Minus-Calculus*. Leibniz developed it mainly in the famous essay "A not inelegant Specimen of Abstract Proof"[5]. This system is inferior to the full algebra $L1$ in two respects. First, it is conceptually weaker than the latter; i.e. not every conceptual operator of $L1$ is present (or at least definable) in $L0.8$. Second, unlike the case of $L1$, the axioms or theorems discovered by Leibniz fail to axiomatize the Plus-Minus-Calculus in a complete way. The decimal in '$L0.8$' can be understood to express the degree of *conceptual incompleteness* − just 80 percent of the operators of $L1$ are able to be handled in the Plus-Minus-Calculus. In the same sense, the weakest calculus $L0.4$ contains only 40 percent of the conceptual operators available in $L1$. In view of the the main operators of containment and converse containment, i.e. being contained, Leibniz occasionally referred to it as "Calculus of containing and being contained" [Calculus de Continentibus et Contentis]. He began to develop it as early as in 1676; and he obtained the final version in the "Specimen Calculi Universalis" (plus "Addenda") dating from around 1679. Leibniz reformulated this calculus some years later in the so-called "Study

---

[5] "Non inelegans specimen demonstrandi in abstractis" − **GP** 7, 228-235; **P**., 122-130.

in the Calculus of Real Addition", i.e. fragment # XX of **GP** 7 [236–247; **P**., 131–144]. In view of the fact that the mere Plus-Calculus is only a weak subsystem of the Plus-Minus-Calculus, it must appear somewhat surprising that many Leibniz-scholars came to regard the former as superior to the latter.[6] Both calculi will be described in some detail in section 5.

Now a characteristic feature of Leibniz's algebra $L1$ (and of its subsystems) is that it is in the first instance *based upon* the propositional calculus, but that it afterwards serves as a *basis for* propositional logic. When Leibniz states and proves the laws of concept logic, he takes the requisite rules and laws of propositional logic for granted. Once the former have been established, however, the latter can be obtained from the former by observing that there exists a strict analogy between *concepts* and *propositions* which allows one to re-interpret the conceptual connectives as propositional connectives. This seemingly circular procedure which leads from the algebra of concepts, $L1$, to an algebra of propositions, *PL1*, will be described in section 6. At the moment suffice it to say that in the 19th century George Boole, in roughly the same way, first presupposed propositional logic to develop his algebra of sets, and only afterwards derived the propositional calculus out of the set-theoretical calculus. While Boole thus arrived at the classical, two-valued propositional calculus, the Leibnizian procedure instead yields a *modal* logic of strict implication. As was shown in Lenzen [1987], *PL1* is deductively equivalent to the so-called Lewis-modal system S2°.

The final extension of Leibniz's logic is achieved by his theory of indefinite concepts which constitutes an anticipation of modern *quantification theory*. To be sure, Leibniz's theory is, in some places, defective and far from complete. But his ideas concerning quantification about concepts (and, later on, also about individuals or, more exactly, about *individual-concepts*) were clear and detailed enough to admit an unambiguous reconstruction, which will be provided in section 7. The resulting system, $L2$, differs from an orthodox second-order logic in the following respect. While normally one begins by quantifying over individuals on the first level and introduces quantification over predicates only in a second step, in the Leibnizian system quantification over *concepts* comes first, and quantifying over *individual(-concept)s* is introduced by definition only afterwards. Within calculus $L2$, there exist various ways of formally representing the categorical forms of the theory of the syllogism. They will be examined in some detail in section 8 where we investigate in particular the so-called theory of "quantification of the predicate" developed in the fragment "Mathesis rationis". Furthermore, in the concluding section 9 it will be indicated how a good portion of Leibniz's metaphysics can be reconstructed in terms of his own logic.

The entire system of Leibniz's logic, then, may be characterized as a

---

[6]Cf., e.g., Loemker's introductory remark to his translation of the Plus-Calculus: "This paper is one of several which mark the most advanced stage reached by Leibniz in his efforts to establish the rules for a logical calculus" (**L** 371).

second-order logic of concepts based upon a sentential logic of strict implication. This is somewhat at odds with the standard evaluation, e.g. by Kneale and Kneale [1962, p. 337], according to which Leibniz "never succeeded in producing a calculus which covered even the whole theory of the syllogism". Some of the reasons for this rather notorious underestimation of Leibniz's logic will be discussed in section 3 below.

2   MANUSCRIPTS AND EDITIONS

Gottfried Wilhelm Leibniz was born in 1646. When he died at the age of 70, he left behind an extraordinarily extensive and widespread collection of papers, only a smallpart of which had been published during his lifetime. The bibliography of Leibniz's printed works [Ravier, 1937] contains 882 items, but only 325 papers had been published by Leibniz himself, and amongst these one finds many brief notes and discussions of contemporary works.

Much more impressive than this group of printed works is Leibniz's correspondence. The Bodemann catalogue (**LH**) contains more than 15,000 letters which Leibniz exchanged with more than 1,000 correspondents all over Europe, and the whole correspondence can be estimated to comprise some 50,000 pages. Furthermore, there is the collection of Leibniz's scientific, historical, and political manuscripts in the Leibniz-Archive in Hannover which was described in another catalogue (**LH**). The manuscripts are classified into fourty-one different groups ranging from Theology, Jurisprudence, Medicine, Philosophy, Philology, Geography and all kinds of historical investigations to Mathematics, the Natural Sciences and some less scientific matters such as the Military or the Foundation of Societies and Libraries. The whole manuscripts have been microfilmed on about 120 reels each of which contains approximately 400-500 pages. This makes all together about 50- to 60,000 pages which are scheduled to be published (together with the letters) in the so-called Akademie-Ausgabe ('**A**'). This edition was started in 1923, and it will probably not be finished, if ever, until a century afterwards.

Throughout his life, Leibniz published not a single line on *logic*, except perhaps for the mathematical Dissertation "De Arte Combinatoria" or the Juridical Disputation "De Conditionibus". The former incidentally deals with some issues in the traditional theory of the syllogism, while the latter contains some interesting observations about the validity of certain principles of what is nowadays called deontic logic. Leibniz's main aim in logic, however, was to extend Aristotelian syllogistics to a "Universal Calculus". And although we know of several drafts for such a logic which had been elaborated with some care and which seem to have been composed for publication, Leibniz appears to have remained unsatisfied with these attempts.

Anyway he refrained from sending them to press. Thus one of his fragments bears the characteristic title "Post tot logicas nondum Logica qualem desidero scripta est"[7] which means: After so many logics the logic that I dream of has not yet been written.

So Leibniz's genuinely logical essays appeared only posthumously. The early editions of his philosophical works by Raspe (**R**), Erdmann (**OP**), and C. I. Gerhardt (**GP**) contained, however, only a very small selection. It was not until 1903 that the majority of the logical works were published in Couturat's valuable edition of the *Opuscules et fragments inédits de Leibniz* (**C**). Some years ago I borrowed from the Leibniz-Archive a copy of those five or six microfilm reels which contain group IV, i.e. the philosophical manuscripts. It took me quite some time to work through the 2,500 pages in search of hitherto unpublished logical material. Though I happened to find some interesting papers that had been overlooked by Couturat, the search eventually turned out less successful than I had thought. I guess that at least 80 percent of the handwritten material relevant for Leibniz's logic are already contained in **C**.

Although, then, Couturat's edition may be considered as rather complete, there is another reason why any serious student of Leibniz's logic cannot be satisfied with these texts alone. The *Opuscules* simply do not fulfil the criteria of a *text-critical* edition as set up by the Leibniz-Forschungsstelle of the University of Münster, i.e. the editors of series VI of the Akademie-Ausgabe. In particular, Couturat all too often suppressed preliminary versions of axioms, theorems, and proofs that were afterwards crossed out and improved by Leibniz., A full knowledge of the gradual ripening of ideas as revealed in a text-critical presentation of the different stages of the fragments, however, is essential for an adequate understanding both of what Leibniz was looking for and of what he eventually managed to find.

Since the recent publication of the important and impressive volume **A** VI, 4 which contains Leibniz's Philosophical Writings from ca. 1676 to 1690[8], the situation for scholars of Leibniz's logic has drastically improved. The majority of the drafts of a "Universal Calculus" now are available in an almost perfect text-critical edition. Just a few works especially on the theory of the syllogism such as "A Mathematics of Reason" [**P**. 95-104; cf. "Mathesis rationis", **C**., 193–202;] and "A paper on 'some logical difficulties"' [**P**., 115–121; cf. "Difficultates Quaedam Logicae" **GP** 7, 211–217] have not yet been included in **A** VI 4 but will hopefully be published in the next (and final?) volume of that series.

As regards English translations of Leibniz's philosophical writings in general, the basic edition still is Loemker [**?**] (**L**, for short). A much more comprehensive selection of Leibniz's *logical* papers was edited by G. H.

---

[7]Cf. **A** VI 4, # 2 (pp. 8-11).

[8]This volume appeared in 1999 and it contains 522 pieces with almost 3,000 pages distributed over three subvolumes (A, B, and C).

R. Parkinson [1965] (**P**). Another translation of the important *General Inquiries about the Analysis of Concepts and of Truths* was given by W. O'Briant in [1968].

3   THE TRADITIONAL VIEW OF LEIBNIZ'S LOGIC

The rediscovery of Leibniz's logical work would not have been possible without the pioneering work Louis Couturat. On the one hand, **C** still is an important tool for all Leibniz scholars; on the other hand, Couturat is also (at least partially) responsible for the underestimation of the value of traditional logic in general and of Leibniz's logic in particular as it may be observed throughout the 20th century. In the "Résumé et conclusion" of chapter 8, Couturat compares Leibniz's logical achievements with those of modern logicians, especially with the work of George Boole:

> Summing up, Leibniz had the idea [. . .] of all logical operations, not only of multiplication, addition and negation, but even of subtraction and division. He knew the fundamental relations of the two copulas [. . .] He found the correct algebraic translation of the four classical propositions [. . .] He discovered the main laws of the logic calculus, in particular the rules of composition and decomposition [. . .] In one word, he possessed almost all principles of the Boole–Schröder-logic, and in some points he was even more advanced then Boole himself. (Cf. Couturat [Couturat, 1901, pp. 385–6])

Despite this apparently very favourable evaluation, Couturat goes on to maintain that Leibniz's logic was bound to fail for the following reason:

> Finally, and most importantly, he did not have the idea of combining logical addition and multiplication and treating them together. This is due to the fact that he adopted the point of view of the comprehension [of concepts]; accordingly he considered only one way of combing concepts: by adding their comprehensions, and he neglected the other way of adding their extensions. This is what prevented him to discover the symmetry and reciprocity of these two operations as it manifests itself in the De Morgan formulas and to develop the calculus of negation which rests on these formulas. (Cf. Couturat [Couturat, 1901, pp. 385–6])

A similar judgement may be found in C. I. Lewis' *A Survey of Symbolic Logic* of 1918. Lewis starts by appreciating:

> The program both for symbolic logic and for logistic, in anything like a clear form, was first sketched by Leibniz [...]. Leibniz left

> fragmentary developments of symbolic logic, and some attempts
> at logistic which are prophetic. [Lewis, 1918, p. 4]

But in the subsequent passage these attempts are degraded as "otherwise
without value", and as regards the comparison of Leibniz's logic and Boolean
logic, Lewis says:

> Boole seems to have been ignorant of the work of his continental
> predecessors, *which is probably fortunate, since his own begin-*
> *ning has proved so much more fruitful.* Boole is, in fact, the
> second founder of the subject, and all later work goes back to
> his. (ibid., my emphasis)[9].

In the introduction of his 1930 monograph *Neue Beleuchtung einer Theorie*
von Leibniz, K. Dürr describes the historical development of logic from
Leibniz to modern times as follows:

> . . . It is well known that Leibniz was the first who attempted to
> create what might be called a logic calculus or a symbolic logic
> [. . .] In the mid of the 19th century the movement aiming at the
> creation of a logic calculus was reanimated by the work of the
> Englishman Boole, and it is beyond every doubt that Boole was
> entirely independent of Leibniz" (Cf. Dürr [1930, p. 5]).

Dürr wants to clarify the relations between Leibniz's logic and modern
logic by providing a formal reconstruction of the Plus-Minus-Calculus, and
he announces that his comparative studies will provide results quite differ-
ent from those of Couturat. Unfortunately, however, Dürr fails to give a
detailed comparison between Leibniz's logic and Boole's logic. Moreover,
as was already mentioned in the preceeding section, unlike Leibniz's "stan-
dard system", $L1$, developed in the *General Inquiries*, the fragments of the
Plus-Minus-calculus in **GP** 7 remain fundamentally incomplete.

In a 1946 paper, ""Uber die logischen Forschungen von Leibniz", H. Sauer
deals with the issue of whether Leibniz or Boole should be considered as
the founder of modern logic. He mentions two reasons why Leibniz's logical
oeuvre was neglected or underestimated for such a long time. First, the
majority of Leibniz's scattered fragments was published only posthumously
– as a matter of fact almost 200 years after having been written. Second,
even after the appearance of **C** the time was not yet ripe for Leibniz's logical
ideas. When Sauer goes on to remark that Leibniz created a logical calculus
which was a precursor of modern propositional and predicate calculus, one
might expect that he wants to throw Boole from the throne and replace him
by Leibniz. However, the following prejudice[10] changes his opinion:

---

[9]Cf. in the same vein chapter I of Lewis and Langford [1932].

[10]Sauer may have adopted this reproach from Couturat [1901], but a similar critique
was already put forward by Kvet [1857].

> [Leibniz's logic calculus] is, however, imperfect in so far as Leibniz, under the spell of Aristotelian logic, fails to get rid of the old error that all concepts can be build up from simple concepts by mere conjunction and that all propositions can be put into the form 'S is P'. (Cf. Sauer [1946, p. 64]).

Thus in the end also Sauer disqualifies Leibniz's logic as inferior to "the essentially more perfect 19th century algebra of logic".

Even more negative is the verdict of W. & M. Kneale in their otherwise competent book *The Development of Logic* published in 1962. After charging Leibniz with the fault of committing "himself quite explicitly to the assumption of existential import for all universal statements [...] which prevented him from producing a really satisfactory calculus of logic", and after blaming him with the "equally fateful" mistake that he "[...] accepted the assimilation of singular to universal statements because it seemed to him there was no fundamental difference between the two sorts" [Kneale and Kneale, 1962, p. 323], they sum up Leibniz's logical achievements as follows:

> When he began, he intended, no doubt, to produce something wider than traditional logic. [...] But although he worked on the subject in 1679, in 168[6] and in 1690, *he never succeeded in producing a calculus which covered even the whole theory of the syllogism.* ([Kneale and Kneale, 1962, p. 337], my emphasis).

The common judgment behind all these views thus has it that Leibniz *in vain* looked for a general logical calculus like Boolean algebra but never managed to find it.

First revisions of this sceptical view were suggested by N. Rescher in a [1954] paper on "Leibniz's interpretation of his logical calculi" and by R. Kauppi's [1960] dissertation *Über die Leibnizsche Logik*. Both authors tried in particular to rehabilitate Leibniz's "intensional" approach. However, it was not until the mid-1980ies when strict proofs were provided to show that – contrary to Couturat's claim —

- the "intensional" interpretation of concepts is equivalent (or isomorphic) to the modern extensional interpretation;

- Leibniz's "algebra of concepts" is equivalent (or isomorphic) to Boole's algebra of sets;

- Leibniz's theory of "indefinite concepts" constitutes an important anticipation of modern quantifier theory;

- Leibniz's "universal calculus" allows in various ways the derivation of the laws of the theory of the syllogism.[11]

---

[11]Cf. Lenzen [1983; 1984a; 1984b] and [1988].

This radically new evaluation of Leibniz's logic was summed up in Lenzen [1990a] which, like the majority of all books about this topic, was written in German.[12] To be sure, there exist many English works on Leibniz's *philosophy* in general. To mention only some prominent examples: Russell [1900], Parkinson [1965], Rescher [1967; 1979], Broad [1975], Mates [1986], Wilson [1989], Sleigh [1990], Kulstad [1991], Mugnai [1992], Adams [1994], and Rutherford [1995]. But these monographs as well as the important selections of papers in Frankfurt [1972], Woolhouse [1981], and Rescher [1989], only occasionally deal with logical issues. As far as I know, only two English studies are devoted to a more detailed investigation of Leibniz's logic, viz. Parkinson's [1965] introduction to his collection **P** and Ishiguro's [1972] book on *Leibniz's Philosophy of Logic and Language*.

4   THE ALGEBRA OF CONCEPTS (L1) AND ITS EXTENSIONAL INTERPRETATION

The starting point for Leibniz' universal calculus is the traditional "Aristotelian" theory of the syllogism with its categorical forms of universal or particular, affirmative or negative propositions which express the following relations between two concepts A and B:

U.A.	Every A is B	U.N.	No A is B
P.A.	Some A is B	P.N.	Some A is not B

Within the framework of so-called "Scholastic" syllogistics[13] negative concepts Not-A are also taken into account, which shall here be symbolized as $\overline{A}$. According to the principle of so-called obversion, the U.N. 'No A is B' is equivalent to a corresponding U.A. with the negative predicate: Every A is Not-B. Thus in view of the well-known laws of opposition – according to which P.N. is the (propositional) negation of U.A. and P.A. is the negation of U.N. – the categorical forms can uniformly be represented as follows:

U.A.	Every A is B	U.N.	Every A is $\overline{B}$
P.A.	$\neg$(Every A is $\overline{B}$)	P.N.	$\neg$(Every A is B).

The algebra of concepts as developed by Leibniz in some early fragments of around 1679 and above all in the **GI** of 1686 grows out of this syllogistic framework by three achievements. First, Leibniz drops the expression 'every' ['omne'] and formulates the U.A. simply as 'A is B' ['A est B'] or also as 'A contains B' ['A continet B']. This fundamental proposition shall here be symbolized as '$A \in B$', and the negation $\neg(A \in B)$ will be abbreviated

---

[12]Cf. Kvet [1857] (written by a Czech author), Dürr [1930], Kauppi [1960] (written by a Finnish author), Poser [1969] and Burkhardt [1980]; in addition cf. the two monographs in French by Couturat [1901] and by the Swiss author Knecht [1981].

[13]Cf. Thom [1981]

as '$A \notin B$'. Second, Leibniz introduces the new operator of conceptual *conjunction* which combines two concepts A and B by juxtaposition to AB. Third, Leibniz disregards all traditional restrictions concerning the number of premisses and concerning the number of concepts in the premisses of a syllogism. Thus arbitrary inferences between sentences of the form $A \in B$ *bzw.* $A \notin B$ will be taken into account, where the concepts A and B may be arbitrarily complex, i.e. they may contain negations and conjunctions of other concepts. Let the resulting language be referred to as $L1$.

One possible axiomatization of $L1$ would take (besides the tacitly presupposed propositional functions $\neg, \wedge, \vee, \rightarrow$, and $\leftrightarrow$) only negation, conjunction and the $\in$-relation as primitive conceptual operators. As regards the relation of conceptual containment, $A \in B$, it is important to observe that Leibniz's formulation 'A contains B' pertains to the so-called *intensional* interpretation of concepts as *ideas*, while we here want to develop an *extensional* interpretation in terms of *sets of individuals*, *viz.* the sets of all individuals that fall under the concepts A and B, respectively. Leibniz explained the mutual relationship between the "intensional" and the extensional point of view in the following passage of the *New Essays on Human understanding*:

> The common manner of statement concerns individuals, whereas Aristotle's refers rather to ideas or universals. For when I say *Every man is an animal* I mean that all the men are included amongst all the animals; but at the same time I mean that the idea of animal is included in the idea of man. 'Animal' comprises more individuals than 'man' does, but 'man' comprises more ideas or more attributes: one has more instances, the other more degrees of reality; one has the greater extension, the other the greater intension. (cf. **GP** 5: 469; my translation).

If 'Int(A)' and 'Ext(A)' abbreviate the "intension" and the extension of a concept A, respectively, then the so-called *law of reciprocity* can be formalized as follows:

(RECI 1)   Int$(A) \subseteq$ Int $(B) \leftrightarrow$ Ext$(A) \supseteq$ Ext(B).

This principle immediately entails that two concepts have the same "intension" if and only if they also have the same extension:

(RECI 2)   Int$(A) =$ Int $(B) \leftrightarrow$ Ext$(A) =$ Ext(B).

But the latter "law" appears to be patently false! On the basis of our modern understanding of intension and extension, there exist many concepts or predicates A, B which have the same extension but which nevertheless differ in intension. Consider, e.g., the famous example in Quine [1953, p. 21], A = 'creature with a heart', B = 'creature with a kidney', or the more recent observation in Swoyer [1995, p. 103] (inspired by Quine and directed against RECI 1):

> For example, it might just happen that all cyclists are mathe-
> maticians, so that the extension of the concept *being a cyclist* is
> a subset of the extension of the concept *being a mathematician*.
> But few philosophers would conclude that the concept *being a*
> *mathematician* is in any sense included in the concept *being a*
> *cyclist*.

However, these examples cannot really refute the law of reciprocity *as un-*
derstood by Leibniz. For Leibniz, the *extension* of a predicate A is not just
the set of all *existing* individuals that (happen to) fall under concept A, but
rather the set of all *possible* individuals that have that property. Thus Leib-
niz would certainly admit that the intension or "idea" of a mathematician is
not included in the idea of a cyclist. But he would point out that even if in
the *real world* the set of all mathematicians should by chance coincide with
the set of all cyclists, there clearly are other possible individuals *in other*
possible worlds who are mathematicians and not bicyclists (or bicyclists but
not mathematicians). In general, whenever two concepts A and B differ in
intension, then it is possible that there exists an individual which has the
one property but not the other. Therefore, given Leibniz's understanding
of what constitutes the extension of a concept it follows that A and B differ
also in extension.[14]

In Lenzen [1983] precise definitions of the "intension" and the extension
of concepts have been developed which satisfy the above law of reciprocity,
RECI 1. Leibniz's "intensional" point of view thus becomes provably equiva-
lent, i.e. translatable or transformable into the more common set-theoretical
point of view, provided that the extensions of concepts are taken from a uni-
verse of discourse, U, to be thought of as a set of *possible individuals*. In
particular, the "intensional" proposition $A \in B$, according to which concept
A *contains* concept B, has to be interpreted extensionally as saying that
the set of all As *is included* in the set of all Bs. The first condition for the
definition of an extensional interpretation of the algebra of concepts thus
runs as follows:

(DEF 1)   Let U be a non-empty set (of possible individuals), and let ϕ
             be a function such that $\phi(A) \subseteq U$ for each concept-letter A.
             Then ϕ is an extensional interpretation of Leibniz's concept
             logic $L1$ if
             (1) $\phi(A \in B) = $ true iff $\phi(A) \subseteq \phi(B)$.

Next consider the identity or *coincidence* of two concepts which Leibniz
usually symbolizes by the modern sign '=' or by the symbol '∞', but which
he sometimes also refers to only informally by speaking of two concepts being

---

[14] As regards the ontological scruples against the assumption of merely possible indi-
viduals, cf. the famous paper "On What There Is" in Quine [1953, pp. 1–19] and the
critical discussion in Lenzen [1980, p. 285 sq.].

the same [idem, eadem]. As stated, e.g., in §30 **GI**, identity or coincidence can be defined as mutual containment: "That A is B and B is A is the same as that A and B coincide", i.e.:

(DEF 2) $A = B \leftrightarrow_{\mathrm{df}} A \in B \land B \in A.$

This definition immediately yields the following condition for an extensional interpretation ϕ:

(2) $\phi(A = B) =$ true iff $\phi(A) = \phi(B).$

In most drafts of the "universal calculus", Leibniz symbolizes the operator of conceptual *conjunction* by mere juxtaposition in the form AB. Only in the context of the Plus-Minus-Calculus, which will be investigated in more detail in section 5 below, he favoured the mathematical '+'-sign (sometimes also '$\oplus$') to express the conjunction of A and B. The intended interpretation is straightforward. The extension of AB is the set of all (possible) individuals that fall under both concepts, i.e. which belong to the intersection of the extensions of A and of B:

(3) $\phi(AB) = \phi(A) \cap \phi(B).$

Let it be noted in passing that the crucial condition (1) which reflects the reciprocity of extension and "intension" would be derivable from conditions (2) and (3) if the relation $\in$ were defined according to §83 **GI** in terms of conjunction and identity: "Generally, 'A is B' is the same as '$A = AB$'" (**P**, 67), i.e. formally:

(DEF 3) $A \in B \leftrightarrow_{\mathrm{df}} A = AB.$

For, clearly, a set $\phi(A)$ coincides with the intersection $\phi(A) \cap \phi(B)$ if and only if $\phi(A)$ is a subset of $\phi(B)$! Furthermore, the relation "A is in B" [A inest ipsi B] may simply be defined as the converse of $A \in B$ according to Leibniz's remark in §16 **GI**: "[...] 'A contains B' or, as Aristotle says, 'B is in A'"

(DEF 4) $A\iota B \leftrightarrow_{\mathrm{df}} B \in A.$

In view of the law of reciprocity, one thus obtains the following condition:

(4) $\phi(A\iota B) =$ true iff $\phi(A) \supseteq \phi(B).$

The next element of the algebra of concepts — and, by the way, one with which Leibniz had notorious difficulties — is *negation*. Leibniz usually expressed the negation of a concept by means of the same word he also used to express propositional negation, viz. 'not' [non]. Especially throughout the **GI**, the statement that one concept, A, contains the negation of another concept, B, is expressed as 'A is not-B' [A est non B], while the related

phrase 'A isn't B' [A non est B] has to be understood as the mere negation
of 'A contains B'. As was shown in Lenzen [1986], during the whole pe-
riod of the development of the "universal calculus" Leibniz had to struggle
hard to grasp the important difference between 'A is not-B' and 'A isn't B'.
Again and again he mistakenly identified both statements, although he had
noted their non-equivalence repeatedly in other places. Here the negation of
concept A will be expressed as '$\overline{A}$', while propositional negation is symbol-
ized by means of the usual sign '$\neg$'. Thus 'A is not-B' must be formulated
as '$A \in \overline{B}$', while 'A isn't B' has to be rendered as '$\neg A \in B$' or '$A \notin B$'.
The intended extensional interpretation of $\overline{A}$ is just the set-theoretical com-
plement of the extension of A, because each individual which fails to fall
under concept A eo ipso falls under the negative concept $\overline{A}$:

$$(5) \ \phi(\overline{A}) = \overline{\phi(A)}.$$

Closely related tp the negation operator is that of *possibility* or self-
consistency of concepts. Leibniz expresses it in various ways. He often
says 'A is possible' [A est possibile] or 'A is [a] being' [A est Ens] or also
'A is a thing' [A est Res]. Sometimes the self-consistency of A is also
expressed elliptically by 'A est', i.e. 'A is'. Here the capital letter '$\mathbf{P}$' will be
used to abbreviate the possibility of a concept A, while the impossibility or
inconsistency of A shall be symbolized by '$\mathbf{I}(A)$'. According to $\mathbf{GI}$, lines 330–
331, the operator $\mathbf{P}$ can be defined as follows: "A not-A is a contradiction.
Possible is what does not contain a contradiction or A not-A":

(DEF 5) $\mathbf{P}(B) \leftrightarrow_{\mathrm{df}} B \notin A\overline{A}$.[15]

It then follows from our earlier conditions (1), (3), and (4) that $\mathbf{P}(A)$ is true
(under the extensional interpretation ϕ) if and only if $\phi(A)$ is not empty:

$$(6) \ \phi(\mathbf{P}(A)) = \text{true iff } \phi(A) \neq \varnothing.$$

At first sight, this condition might appear inadequate, since there are
certain concepts – such as that of a unicorn – which happen to be empty
but which may nevertheless be regarded as possible, i. e. not involving a
contradiction. Remember, however, that the universe of discourse underly-
ing the extensional interpretation of $L1$ does not consist of *actually existing*
objects only, but instead comprises all *possible* individuals. Therefore the
non-emptiness of the extension of A is both *necessary and sufficient* for
guaranteeing the self-consistency of A. Clearly, if A is possible then there
must exist at least one *possible* individual that falls under concept A.

The main elements of Leibniz's algebra of concepts may thus be summa-
rized in the following diagram.

Some further elements will be discussed in the subsequent section 5 when
we investigate the operators and laws of the Plus-Minus-Calculus. Before we

---

[15]This definition might be simplified as follows: $\mathbf{P}(B) \leftrightarrow_{\mathrm{df}} B \notin \overline{B}$.

Table 1. "Existential" propositions

Element of Algebra of Concepts L1	Symbolization	Leibniz's Notation	Set-theoretical Interpretation
Identity	$A = B$	$A \infty B$; $A = B$; coincidunt A et B; ...	$\phi(A) = \phi(B)$
Containment	$A \in B$	A est B; A continet B	$\phi(A) \subseteq \phi(B)$
Converse Containment	$A \iota B$	A inest ipsi B	$\phi(A) \supseteq \phi(B)$
Conjunction	AB	AB; $A + B$	$\phi(A) \cap \phi(B)$
Negation	$\overline{A}$	Non-A	$\overline{\phi(A)}$
Possibility	$\mathbf{P}(A)$	A est Ens; A est res; A est possibile	$\phi(A) \neq \emptyset$

do this, however, let us have a look at some *axioms* and *theorems* of L1! The subsequent selection of principles, all of which (with the possible exception of the last one) were stated by Leibniz himself, is more than sufficient to derive the laws of the Boolean algebra of sets:

Axioms and Theorems of L1	Formal version	Leibniz's version
Containment 1	$A \in RA$	"B is B" (bf GI, §37)
Containment 2	$A \in B \land B \in C \to A \in C$	"[...] if A is B and B is C, A will be C" (**GI**, §19)
Containment 3	$A \in B \leftrightarrow 3A = AB$	"Generally 'A is B' is the same as '$A = AB$'" (**GI**, §83)
Conjunction 1	$A \in BC \leftrightarrow A \in B \land A \in C$	"That A contains B and A contains C is the same as that A contains BC" (**GI**, §35; cf. **P** 58, note 4)
Conjunction 2	$AB \in A$	"AB is A" (**C**, 263)
Conjunction 3	$AB \in B$	"AB is B" (**GI**, §38)
Conjunction 4	$AA = A$	"$AA = A$" (**GI**, §171, Third)
Conjunction 5	$AB = BA$	"$AB \infty BA$" (**C**. 235, # (7))
Negation 1	$\overline{\overline{A}} = A$	"Not-not-$A = A$" (**GI**, §96)
Negation 2	$A \neq \overline{A}$	"A proposition false in itself is 'A coincides with not-a'" (**GI**, §11)
Negation 3	$A \in B \leftrightarrow \overline{B} \in \overline{A}$	"In general, 'A is b' is the same as 'Not-B is not-A'" (**GI**, §77)
Negation 4	$\overline{A} \in \overline{AB}$	"Not-A is not-AB" (**GI**, §76a)
Negation 5	$[\mathbf{P}(A)\land]A \in B \to A \notin \overline{B}$	"If A is B, therefore A is not not-B" (**GI**, §91)
Possibility 1	$\mathbf{I}(A\overline{B}) \leftrightarrow A \in B$	"if I say 'A not-b is not', this is the same as if I were to say [...] 'A contains B'" (**GI**, §200).[16]
Possibility 2	$A \in B \land \mathbf{P}(A) \to \mathbf{P}(B)$	"If A contains B and A is true, B is also true" (**GI**, §55)[17]
Possibility 3	$\mathbf{I}(A\overline{A})$	"A not-A is not a thing" (**GI**, §171, Eighth)
Possibility 4	$AA \in B$	"[...] the round square is a quadrangle with null-angles. For this proposition is ture in virtue of an impossible hypothesis" (**GP** 7, 224/5)[18]

CONT 1 and CONT 2 show that the relation of *containment* is reflexive and transitive: Every concept contains itself; and if A contains B which in turn contains C, then A also contains C. CONT 3 shows that the funda-

mental relation $A \in B$ might be defined in terms of conceptual conjunction (plus identity).

CONJ 1 is the decisive characteristic axiom for *conjunction*, and it establishes a connection between *conceptual* conjunction on the one hand and *propositional* conjunction on the other: concept A contains 'B and C' iff A contains B and A also contains C. The remaining theorems CONJ 2–CONJ 5 may be derived from CONJ 1 with the help of corresponding truth-functional tautologies.

Negation is axiomatized by means of three principles: the law of double negation NEG 1, the law of consistency NEG 2, which says that every concepts differs from its own negation, and the well known principle of contraposition, NEG 3, according to which concept A contains concept B iff $\overline{B}$ contains $\overline{A}$. The further theorem NEG 4 may be obtained from NEG 3 in virtue of CONJ 2.

The important principle POSS 1 says that concept A contains concept B iff the conjunctive concept A Not-B is impossible. This principle also characterizes negation, though only indirectly, since according to DEF 4 the operator of *self-consistency* of concepts is definable in terms of negation and conjunction. POSS 2 says that a term B which is contained in a self-consistent term A will itself be self-consistent. POSS 3 easily follows from POSS 1 in virtue of CONT 1. POSS 4 is the counterpart of what one calls "ex contradictorio quodlibet" in propositional logic: an inconsistent concept contains every other concept! This law was not explicitly stated by Leibniz but it may yet regarded as a genuinely Leibnitian theorem because it follows from POSS 1 and POSS 3 in conjunction with the observation that, since $A\overline{A}$ is inconsistent, so is, according to POSS 2, also $A\overline{AB}$.

As was shown in Lenzen [1984b, p. 200], the set of principles {CONT 1, CONT 2, CONJ 1, NEG 1, POSS 1, POSS 2} provides a complete axiomatization of the algebra of concepts which is isomorphic to the Boolean algebra of sets.

5   THE PLUS-MINUS-CALCULUS

The so-called Plus-Minus-Calculus (together with its subsystem of the mere Plus-Calculus) was developed mainly in two essays of around 1686/7[19] which have been published in various editions and translations of widely varying quality. The first and least satisfactory edition is Erdmann's **OP** (# XIX), the last and best, indeed almost perfect one may be found in vol. VI, 4

---

[19]This dating by the editors of **A** VI, 4 rests basically on extrinsic factors such as the type of paper and watermarks. Other authors suspect these fragments to have been composed during a much later period. Cf., e.g., Parkinson's classification "after 1690" in the introduction to **P** (p. lv) and the references to similar datings in Couturat [1901, p. 364] and Kauppi [1960, p. 223].

of **A** (## 177, 178). The most popular and most easily accessible edition, however, still is Gerhardt's **GP** 7 (## XIX, XX). English translations have been provided in an appendix to Lewis [1918], in Loemker's **L** (# 41), and in Parkinson's **P** (## 15, 16).

The Plus-Minus-Calculus offers a lot of problems not only concerning interpretation, meaning and consistency of these texts, but also connected with editorial and translational issues. Since the latter have been discussed in sections 2 and 3 of Lenzen [2000], it should suffice here to point out that an adequate understanding of the Plus-Minus-Calculus can hardly be gained by the study of the two above-mentioned fragments alone. On the one hand, some additional short but very important fragments such as **C**. 250–251, **C**. 251, **C**. 251–252 and **C**. 256 (i.e., ## 173, 174, 175, 180, 181 of **A** VI, 4) have to be taken into account. Second, both the genesis and the meaning of the Plus-Minus-Calculus will become clear only if one also considers some of Leibniz's mathematical works, in particular his studies on the foundations of arithmetic.

After sketching the necessary arithmetical background in section 5.1, I will examine in 5.2 how Leibniz gradually develops his ideas of "real addition" and "real subtraction" from the ordinary theory of mathematical addition and subtraction. Strictly speaking, the resulting Plus-Minus-Calculus is not a *logical* calculus but a much more general calculus which allows of quite different applications and interpretations. In its abstract form, it is best viewed as a theory of set-theoretical containment, $\subseteq$, set-theoretical "addition", $A \cup B$, and set-theoretical subtraction, $A - B$, while it comprises neither set-theoretical "negation", $\overline{A}$, nor the elementship-relation, $A \varepsilon B$! Furthermore, Leibniz's drafts exhibit certain inconsistencies which result from his vacillating views concerning the laws of "real" subtraction. These inconsistencies can be removed basically in three ways. The first possibility would consist in dropping the entire theory of "real subtraction", $A - B$, thus confining oneself to the mere Plus-Calculus. Second, one might restrict $A - B$ to the case where B is contained in A — a reconstruction of this conservative version of the Plus-Minus-Calculus was given by Dürr [1930]. The third and logically most rewarding alternative consists in admitting "real subtractions" $A - B$ also if $B \not\subseteq A$; in this case, however, one has to dispense with Leibniz's idea that there might exist "privative" entities which are "less than nothing" in the sense that, when $-A$ is added to A, the result will be 0.

In section 5.3 the application of the Plus-Minus-Calculus to the "intensions" of concepts is considered. One thus obtains two *logical calculi*, L0.4 and L0.8, which are subsystems of the full algebra of concepts, $L1$, and which can accordingly be given an extensional interpretation as developed in section 4 above.

5.1 Arithmetical Addition and Subtraction

From a modern point of view, the operators of elementary arithmetic should
be characterized axiomatically by a set of general principles such as:

(ARITH 1) $a = b \vdash \tau(a) = \tau(b)$

(ARITH 2) $a = a$

(ARITH 3) $a + b = b + a$

(ARITH 4) $a + (b + c) = (a + b) + c$

(ARITH 5) $a + 0 = a$

(ARITH 6) $aa = 0$

(ARITH 7) $a + (bc) = (a + b)c.$

Guided by the idea that only identical propositions are genuinely axiomatic
while all other basic principles in mathematics (as well as in logic) should
be derivable from the definitions of the operators involved, Leibniz tried to
reduce the number of axioms to an absolute minimum. Thus in a fragment
on "The First Elements of a Calculus of Magnitudes" ["Prima Calculi Mag-
nitudinum Elementa", PCME, for short] only ARITH 2 receives the status
of an "*Axiom $a = a$*" (**GM** 7, 77). The rule of substitutivity, ARITH 1, is
presented as a definition: "Those are *equal* which can be substituted for one
another salva magnitudine" (ibid.). The axiom of commutativity, ARITH 3,
appears as a "*Theorem $+a + b = +b + a$*" (**GM** 7, 78).[20] The characteristic
axiom of the neutral element 0, ARITH 5, is conceived as an "*Explication
$+0 + a = a$, i.e. 0 is the sign for nothing,* which adds nothing" (ibid.). The
subtraction axiom ARITH 6 is introduced as a logical consequence of the
definition of the '-'operation: "Hence [...] $+b - b = 0$" (ibid.). And the
structural axiom ARITH 7 is put forward as a "*Theorem* Those to be added
are written down with their original signs, i.e. $f + (a - b) = [\ldots]f + a - b$."
(**GM** 7, 80).

The latter, unbracketed formulation of the term '$(f + a) - b$', already
indicates that Leibniz never took very much care about bracketing. This is
not only confirmed by the fact that he habitually "forgot" to state the law

---

[20]Leibniz sometimes conceives arithmetic as a theory of positive $(+a)$ and negative
$(-b)$ magnitudes which can be conjoined by the operation of "positing" (denoted by
juxtaposition) so as to yield the sum $+a + b$ or the difference $+a - b$: cf. **GM** 7, 78. If
the operation of positing itself is assumed to be commutative ("... nihil refert, quo ordine
collocentur"), then not only '+' is provably commutative, but so is also '−' in the sense
of: "$a - b = -b + a$" (AEAS, 19 v.); or "$-a - b = -b - a$ seu transpositio" (AEAS, 20
v.). In "Conceptus Calculi" Leibniz mistakenly claimed subtraction to be symmetric in
the stronger sense: "In additione et subtractione [...] ordo nihil facit, ut $+b + a$ aequ.
$+a + b, b - a$ aequ. $a - b$" (**GM** 7, 84).

of associativity, ARITH 4, but also by various other examples. For example, the theorems:

(ARITH 8) $(a + b) - b = a$

(ARITH 9) $(a - b) + b = a$

were stated by Leibniz in an hitherto unpublished manuscript "De Aequalitate; Additione; Subtractione" (**LH** XXXV, 1, 9, 18–21 — AEAS, for short) quite ambiguously as "$a + b - b = a$" (AEAS, 21 r.) and "$+a - b + b$ will be equiv. to a".[21] This unbracketed formulation seduced him to think that ARITH 8 might be proved as follows: "for $b - b$ putting 0 gives $a + 0 = a$" (AEAS, 21 r.). Actually, however, ARITH 7 has to be presupposed to guarantee that $(a + b) - b$ equals $a + (b - b)$. That Leibniz really had ARITH 8 and 9 in mind is evidenced by the fact that he considered

(ARITH 10)   "If $a + b = c$ then $c - b = a$" (AEAS, 21 r.)

(ARITH 11)   "If $a - b = c$ then $a = c + b$" (AEAS, 20r)

as immediate corollaries of the former theorems. The subsequent two principles are special instances of the rule ARITH 1:

(ARITH 12)   "If you add equals to equals, the results will be equal, i.e. if $a = l$ and $b = m$, then $a + b = l + m$" (**GM** 7, 78)

(ARITH 13)   "If you subtract equals from set-theoretical equals, the rest will be equal, i.e. if $a = l$ and $b = m$, then $a - b = l - m$" (**GM** 7, 79)

By contrast, the converse inference

(ARITH 14)   "Si $a = l$ et $a + b = l + m$ erit $b = m$" (AEAS, 19 v.)

(ARITH 15)   "Si $a - b = lm$ et sit $b = m$ erit $a = l$" (*ibid.*)

cannot be derived from the axioms of equality, ARITH l and 2, alone. Leibniz's negligent attitude towards bracketing veils that the "proof" of, e.g., ARITH 14: "For $b + a = m + l$ (by transpos. of add.) therefore (by the preced.) $b + a - a = m + l - l$. Hence $b = m$" (AEAS, 20 v.) makes use not only of ARITH 3 ("transpos. of add.") and ARITH 13 ("preced."), but also presupposes either ARITH 8 or ARITH 7 when $(b + a) - a$ is tacitly equated with $b + (a - a)$.

It may be interesting to note that in the unpublished fragment, "Fundamenta Calculi Literalis", Leibniz came to recognize the axiomatic status of ARITH 1, 2, 3, 5, and 6. After stating the usual principles of the equality relation, he listed the relevant

---

[21] The latter quotation is not from AEAS but from Knobloch [1976, p. 117].

Axioms in which the meaning of the characters is contained [...]

(4) $+a + b = +b + a$[...]

(5) $a + 0 = a$[...]

(9) $a - a = 0$[...] (**LH** XXXV, XII, 2, 72 r.)

Originally he had also included "(2) $a = c$ is equivalent to $a + b = c + b$" (*ibid.*); but later on he thought that this equivalence "can be proved [...] by the Def. of equals" (*ibid.*). Once again his negligence concerning brackets may have been due to his recognizing that only one half of the equivalence, viz. ARITH 12, follows from the above axioms while the other implication, ARITH 14, additionally presupposes the crucial axiom ARITH 7. Anyway, it is quite typical of Leibniz that he "forgot" to state just those two basic principles, ARITH 4 and 7, which involve brackets.

For the sake of the subsequent discussion it should be pointed out that (on the basis of the remaining axioms ARITH 16) ARITH 7 can be replaced equivalently by the conjunction of ARITH 8 and 9.[22] Furthermore the related structural laws

(ARITH 16) $a - (b + c) = (a - b) - c$

(ARITH 17) $a - (b - c) = (a - b) + c$

can be derived either from ARITH 7 or from ARITH 8 + 9.[23] ARITH 17 was formulated by Leibniz as the rule: "Those to be subtracted will be written down with signs changed, + in −, and − in +, i.e. $f - (a - b) = f - a + b$" (**GM** 7, 80). And in AEAS he presented an elliptic version of ARITH 16 in a way that indicates that here at least he became aware of the logical function of brackets: "$-(a + b) = -a - b$. This is the meaning of brackets" (o.c., 19 r.) It will turn out in the next section that it is just axiom ARITH 7 (and the theorems that depend on it) which lead into difficulties when one tries to transfer the mathematical theory of '+' and '−' to the field of "real entities".

5.2   "Real" Addition and Subtraction

Already in PCME Leibniz envisaged to apply the arithmetical calculus to "things", e.g. to "straight lines to be added or subtracted" (o.c., # (25)). In the fragments # XIX and XX of **GP** 7, he mentions two further applications: the addition or composition, i.e. conjunction, of concepts, or the

---

[22] According to ARITH 4 and 9 (a+(b-c))+c=a+((b-c)+c)=a+b; from this it follows by ARITH 10 which is an immediate corollary of ARITH 8 that (a+b)-c=a+(b-c).

[23] According to ARITH 3, 4, 9: $((a - b) - c) + (b + c) = ((a - b) - c) + (c + b) = (((a - b) - c) + c) + b = (ab) + b = a$; hence it follows by ARITH 10: $a - (b+c) = (a-b) - c$. Similarly, according to ARITH 16 and 9: $(a - (b-c)) - c = a - ((b-c) + c) = a - b$, from which it follows by ARITH 11 that $(a - b) + c = a - (b - c)$.

addition, i.e. union, of sets. In what follows we will concentrate upon the latter interpretation where accordingly '$-$' represents set-theoretical sub-traction and '0' stands for the empty set which shall therefore be symbolized as '$\emptyset$'. The underlying theory of '$=$' now, of course, no longer refers to the relation of numerical *equality* but to the stricter relation of *identity* or coincidence. Thus, e.g., the basic rule of substitutivity, $A = B \vdash \tau(A) = \tau(B)$, has to be reformulated with 'salva veritate' replacing 'salva magnitudine' (cf. **GP** 7, 236, Def. 1). Accordingly ARITH 12 and 13 now reappear as "If coinciding [terms] are added to coinciding ones, the results coincide" (**GP** 7, 238) and "If from coinciding [terms] coinciding ones are subtracted, the rests coincide" (**GP** 7, 232). The law of reflexivity, $A = A$, can be adopted without change. The law of symmetry of set-theoretical addition now is presented as "*Axiom. 1* $B + N = N + B$, i.e. transposition here makes no difference" (**GP** 7, 237). The "real nothing", i.e. the empty set $\emptyset$, is characterized as follows "It does not matter whether Nothing [nihil] is put or not, i.e. $A+$Nih.$= A$" (**C**. 267),

(NIHIL 1) $A + \emptyset = A$.

The subtraction of sets is again conceived in analogy to the arithmetical case as the converse operation of addition: "If the same is put and taken away [...] it coincides with Nothing. I.e. $A[\ldots] - A[\ldots] = N$" (**GP** 7, 230), formally:

(MINUS 1) $A - A = \emptyset$.

The main difference between arithmetical addition on the one hand and "real addition" on the other is that, whereas for any number $a \neq 0, a + a$ is unequal to a, the addition of one and the same set A does not yield anything new:

(PLUS 1)   "$A + A = A[\ldots]$ or the repetition here makes no difference" (**GP** 7, 237).

However, this new axiom cannot simply be added to the former collection without creating inconsistencies. As Leibniz himself noticed, it would otherwise follow that there is no real entity besides $\emptyset$: "For e.g. [by PLUS 1] $A + A = A$, therefore one would obtain [by the analogue of ARITH 10] $A - A = A$. However (by [MINUS 1]) $A - A =$ Nothing, hence A would be $=$ Nothing" (**C**. 267, # 29). Thus any non-trivial theory of real addition satisfying PLUS 1 has to reject as least the counterparts of the laws ARITH 10 (or ARITH 8) and ARITH 7.

As was suggested by Leibniz, ARITH 10 should be restricted to the special case where A and B are uncommunicating or have nothing in common: "Therefore if $A + B = C$, then $A = CB[\ldots]$ But it is necessary that A et B

have nothing in common" (**C**. 267, # 29).[24] A precise definition of this new relation presupposes that one first introduces the more familiar relation 'A contains B' or its converse 'A is contained in B', formally $A \subseteq B$, as follows:

> $A + Y = C$ means 'A is in C', or 'C contains A'. (cf. **C**. 265, ## 9, 10).

That is, C contains A iff there is some set Y such that the union of A and Y equals C. As Leibniz noted in Prop. 13 and Prop. 14 of fragment XX, this definition may be simplified by replacing the variable 'Y' by 'C':

(DEF 6) $\qquad A \subseteq B \leftrightarrow_{\mathrm{df}} A + B = B$.

It is now possible to define:

> If some term, M, is in A, and the same term is in B, this term is said to be ,common' to them, and they will be said to be 'communicating'.[25]

I.e., two sets A and B have something in common iff there exists some Y such that $Y \subseteq A$ and $Y \subseteq B$. Since, trivially, the empty set is included in any set A (cf. NIHIL 1)

(NIHIL 2) $\qquad \emptyset \subseteq A$,

one has to add the qualification that Y is not empty:

(DEF 7) $\qquad \mathrm{Com}(A, B) \leftrightarrow_{\mathrm{df}} \quad \exists Y (Y \neq \emptyset \quad \wedge Y \subseteq A \wedge Y \subseteq B)$.

The necessary restriction of ARITH 8 can then be formalized as

(COM 1) $\qquad \neg \mathrm{Com}(A, B) \to (A + B) - B = A$.

According to Leibniz this implication may be strengthened into a biconditional:

> Suppose you have A and B one you want to know if there exists some M which is in both of them. Solution: combine those two into one, $A + B$, which shall be called L [...] and from L one of the constituents, A, shall be subtracted [...] let the rest be N; then, if N coincides with the other constituent, B, they have nothing in common. But if they do not coincide, they have something in common which can be found by subtracting the

---

[24]Leibniz also recognized that the same restriction was necessary in the case of ARITH 14: "Si $A + B = D + C$ et $A = D$, erit $B = C$.[...] Imo non sequitur nisi in incommunicantibus" (**C**., 268).

[25]**P**., 123; cf. **GP** 7, 229: "Si aliquid M insit ipsi A, itemque insit ipsi B, id dicetur ipsis commune, ipsa autem dicentur communicantia".

rest N, which necessarily is in B, from $B[\ldots]$ and there remains M, the commune of A and B, which was looked for.[26]

What is particularly interesting here is that Leibniz not only develops a criterion for the relation $\mathrm{Com}(A, B)$ in terms of whether $(A+B)-B$ coincides with A or not, but that he also gives a formula for "the commune" of A and B in terms of addition and subtraction. If '$A \cap B$' denotes the commune, i.e. the intersection of A and B, Leibniz's formula takes the form:

(Com 2) $A \cap B = B - ((A + B) - A)$.

Closely related with Com 2 is the following theorem: "If, however, two terms, say A and B, are communicating, and A shall be constituted by B, let again be $A + B = L$ and suppose that what is common to A and B is N, one obtains $A = L - B + N$";[27] formally:

(Com 3) $A = ((A + B) - B) + (A \cap B)$.

The subsequent theorems also may be of interest: "What has been subtracted and the remainder are uncommunicating" (**P.**, 128; cf. **GP** 7, 234), formally:

(Com 4) $\neg \mathrm{Com}(A - B, B)$.

"Case 2. If $A + B - B - G = F$, and everything which both A and B and B and G have in common is M, then $F = A - G$"[28], formally:

(Com 5) $A \cap B = A \cap C \rightarrow ((A + B) - B) - C = A - C$.

Furthermore one gets the following necessary restriction of Arith 14: "In symbols: $A + B = A + N$. If A and B are uncommunicating, then $B = N$" (**P.**, 130; cf. **GP** 7, 235), formally:

(Minus 2) $\neg \mathrm{Com}(A, B) \wedge \quad \neg \mathrm{Com}(A, C) \rightarrow (A + B = A + C \rightarrow B = C)$.

Finally, when Leibniz remarks: "Let us assume meanwhile that E is everything which A and G have in common — if they have something in common, so that if they have nothing in common, $A = $ Nothing",[29] he

---

[26]Cf. **C.**, 250: "Sint A et B, quaeritur an sit aliquod M quod insit utrique. Solutio: fiat ex duobus unum $A + B$ quod sit $L[...]$ et ab L auferatur unum constituentium $A[...]$ residuum sit N, tunc si N coincidit alteri constituentium B, nihil habebunt commune. Si non coincidant, habebunt aliquid commune, quod invenitur, si residuum N quod necessario inest ipsi B detrahatur a $B[...]$ et restabit M quaesitum commune ipsis A et B."

[27]Cf. **C.**, 251: "Sin communicantia sint duo, ut A et B, et A constitui debeat per B, fiat rursus $A + B = L$ et posito ipsis A et B commune esse N, fiet $A = L - B + N$".

[28]**P.**, 127; cf. **GP** 7, 233: "Si $A + B - B - G = F$, et omne quod tam A et B, quam G et B commune habent, sit M, erit $F = A - G$."

[29]**P.**, 127; cf. **GP** 7, 233: "Ponamus praeterea omne quod A et G commune habent esse E [...] ita ut si nihil commune habent, E sit $=$ Nih.".

thereby incidentally formulates the following law which expresses the obvious connection between the relation of communication and the operator of the commune:

(Com 6) $(A \cap B) = \emptyset$ $\leftrightarrow$ $\neg \text{Com}(A, B)$.

In this way Leibniz gradually transforms the theory of mathematical addition and subtraction into (a fragment of) the theory of sets. It is interesting to see how the problem of incompatibility between the arithmetical axiom ARITH 7 and the new characteristic axiom of set-theoretical union, PLUS 1, leads him to the discovery of the new operators '$\subseteq$', 'Com', and '$\cap$' which have no counterpart in elementary arithmetic.

It cannot be overlooked, however, that the theory of real addition and subtraction is incomplete in two respects. First, the axioms and theorems actually found by Leibniz are insufficient to provide a complete axiomatization of the set of operators $\{=, +, \emptyset, , \subseteq, \text{Com}, \cap\}$; second, when compared to the full algebra of sets, Leibniz's operators turn out to be conceptually weaker. In particular, it is not possible to define negation or complementation in terms of subtraction (plus the remaining operators listed above). Leibniz only pointed out that there is a *difference* between negation (i.e., set-theoretical complement) and subtraction:

> *Not* or the negation differs from *Minus* or the subtraction in so far as a repeated 'not' destroys itself while a repeated subtraction does not destroy itself.[30]

Furthermore he believed that just as the "negation" of a positive number a is the negative number $(-a)$, i.e. $(0 - a)$, so also in the domain of real things the "negation" of a set A should be conceived of as a "privative" thing $(\emptyset - A)$:

> If from $aBaC$ shall be subtracted which is not in B, the rest A or $B - C$ will be a semi-privative thing, and is a D is added, then $D + A = E$ means that in a way D and B have to be put in E, yet first C has to be removed from D [...] Thus let be [...] $E = L - M$ where L and M have nothing else in common; now if L and M (uncommunicating) are both positive, then E will be a *semi-privative thing*. If $M =$ Nothing, then $E = L$ and E will be a *positive thing* [...]; finally, if L is $=$ Nothing, then

---

[30]Cf. **C.**, 275: "Differunt *Non* seu negatio a [...] *Minus* seu detractione, quod 'non' repetitum tollit se ipsum, at vero detractio repetita non seipsam tollit." Leibniz goes on to explain that "non-non B est B, sed $-B$ idem est quod Nihilum. Verbi gratia [...] $A-B$ est A." This happens to be true, though, in the sense that $A - (-B) = A - (0 - B) = A - 0 = A$; but this equation is based upon the non-existence of "privative sets" which contradicts Leibniz's explicit statements some lines earlier.

$E = M$ and E will be a *privative thing*.[31]

To be sure, if ARITH 7, 9, or 11 would also hold in the case of real addition and subtraction, then it might be shown that there exist privative sets which are "less than nothing" in the sense that when $(-M)$ is added to M, the result equals the empty set $\emptyset$. E.g., letting be $A = \emptyset$ in ARITH 9, one immediately obtains $(\emptyset - B) + B = \emptyset$; and ARITH 7 analogously entails that $B + (\emptyset - B) = (B + \emptyset) - B = B - B = \emptyset$. However, the existence of a privative set $-B$ which is "less than nothing" is inconsistent with the rest of Leibniz's theory of sets, in particular with the characteristic axiom PLUS 1. Since $B = B + B$, it follows that $B + (-B) = (B + B) + (-B) = B + (B + (-B))$; hence if $B + (-B)$ were equal to $\emptyset$, one would obtain that $\emptyset = B + \emptyset = B$, i.e. each set B would coincide with $\emptyset$.[32]

It is somewhat surprising to see that, although Leibniz clearly recognized that the first half of ARITH 7, viz. ARITH 8 or 10, is no longer valid in the field of real entities, he failed to recognize that the other half, i.e. ARITH 9 or 11, which involves the existence of "privative sets", also has to be abandoned. In fragment XIX of **GP** 7, which may be considered as an attempt to give a final form of the theory of real addition and subtraction, Leibniz "solved" the problems at hand by just restricting subtractions $(A - B)$ to the case where $B \subseteq A$:

> Postulate 2. Some term, e.g. A, can be subtracted from that in which it is – e.g., from $A + B$. (**P**. 124; cf. **GP** 7, 230).

Leibniz still stuck to the idea that otherwise "privative sets" would result[33], and he failed to see that ARITH 16 (which he had tacitly presupposed in several other places[34]) is set-theoretically valid and entails that

(MINUS 3) $\emptyset - B = \emptyset$.[35]

---

[31]Cf. **C**., 267–8: "Si ab aliquo B detrahi jubeatur C quod ipsi non inest, tunc residuum A seu $B - C$ erit res semi-privativa et si apponatur alicui D, tunc $D + A = E$ significat D quidem et B esse ponenda in E, sed tamen a D prius esse removendum C [...] sic ut sit [...] $E = L - M$ et L atque M nihil amplius habebunt commune; quodsi jam L et M (incommunicantia) ambo sint aliquid positivum, erit E *res semiprivativa*. Sin sit $M =$Nih. erit $E = L$, seu E erit *res positiva* [...]; denique sin sit $L =$Nih. erit $E = M$, seu E erit *res privativa*." Cf. also **C**., 275: "Hinc si ponatur $D - B$, et D non contineat B, non ideo putandum est notam omissivam nihil operari. Saltem enim significat provisionaliter, ut ita dicam, et in antecessum, si quando contingat augeri $D - B$ per adjectionem alicujus cui insit B, tunc saltem sublationi illi locum fore. Exempli causa si $A = B + C$ erit $A + D - B = D + C$."

[32]This proof, by the way, presupposes the axiom of associativity, ARITH 4, $A + (B + C) = (A + B) + C$.

[33]**P**. 127, fn. 1; cf. **GP** 7, 233: "[...] hinc detractiones possunt facere nihilum [...] imo minus nihilo".

[34]Cf. his "proof" of "Theor. IX" in **GP** 7, 233.

[35]According to MINUS 1, ARITH 16, and 8: $0 - B = (B - B) - B = B - (B + B) = B - B = 0$!

Hence real subtractions never yield "less than Nothing".

To conclude this section let me point to some modifications of Leibniz's theory of real addition which are (necessary and) sufficient for obtaining a *complete* version of the algebra of sets. First, one has to introduce a new constant, U, denoting the universal set (or the universe of discourse). This set may be characterized axiomatically by the principle that U contains any set A:

(UD 1) $A \subseteq U$.

Second, the commune of A and B will have to be characterized by the axiom

(COM 7) $C \subseteq A \cap B \leftrightarrow C \subseteq A \wedge C \subseteq B$.

Leibniz put forward this defining principle only indirectly when he referred to the commune of two sets as "that in which there is whatever is common to each"[36]. Third, instead of ARITH 7, which becomes invalid in the area of set-theory, one has to adopt former theorem ARITH 16:

(MINUS 4) $A - (B + C) = (A - B) - C$,

plus the following refinement of ARITH 17:

(MINUS 5) $A - (B - C) = (A - B) + (A \cap C)$.

It may then be shown that the resulting collection of principles[37] forms a complete axiomatization of the algebra of sets, where negation is definable by $\overline{A} = \mathrm{df}\, U - A$.

5.3   Application of the Plus-Minus-Calculus to Concepts

The main draft of the Plus-Minus-Calculus was aptly called by Leibniz "A not inelegant specimen of *abstract* proof". This led some commentators to attribute to him the insight:

> [...] that logics can be viewed as abstract formal systems that are amenable to alternative interpretations. [...] In Leibniz's intensional interpretations of his system, $\oplus$ is a *conjunction*-like operator on concepts, but in his extensional interpretations, it becomes a *disjunction*-like operation on extensions (in effect, it becomes set-theoretic union).[38]

---

[36]**P.**, 128; cf. **GP** 7, 234: "id cui inest quicquid utrique commune est".

[37]I.e., the counterparts of ARITH 1–6, and the "new" principles UD 1, COM 7, MINUS 4 and 5. For details cf. Lenzen [1989a].

[38]Swoyer [1995, p. 104]. Cf. also Schupp [2000, LII].

This view of the dual interpretability of '+' as conjunction and as disjunction is, however, misleading. It is true, though, that *if* the Plus-Calculus is considered as an abstract structure whose operators $\langle +, \subseteq \rangle$ are only implicitly defined by the axioms, then there exist different *models* for this system. As was shown, e.g., in Dürr [1930], in a first model '$A + B$' may be interpreted as the conjunction (or intersection) of A and B, while in a second model '$A + B$' is interpreted as the disjunction (or union) of A and B. However, these models will satisfy the axioms of the Plus-Minus-Calculus only if the interpretation of the remaining operators of the abstract structure also are duly adjusted. Thus in view of the equivalence expressed in "Theorem VII" + "Converse of the preceding Theorem":

> [...] if B is in A, then $A + B = A$. [...] If $A + B = A$, then B will be in A. (**P.**, 126/7; cf. **GP** 7, 232)

in the first model (with '+' taken as '∩') the fundamental inesse-relation would have to be interpreted as the *super*set-relation $B \supseteq A$; while only in the second model (with '+' taken as '∩') "B is in A" might be interpreted like in DEF 1 as the *subset*-relation $B \subseteq A$.

Dürr [1930, p. 42] holds that Leibniz himself had envisaged the dual interpretation of the abstract structure either as $\langle \cap, \supseteq \rangle$ or as $\langle \cup, \subseteq \rangle$ because he thought that Leibniz had used the expression "A is in B" *alternatively* in the sense of $A \subseteq B$ or in the sense of $B \subseteq A$. Dürr quotes the remark that "the concept of the genus is in the concept of the species, the individuals of the species in the individuals of the genus" (**P** 141) as evidence for Leibniz's allegedly vacillating interpretation of the phrase "A is in B" [A inest ipsi B]. But this is untenable. For Leibniz, the logical operator "A is in B" *always* means exactly what it literally says, namely that A *is contained in B* The crucial quotation only expresses the law of reciprocity, RECI 1, according to which the intension of the concept of the genus *is contained* in the intension of the concept of the species, while at the same time the extension of the concept of the species *is contained in* the extension of the concept of the genus. In both cases one and the same logical (or set-theoretical) relation of containment, $\subseteq$, is involved.

There is one further, elementary point which proves that Leibniz's addition $A + B$ always has to be interpreted as the *union* of A and B. Within the framework of the Plus-Minus-Calculus, the operators $\langle +, \subseteq \rangle$ are only part of a larger structure which contains in particular also the distinguished element '0' ("Nothing"). Thus, if $\langle \cap, \supseteq \rangle$ would constitute a model of the Plus-Minus-Calculus, then the defining axiom AX 5, $A + 0 = A$, would have to hold. But with '+' interpreted as '∩', this would mean that '0' is not the empty but the *universal* set! Such an interpretation, however, is entirely incompatible with Leibniz's characterization of '0' as "Nihilum"![39]

---

[39]**C.**, 267, # 28: "*Nihilum* sive ponatur sive non, nihil refert. Seu $A + Nih. \infty A$". Dürr [1930: 96] was well aware of this axiom and pointed out that in the second model

What is at issue, then, is not a dual (or multiple) interpretation in the sense of Dürr's different *models*, but rather, as Leibniz himself stressed, different *applications* of the Plus-Minus-Calculus.[40] One particularly important application concerns the realm of:

> [...] absolute concepts, where no account is taken of order or of repetition. Thus it is the same to say 'hot and bright' as to say 'bright and hot', and [...] 'rational man' — i.e. 'rational animal which is rational' — is simply 'rational animal'. (*ibid.*).

Let us now take a closer look at this interpretation of the Plus-Minus-Calculus, where the entitites A, B are viewed as (intensions of) *concepts* and where the sum $A + B$ therefore corresponds to (the intension of) the *conjunction AB* in accordance with Leibniz's remark: "For $A + B$ one might put simply AB".[41] Hence the extensional interpretation of $A + B$ coincides with our earlier requirement:

$$(4) \quad \phi(A \oplus B) = \phi(AB) = \phi(A) \cap \phi(B).$$

Most of the basic theorems for conjunction mentioned in section 4 now reappear in the Plus-Minus-Calculus as theorems of conceptual addition. For example, one half of the equivalence CONJ 1 is put forward as "*Theorem V* [...] If A is in C and B is in C, then $A + B$ [...] is in C" (**P**, 126). CONJ 2 is formulated in passing when Leibniz notes that "N is in $A \oplus N$ (by the definition of 'inexistent')" (**P**, 136). CONJ 4 simply takes the shape of "*Axiom 2* [...] $A + A = A$" (**P**, 132); and CONJ 5 is similarly formulated as "*Axiom 1 $B \oplus N = N \oplus B$*".

The axiom of the reflexivity of the $\in$-relation, CONT 1, reappears as "Proposition 7. A is in A" which, interestingly, is proven by Leibniz as follows: "For A is in $A \oplus A$ (by the definition of 'inexistent' [...]), and $A \oplus A = A$ (by axiom 2). Therefore [...] A is in A" (**P**, 133). The counterpart of the law of transitivity of the $\in$-relation, CONT 2, is formulated straightforwardly as "THEOREM IV [...] if A is in B and B is in C, A will also be in C" (**P**, 126). And the analogue of CONT 3, $A \in B \leftrightarrow A = AB$, is formulated in two parts as "THEOREM VII [...] if B is in A, then $A + B = A$" and as "*Converse of the preceding theorem* [...] If $A + B = A$, then B will be in A" (**P**, 126-7). Here, of course, 'A is in B' is taken to hold if and only if, in the terminology of $L1$, "B contains A".

The mere Plus-Calculus, $L0.4$, as developed in the "Study in the Calculus of Real Addition" is the logical theory of the operators 'ι' (or '$\in$'), '$\oplus$', and '$=$'. Although the theorems for identity (coincidence) are developed there

---

"Nihil" corresponds to the "allumfassende Klasse".

[40]Cf. **P**., 142: "[...] whenever these laws [$A + B = B + A$ and $A + A = A$] are observed, the present calculus can be appplied".

[41]Cf. **C**., 256: "Pro $A + B$ posset simpliciter poni AB".

in rather great detail, it remains a very weak and uninteresting system (at least in comparison with the full algebra of concepts, $L1$); thus it shall no longer be considered here. Much more interesting, however, is the Plus-*Minus*-Calculus, $L0.8$, which contains many challenging laws for conceptual subtraction and for the auxiliary notions of the empty concept 0, the relation of communication among concepts, $\text{Com}(A, B)$, and for the commune of A and B, $A \otimes B$, which comprises all what two concepts A and B have in common.

The "empty concept"

When the Plus-Minus-Calculus is applied to (intensionally conceived) concepts, the empty *set* "Nihil" corresponds to the empty *concept*, i.e. the concept which has an (almost) empty *intension*. Leibniz tried to define or to characterize this concept as follows:

> *Nothing* is that which is capable only of purely negative determination, namely if N is not A, neither B, nor C, nor D, and so forth, then N can be called *Nothing*.[42]

The 'and so forth'-clause should be made more precise by postulating that *for no concept Y, N contains Y*. Within the framework of Leibniz's quantifier logic (to be developed systematically in section 6 below), this definition would take the form $N = 0 \leftrightarrow \neg \exists Y (N \in Y)$. However, according to CONT 1, each concept contains itself; hence the empty concept always contains at least one concept, namely 0. Therefore one has to amend Leibniz's definition by adding the restriction that 0 contains no *other* concept Y (different from 0):

(DEF 8) $A = 0 \leftrightarrow_{\text{df}} \neg \exists Y (A \in Y \land Y \neq A)$.

As we saw earlier, the "addition" of 0 to any concept A leaves A unchanged, i.e. $A + 0 = A$ or, equivalently, $A0 = A$. According to CONT 3 this means that 0 is contained in each concept A:

(NIHIL 1) $A \in 0$.

Furthermore it is easy to prove that the empty concept 0 coincides with the *tautological* concept:

(NIHIL 2) $0 = \overline{A\overline{A}}$

---

[42]Cf. **A** VI, 4, 625: "*Nihil* est cui non competit nisi terminus mere negativus, nempe si N non est A, nec est B, nec C, nec D, et ita porro, tunc N dicitur esse *Nihil*". Cf. also **A** VI, 4, 551: "Si N non est A, et N non est B, et N non est C, et ita porro; N dicetur esse Nihil" or **C**., 252: "Esto N non est A, item N *non est B*, item N *non est C*, et ita porro, tunc dici poterit N *est Nihil*".

For according to the Poss 4, $A\overline{A} \in Y$ for every Y. Hence by the law of contraposition, the negation of $A\overline{A}$, i.e. the tautological concept, *is contained* in every Y. Thus if there exists some Y such that $\overline{A\overline{A}}$ *contains* Y, it follows by Def 2 that $Y = \overline{A\overline{A}}$.

If it is further observed that, according to Reci 1, a concept with minimal intension must have maximal extension, we obtain the following requirement for the extensional interpretation of the empty (or tautological) concept 0:

$$(7)\ \phi(0) = U \text{ (universe of discourse).}$$

(Un)communicating concepts and their commune

Under the present application of the Plus-Minus-Calculus, the relation of communication no longer expresses the fact that two *sets* A and B are overlapping, but $\mathrm{Com}(A, B)$ means that the *concepts* A and B "have something in common" [A et B habent aliquid commune; A et B sunt communicantia]. This relation can be defined as follows:

> If some term, M, is in A, and the same term is in B, this term will be said to be 'common' to them, and they will be said to be 'communicating'. If, however, they have nothing in common [...], they will be called 'uncommunicating'. (**P**, 123)

This explanation might be formalized straightforwardly as $\mathrm{Com}(A, B) \leftrightarrow \exists X (A \in X \wedge B \in X)$. But since the empty, tautological concept 0 is contained in each A, it has to be modified as follows:

(Def 9) $\mathrm{Com}(A, B) \leftrightarrow_{\mathrm{df}}$ $\exists X (X \neq 0 \wedge A \in X \wedge B \in X)$.

Now, whenever A and B are communicating, Leibniz refers to what they have in common as "quod est ipsis A et B commune", and he explained the meaning of this operator quite incidentally as follows:

> In two communicating terms [A and B, M is] that in which there is whatever is common to each [iff ...] $A = P + M$ and $B = N + M$, in such a way that whatever is in A and [in] B is in M but nothing of M is in P or N. (**P**, 128).

The first equation, $A = P + M$, says that the commune of A and B, M, together with some other concept P constitutes A, i.e. M is contained in A. If we symbolize the commune of A and B, i.e. the "greatest" concept C that is contained both in A and in B, by '$A \otimes B$', this condition amounts to the law:

(Comm 1) $A \in A \otimes B$.

Similarly, the second equation, $B = N + M$, entails that

(COMM 2) $B \in A \otimes B$.

Moreover, "whatever is in A and [in] B is in M", i.e. whenever some concept C is contained both in A and in B, it will also be contained in the commune:

(COMM 3) $A \in C \land B \in C \to A \otimes B \in C$.

Thus in sum the commune may be defined as that concept C which contains all and only those concepts Y that are contained both in A and in B:

(DEF 10) $A \otimes B = C \leftrightarrow_{\mathrm{df}}$ $\forall Y (C \in Y \leftrightarrow A \in Y \land B \in Y)$.

Now it is easy to prove (although Leibniz himself never realised this) that the commune of A and B coincides with the disjunction, i.e. the 'or-connection' of both concepts:

(COMM 4) $A \otimes B =_{\mathrm{df}} \overline{\overline{A}\ \overline{B}}$.

According to DEF 10, it only has to be shown that for any concept $Y : \overline{\overline{A}\ \overline{B}} \in Y$ and $B \in Y$. Now if (1) $A \in Y \land B \in Y$, then by the law of contraposition, NEG 3, $\overline{Y} \in \overline{A} \land \overline{Y} \in \overline{B}$, hence by CONJ 1 $\overline{Y} \in \overline{AB}$, from which one obtains by another application of NEG 3 that $\overline{\overline{A}\ \overline{B}} \in Y$; (2) if conversely for any Y $\overline{\overline{A}\ \overline{B}} in Y$, then the desired conclusion $A \in Y \land B \in Y$ follows immediately from the laws

(DISJ 1) $A \in \overline{\overline{A}\ \overline{B}}$

(DISJ 2) $B \in \overline{\overline{A}\ \overline{B}}$

in virtue of CONT 2. The validity of DISJ 1, 2 in turn follows from the corresponding laws of conjunction (CONJ 2, 3), $\overline{A}\ \overline{B} \in \overline{A}$ and $\overline{A}\ \overline{B} \in \overline{B}$ by means of contraposition, NEG 3, plus double negation, NEG 1. In view of COMM 4, then, one obtains the following condition for the extensional interpretation of the commune of A and B:

$$(8)\ \phi(A \otimes B) = \phi(A) \cup \quad \phi(B).$$

Furthermore, as Leibniz noted in passing[43], two concepts are communicating iff the commune of A and B is not the empty concept:

(COMM 5) $\mathrm{Com}(A, B) \leftrightarrow A \otimes B \neq 0$.

Hence the extensional interpretation for the relation $\mathrm{Com}(A, B)$ amounts to the condition that the extensions of A and B are non-exhaustive:

$$(9)\ \phi(\mathrm{Com}(A, B)) = \text{true iff } \phi(A) \cup \quad \phi(B) \neq U.$$

---

[43]Cf. **P**, 127, Theorem IX: "Let us assume meanwhile that E is everything which A and [B] have in common — if they have something in common, so that if they have nothing in common, $E = $ Nothing".

Conceptual subtraction

To conclude our discussion of the Plus-Minus-Calculus, we have to (re)consider the operation of real subtraction, $A - B$, as applied to (intensionally conceived) *concepts.* Leibniz tried to define this operation as follows:

> *Definition 5.* If $[B]$ is in $[A]$, and some other term, $[C]$, should be produced in which there remains everything which is in $[A]$ except what is also in $[B]$ (of which nothing must remain in $[C]$), B will be said to be subtracted or removed from $[A]$, and C will be called the 'remainder'. (**P**, 124).

Thus $(A - B)$ is said to contain all and only those (non-empty) concepts Y which are contained in A but which are not contained in B:

(DEF 11*) $A - B = C \leftrightarrow_{\mathrm{df}}$ $\forall Y (Y \neq 0 \rightarrow (C \in Y \leftrightarrow A \in Y \wedge B \notin Y))$.

This definition entails, firstly, that, as Leibniz postulated in an extra "*Axiom 2*: If the same term is added and subtracted, then [. . .] this coincides with Nothing. That is A [. . .] — A [. . .] = Nothing" (**P**, 124):

(MINUS 1) $A - A = 0$.[44]

Second, a concept Y can remain in the "remainder" $A - B$, only if Y was originally contained in A itself: $\forall Y ((A - B) \in Y \rightarrow A \in Y)$. Substituting $(A - B)$ for Y, one thus obtains (in view of the trivial law CONT 1):

(MINUS 2) $A \in (A - B)$.

Third, whenever some (non-empty) concept C is contained both in A and in B, then C is no longer contained in the remainder $(A - B) : A \in C \wedge B \in C \wedge C \neq 0 \rightarrow (A - B) \notin C$. Thus in particular there does not exist a (non-empty) concept C which is contained both in B and in $(A - B)$, or, as Leibniz put it: "What has been subtracted and the remainder are uncommunicating. If $L - A = N$, I assert that A and N have nothing in common" (**P**, 128):

(MINUS 3) $\neg \mathrm{Com}(A - B, B)$.

Fourth, the above version of DEF 11* would allow to infer that any (non-empty) concept C which is contained in A but not in B will therefore be contained in $(A - B)$:

(MINUS 4*) $A \in C \wedge B \notin C [\wedge C \neq 0]^{45} \rightarrow (A - B) \in C$.

---

[44]For according to DEF 11* $A - A$ would contain a non-empty concept Y only if both $A \in Y$ and $A \notin Y$!

[45]Unlike in MINUS 1, this restriction now is redundant since in view of NIHIL 1 $B \notin C$ already entails that $C \neq 0$.

But this is incompatible with certain other basic principles of the Plus-Minus-Calculus. Consider, e.g., the case where A is the sum of two uncommunicating (non-empty) concepts B and C, $A = B + C$, or $A = BC$. Clearly, A contains B, but not conversely. Hence one could derive from Minus 4* (with 'A' substituted for 'C') that $(A - B) \in A$ which, in view of Minus 2, would mean that $(A - B) = A$! But this is absurd, since if you subtract from $A = BC$ one of the (uncommunicating) components, B, then, as Leibniz's himself noted elsewhere[46], the remainder will be just the other component, C:

(Minus 5) $A = BC \wedge \quad \neg\mathrm{Com}(B, C) \to (A - B) = C.$

The problem behind Minus 4* becomes clearer if one considers another (slightly more complicated) counterexample. Let A contain B which in turn contains some $D(\neq 0)$, and suppose that A contains another concept $E(\neq 0)$ such that $\neg\mathrm{Com}(B, E)$; let C be the "sum" of D and E. Since B and E are uncommunicating, it follows a fortiori that B does not contain E. Hence B does not contain the "larger" concept $C(= DE)$ either. According to Minus 4*, however, the premisses $A \in C \wedge B \notin C$ would entitle us to conclude that C remains (entirely) in $(A - B)$ while, intuitively, only a part of C, namely E, should remain in $(A - B)$ since everything that was contained in B, in particular D, must be removed from A in order to yield $(A - B)$.

Generalizing from this example, one finds that Leibniz's requirement $B \notin Y$ (in Def 11*) is too weak to warrant that a concept Y which was originally contained in A may remain in $(A - B)$. This inference is valid only if Y does not itself contain a component X which is also contained in B. In other words, Y must be *entirely outside* B, i.e. Y and B may have *nothing in common*. Principle Minus 4*, and the corresponding clause of Def 11*, therefore have to be corrected as follows:

(Minus 4) $A \in C \wedge \quad \neg\mathrm{Com}(B, C) \to (A - B) \in C$

(Def 11) $A - B = C \leftrightarrow_{\mathrm{df}} \quad \forall Y (Y \neq 0 \to (C \in Y \leftrightarrow A \in Y \wedge \neg\mathrm{Com}(B, Y))).$

It may then be shown that conceptual subtraction $(A - B)$ might alternatively (and much more simply) be defined as the commune of A and $Non-B$:

(Minus 6) $(A - B) = A \otimes \overline{B}.$

All that has to be proved, according to Def 11, is that for each (non-empty) concept Y : $A \otimes \overline{B} \in Y$ iff $A \in Y \wedge \quad \neg\mathrm{Com}(B, Y)$. Suppose

---

[46]Cf. **C.** 267, # 29: : "[...] if $A + B = C$, then $A = C - B$ [...] but it is necessary that A and B have nothing in common".

(1) that $A \otimes \overline{B} \in Y$. Then COMM 1 immediately gives us $A \in Y$, while $\neg \text{Com}(B, Y)$ is obtained indirectly as follows. Assume that B and Y would have something in common, i.e. there exists some $X (\neq 0)$ such that $B \in X \wedge Y \in X$; premiss (1) by way of COMM 1 entails that $\overline{B} \in Y$, hence because of $Y \in X$ also $\overline{B} \in X$. Together with $B \in X$ one thus obtains by COMM 3 that $\overline{B} \otimes B \in X$, hence by COMM 4 $\overline{B} \ \overline{\overline{B}} \in X$ i.e. $\overline{\overline{B}} \ B \in X$.. But this is a contradiction since any concept contained in the empty or tautological concept must itself be tautological while it was presupposed that $X \neq 0$!

For the proof of the converse implication suppose (2) that $A \in Y \wedge \neg \text{Com}(B, Y)$. In view of COMM 3 it suffices to show that $\overline{B} \notin Y$. Again, this shall be proved indirectly. So if one assumes that $\overline{B} \notin Y$, it follows by POSS 1 that $\mathbf{P}(\overline{BY})$, i.e. $\overline{BY}$ doesn't coincide with the contradictory concept $A\overline{A}$. Hence by contraposition its negation, i.e. according to COMM 4 the commune of B and Y, $B \otimes Y$, does not coincide with the negation of $A\overline{A}$, i.e. with the tautological concept $\overline{A\overline{A}}$. But according to COMM 5 this means that B and Y are communicating, which contradicts our premiss $\neg \text{Com}(B, Y)$.

In the end, then, conceptual subtraction $(A - B)$ turns out to be tantamount to the disjunction of A and $\overline{B}$, and this gives rise to the subsequent condition for the extensional interpretation of $A - B$:

$$(10) \ \phi(A - B) = \phi(A) \cup \overline{(\phi(B))}.$$

We are now in a position to sum up our definition of an extensional interpretation of Leibniz's algebra of concepts which at the same time serves also as an instrument for the extensional interpretation of the Plus-Minus-Calculus (as applied to intensions of concepts):

(DEF 1)   Let U be a non-empty set (of possible individuals). Then the function ϕ is an *extensional interpretation* of the algebra of concepts, $L1$, and of the Plus-Minus-Calculus, $L0.8$, if and only if:

(I) $\phi(A) \subseteq U$ for each concept-letter A, and

(II)   (1) $\phi(A \in B) = $ true iff $\phi(A) \subseteq \phi(B)$

        (2) $\phi(A = B) = $ true iff $\phi(A) = \phi(B)$

        (3) $\phi(A \iota B) = $ true iff $\phi(A) \supseteq \phi(B)$

        (4) $\phi(A \oplus B) = \phi(AB) = \phi(A) \cap \phi(B)$

        (5) $\phi(\overline{A}) = \overline{\phi(A)}$

        (6) $\phi(\mathbf{P}(A)) = $ true iff $\phi(A) \neq \emptyset$

        (7) $\phi(0) = U$

$$(8)\ \phi(A \otimes B) = \phi(A) \cup \phi(B)$$

$$(9)\ \phi(\mathrm{Com}(A, B)) = \text{true iff } \phi(A) \cup \phi(B) \neq U$$

$$(10)\ \phi(A - B) = \phi(A) \cup \overline{\phi(B)}.$$

This summary also allows me to explain why the Plus-Minus-Calculus and the mere Plus-Calculus have been dubbed '$L0.8$' and '$L0.4$', respectively. While the full algebra of concepts, $L1$, contains all of the above ten elements either as primitive or as defined operators, in $L0.4$ only 40 %, namely $\{\in, \iota, =, \oplus\}$, and in $L0.8$ only 80 %, namely $\{\in \{\in, \iota, =, \oplus, 0, \otimes, \mathrm{Com}, -\}$, are available.[47]

To conclude this section let me add some further interesting theorems involving subtraction $(A - B)$ plus the commune of A and B:

Formalzation	Leibniz's formulation
$A = ((A + B) - B) + (A \otimes B)$	"[...] if $A+B = L$ and it is assumed that what A and B have in common is N, then $A = L-B+N$" (**D.**, 251)
$A \otimes B = B - ((B + A) - A)$	"[...] let $A + B$ be L [...] and from L one of the constituents A, is subtracted [...] let the remainder be N [...] if the remainder is subtracted from B [...] there remains M, the common part of A and B;; (**C.**, 250)
$A \otimes B = \{(A + B) - [((A+B)-A)+((A+B)-B]\}$	"From $A + B$ one subtracts A, remains L; from the same one subtracts B, remains M. Now the given $L + M$ is subtractedfrom $A + B$; remains the commune" (**D.**, 251/2).

6 ALETHIC AND DEONTIC MODAL LOGIC

Although Leibniz never spent much time for the investigation of the proper laws of (ordinary or modal) propositional logic, he may yet be credited with three important discoveries in this field:

1. By means of a simple, ingenious device Leibniz transformed the algebra of concepts into an algebra of propositions;

2. Leibniz developed the basic idea of possible-worlds-semantics for the interpretation of the modal operators;

3. Leibniz not only discovered the strict analogy between the logical laws for deontic operators ('forbidden', 'obligatory', 'allowed') on the one hand and the alethic operators ('impossible', 'necessary', 'possible')

---

[47] Neither negation nor the (Im-)Possibility operator can be defined in terms of "Nihil" and/or subtraction!

on the other hand; but he even anticipated A. R. Anderson's [1958] idea of "defining" the former in terms of the latter.

6.1   Leibniz's Calculus of Strict Implication

In the fragment *Notationes Generales*, probably written between 1683 and 1685[48], Leibniz pointed out to the parallel between the containment relation among concepts and the implication relation among propositions. Just as the simple proposition 'A is B' (where A is the "subject", B the "predicate") is true, "when the predicate is contained in the subject", so a conditional proposition 'If A is B, then C is D' (where 'A is B' is designated as 'antecedent', 'C is D' as 'consequent') is true, "when the consequent is contained in the antecedent" (cf. **A** VI, 4, 551). In later works Leibniz compressed this idea into formulations such as "a proposition is true whose predicate is contained in the subject *or more generally* whose consequent is contained in the antecedent".[49] The most detailed explanation of the basic idea of deriving the laws of the algebra of *propositions* from the laws of the algebra of *concepts* was sketched in §§75, 137 and 189 **GI** as follows:

> If, as I hope, I can *conceive all propositions as terms, and hypotheticals as categoricals* [...] this promises a wonderful ease in my symbolism and analysis of concepts, and will be a discovery of the greatest importance. [**P**, 66 ...]

> We have, then, discovered many secrets of great importance for the analysis of all our thoughts and for the discovery and proof of truths. We have discovered [...] how *absolute and hypothetical truths have one and the same laws* and are contained in the same general theorems. [**P**, 78 ...]

> Our principles, therefore, will be these [...] Sixth, *whatever is said of a term which contains a term can also be said of a proposition from which another proposition follows.* (**P**, 85, all italics are mine).

To conceive all propositions in analogy to concepts ("instar terminorum") means in particular that the hypothetical proposition 'If α then β' will be logically treated exactly like the fundamental relation of containment between concepts, 'A contains B'. Furthermore, as Leibniz explained elsewhere, negations (and conjunctions) of propositions are to be conceived just as negations (and conjunctions) of concepts:

---

[48] **A** VI, 4, # 131.

[49] Cf. **C**. 401: "vera autem propositio est cujus praedicatum continetur in subjecto, *vel generalius* cujus consequens continetur in antecedente" (my emphasis); cf. also **C**. 518: "Semper igitur praedicatum seu consequens inest subjecto seu antecedenti".

> If A is a proposition or statement, by non-A I understand the
> proposition A to be false. And if I say 'A is B', and A and B
> are propositions, then I take this to mean that B follows from
> A [...] This will also be useful for the abbreviation of proofs;
> thus if for 'L is A' we would say 'C' and for 'L is B' we say 'D',
> then for this [hypothetical] 'If L is B, it follows that L is B' one
> could substitute 'C is D'.[50]

One thus obtains the following "mapping" of the primitive formulas of
the algebra of concepts into primitive formulae of an algebra of propositions:

$$
\begin{array}{ll}
A \in B & \alpha \to \beta \\
\overline{A} & \neg \alpha \\
AB & \alpha \wedge \beta
\end{array}
$$

As Leibniz himself mentioned, the fundamental law Poss 1 does not only
hold for the containment-relation between concepts but equally for the en-
tailment relation between propositions:

> A contains B is a *true proposition* if A non-B entails a con-
> tradiction. *This applies both to categorical and to hypothetical
> propositions*, e.g., 'If A contains B, C contains D' can be formu-
> lated as follows: 'That A contains B contains that C contains
> D'; therefore 'A containing B and at the same time C not con-
> taining D' entails a contradiction.[51]

Hence $A \in B \leftrightarrow \mathbf{I}(A\overline{B})$ may be "translated" into $(\alpha \to \beta) \leftrightarrow \neg\Diamond(\alpha \wedge \neg\beta)$.
This formula shows that Leibniz's implication is not a material but rather a
strict implication. As was already noted by Rescher [1954, p. 10], Leibniz's
account provides a definition of "entailment in terms of negation, conjunc-
tion, and the notion of possibility", for α implies β iff it is *impossible* that
α is true while β is false. This definition of strict implication "re-invented",
e.g., by C. I. Lewis[52] was formulated also in the "Analysis Particularum":

> Thus if I say 'If L is true it follows that M is true', this means
> that one cannot suppose at the same time that L is true and

---

[50]Cf. **C.**, 260, # 16: "Si A sit propositio vel enuntiatio, per non-A intelligo proposi-
tionem A esse falsam. Et cum dico A est B, et A et B sunt propositiones, intelligo ex
A sequi B. [...] Utile etiam hoc ad compendiose demonstrandum, ut si pro L est A
dixissemus C et pro L est B dixissemus D pro ista si L est A sequitur quod L est B,
substitui potuisset C est D."

[51]Cf. **C.**, 407: "*Vera propositio* est A continet B, si A non-B infert contradictionem.
Comprehenduntur et categoricae et hypotheticae propositiones, v.g. si A continet B, C
continet D, potest sic formari: A continere B continet C continere D; itaque A continere
B, et simul C non continere D infert contradictionem" (second emphasis is mine).

[52]Cf. e.g., [Lewis and Langford, 1932, p. 124]: "The relation of strict implication can
be defined in terms of negation, possibility, and product [...] Thus "p implies q" [...] is
to mean "It is false that it is possible that p should be true and q false".

that M is false.[53]

As regards the other, non-primitive elements of $L1$, the relation 'A is in B' represents, according to DEF 4, the converse of $A \in B$. Hence its propositional counterpart is the "inverse implication", $\alpha \leftarrow \beta$. According to DEF 2, the coincidence relation $A = B$ is tantamount to mutual containment, $A \in B \wedge B \in A$, which will thus be translated into a mutual implication between propositons, $\alpha \to \beta \wedge \beta \to \alpha$, i.e. into strict equivalence, $\alpha \leftrightarrow \beta$. Next, according to DEF 5, the possibility or self-consistency of a concept B amounts to the conditions $B \notin A\overline{A}$. In the field of propositions one hence obtains that α is possible, $\Diamond\alpha$, if and only if α does not entail a contradiction: $\neg(\alpha \to \beta \wedge \neg\beta)$.

$$
\begin{array}{lll}
A\iota B & (\alpha \leftarrow \beta) & [\leftrightarrow_{\mathrm{df}} (\beta \to \alpha)] \\
A = B & \alpha \leftrightarrow \beta & [\leftrightarrow_{\mathrm{df}} (\alpha \to \beta) \wedge (\beta \to \alpha)] \\
\mathbf{P}(A) & \Diamond\alpha & [\leftrightarrow_{\mathrm{df}} \neg(\alpha \to (\beta \wedge \neg\beta))]
\end{array}
$$

Finally one could also map the specific elements of the Plus-Minus-Calculus into the following somewhat unorthodox propositional operators:

$$
\begin{array}{ll}
0 & \neg(\alpha \wedge \neg\alpha) \\
\mathrm{Com}(A, B) & \Diamond(\neg\alpha \wedge \neg\beta)^{54} \\
A \otimes B & \alpha \vee \beta \\
A - B & \alpha \vee \neg\beta.
\end{array}
$$

Given this "translation", the basic axioms and theorems of the algebra of concepts listed at the end of section 4 may be transformed into the following set of laws of an algebra of propositions:

---

[53]Cf. **A** VI, 4, 656: "Itaque si dico *Si L est vera sequitur quod M est vera*, sensus est, non simul supponi potest quod L est vera, et quod M est falsa"'.

	Basic Principles of PL1
IMPL 1	$(\alpha \to \alpha)$
IMPL 2	$((\alpha \to \beta) \wedge (\beta \to \gamma)) \to (\alpha \to \gamma)$
IMPL 3	$(\alpha \to \beta) \leftrightarrow (\alpha \leftrightarrow \alpha \wedge \beta)$
CONJ 1	$(\alpha \to \beta \wedge \gamma) \leftrightarrow ((\alpha \to \beta) \wedge (\alpha \to \gamma))$
CONJ 2	$\alpha \wedge \beta \to \alpha$
CONJ 3	$\alpha \wedge \beta \to \beta$
CONJ 4	$\alpha \wedge \alpha \leftrightarrow \alpha$
CONJ 5	$\alpha \wedge \beta \leftrightarrow \beta \wedge \alpha$
NEG 1	$(\neg\neg\alpha \leftrightarrow \alpha)$
NEG 2	$\neg(\alpha \leftrightarrow \neg\alpha)$
NEG 3	$(\alpha \to \beta) \leftrightarrow (\neg\beta \to \neg\alpha)$
NEG 4	$\neg\alpha \to \neg(\alpha \wedge \beta)$
NEG 5	$[\Diamond\alpha\wedge](\alpha \to \beta) \to \neg(\alpha \to \neg\beta)$
POSS 1	$(\alpha \to \beta) \leftrightarrow \neg\Diamond(\alpha \wedge \neg\beta)$
POSS 2	$(\alpha \to \beta \wedge \Diamond\alpha \to \Diamond\beta$
POSS 3	$\neg\Diamond(\alpha \wedge \neg\alpha)$
POSS 4	$(\alpha \wedge \neg\alpha) \to \beta$

Although Leibniz didn't care very much about propositional logic, he happened to put forward at least some of these laws in scattered fragments. For instance, in the first juridical disputation *De Conditionibus* the transitivity of the inference relation, IMPL 2, is characterized as follows: "The Co[ndition] of the co[ndition] is the co[ndition] of the co[nditioned]. If by positing AB will be posited and by positing BC will be posited, then also by positing AC will be posited".[55] As regards IMPL 1 and CONJ 2, 3, Leibniz mentions in the fragment "De Calculo Analytico Generale" the "Primary Consequences: A is B, therefore A is B [...] A is B and C est D, therefore A is B, or as well [therefore] C is D",[56] and the corresponding "Axioms [...] 3) If A is B, also A is B. If A is B and B is C, also A is B". Furthermore the definition of strict implication in terms of strict equivalence (and conjunction), IMPL 2, is exemplified in another fragment as follows:

> *A true hypothetical proposition* of first degree is 'If A is B, and from this it follows that C is D' [...] Let the state of affairs 'A is B' be called L, and the state of affairs 'C is D' be called M. Then one obtains $L = LM$; *in this way the hypothetical [proposition] is reduced to a categorical.* (cf. **C**. 408, second emphasis is mine).

---

[55]Cf. **A** VI, 1, 110: "C[onditi]o C[onditio]nis est C[onditi]o C[onditiona]ti. Si posito A positur B, et posito B positur C; etiam posito A positur C." For a discussion of Leibniz's early work on juridic (or deontic) logic cf. Schepers [1975].

[56]Cf. **A** VI, 4, 149; "PRIMAE CONSEQUENTIAE A *est* B *ergo* A *est* B. [...] A *est* B *et* C *est* D *ergo* A *est* B vel *ergo* C *est* B".

Moreover in "De Varietatibus Enuntiationum" Leibniz forwards principle CONJ 1 for the special case $A = $ 'a is b', $B = $ 'e is d' and $C = $ 'l is m' by maintaining that the proposition "If a is b it follows that e is d and l is m" can be resolved into the conjunction of the propositions "If a is b it follows that e is d" and "If a is b it follows that l is m" (cf. **A** VI, 4, 129). Versions of the principle of double negation, NEG 1, may be found in §4 **GI** or, for the special cases of propositions of the type '$A = B$' and '$A \in B$', more formally in **C**. 235[57]. Finally the "Analysis particularum" contains besides the above quoted paraphrase of POSS 1 also the law of (propositional) contraposition NEG 3: "If a proposition M [...] follows from a proposition L [...], then conversely the falsity of the proposition L follows from the falsity of the proposition M".[58]

The above collection of basic principles does not yet, however, constitute a genuine *calculus* of (modal) propositional logic. At least some additional *rules of deduction* are needed which allow one to derive further theorems from these "axioms". As was shown elsewhere, Leibniz was well aware at least of the validity of the rule of (strict) *modus ponens*:

(MP) $\qquad (\alpha \rightarrow \beta), \alpha \vdash \beta$

and of the rule of conjunction:

(RC) $\qquad \alpha, \beta \vdash \alpha \wedge \beta.$

Furthermore it was argued there that the mapping of $L1$ into $PL1$ yields a calculus of strict implication in the vincinity of Lewis' system S2°. This does not mean, however, that Leibniz would have favoured such a weak system as the proper calculus of (alethic) modal logic. For example, Leibniz would certainly have subscribed to the validity of the truth-axiom $\Box \alpha \rightarrow \alpha$ (or, equivalently, $\alpha \rightarrow \Diamond \alpha$). But, for purely syntactical reasons, these laws can never be obtained by Leibniz's consideration of propositions "instar terminorum" from corresponding theorems of $L1$.[59] For reasons of space, this issue shall not be discussed here further — the reader is referred to the detailed exposition in [Lenzen, 1987]. Only a few more theorems for the modal operators $\Box$ and and $\Diamond$ shall be considered in the subsequent section where Leibniz's version of a possible worlds semantics is represented.

6.2 Leibniz's Possible Worlds Semantics

The fundamental logical relations between necessity, $\Box$, possibility, $\Diamond$, and impossibility can be expressed, e.g., by:

---

[57]"Idem sunt $A \infty B$ [...] et A non non ∞B"; cf. also **C**. 262: "A non non est B, idem est quod A est B"

[58]Cf. **A** VI, 4, 655/6: "Si ex propositione L [...] sequitur propositio M [...] tunc contra ex falsitate propositionis M sequitur falsitas propositionis L".

[59]E.g., $\alpha \rightarrow \Diamond \alpha$, could only result from mapping the formula $A \in \mathbf{P}(A)$ or $A \rightarrow \mathbf{P}(A)$ into PL1; but none of these is syntactically well-formed!

(Nec 1) $\Box(\alpha) \leftrightarrow \neg\Diamond(\neg\alpha)$

(Nec 2) $\neg\Diamond(\alpha) \leftrightarrow \Box(\neg\alpha)$.

Of course, these laws were familar already to logicians long before Leibniz. However, Leibniz not only formulated, e.g., Nec 1 already as a youth, at the age of 25, as follows:

> Whenever the question is about necessity, the question is also about possibility, for if something is called necessary, then the possibility of its opposite is negated.[60]

but he also "proved" these relations by means of an admirably clear semantic analysis of modal operators in terms of "possible cases", i.e. possible worlds:

"Possible   is whatever can   happen   or what is true in some cases
Impossible is whatever cannot happen   or what is true in no [...] case
Necessary   is whatever cannot not happen or what is true in every [...] case
Contingent is whatever can   not happen or what is [not] true in some case".[61]

Hence a proposition α is *possible* iff α is true in at least one case; α is *impossible*, iff α is true in no case; α is *necessary* iff α is true in each case; and, finally, α is *contingent*, i.e. non-necessary, iff α is not true in at least one case.[62] Now this analysis of the truth-conditions for modal propositions not only entails the above mentioned laws Nec 1 and 2, but it also gives rise to the principle that whenever α is necessary, α will be possible as well, and by contraposition: "Because all that is necessary is possible, all that is impossible is contingent":[63]

(Nec 3) $\Box\alpha \rightarrow \Diamond(\alpha)$,

(Nec 4) $\neg\Diamond(\alpha) \rightarrow \neg\Box(\alpha)$.

Leibniz "demonstrates" these laws by reducing them to corresponding laws for (universal and existential) quantifiers such as: "If α is true in each case, then α is true in at least one case". These quantificational principles were tacitly presupposed by Leibniz who only mentioned them in passing by maintaining (very elliptically), e.g.: "'All' is the same as 'none not"'or "'All not' is the same as 'none"'. Cf. the following "proof" of Nec 2:

---

[60] Cf. **A** VI, 1, 460: "Quoties autem de necessitate quaestio est, de possibilitate quaestio est, nam quid necessarium dicitur, possibilitas oppositi negatur".

[61] Cf. **A** VI, 1, 466:
"Possibile   est quicquid potest   fieri seu quod verum est quodam   casu
Impossibile   est quicquid non potest   fieri seu quod verum est nullo [...]   casu
Necessarium est quicquid non potest non fieri seu quod verum est omni [...]   casu
Contingens   est quicquid potest non   fieri Seu quod verum est quodam non casu."

[62] As this quotation shows, Leibniz uses the notion of contingency not in the modern sense of ,neither necessary nor impossible' but as the simple negation of ,necessary'.

[63] Cf. **A** VI, 4, 2759: "Quia omne necessarium est possibile omne impossibile est contingens seu potest non fieri".

> [...] 'necessarily not happen' and 'impossible' coincide. For also
> 'none' and 'everything not' coincide. Why so? Because 'none'
> is 'not something'. 'Every' is 'not something not'. Therefore
> 'everything not' is 'not something not not'. The two latter 'not'
> destroy each other, thus remains 'not something'.[64]

On the background of certain rules for the negation of the quantifier
expressions 'all', 'some', and 'none', which reflect the core ideas of the tra-
ditional theory of opposition of categorical forms, Leibniz thus argues that
an impossible proposition which is false in every case is the same as a propo-
sition which is not true in any case. Let it be mentioned in passing that the
analogue "proof" of NEC 3 contains a minor mistake which is quite typical
of Leibniz:[65]

> [...] everything which is necessary is possible. For always, when
> 'everything is', also 'something is' [the case]. Thus if 'everything
> is', 'not something is not', or 'something is not not'. Hence
> 'something is'.[66]

To be sure, a necessary proposition α which is true in every case a fortiori
has to be true in at least one case, hence α is possible. But this principle
— or the corresponding quantificational law $(\forall x \alpha \rightarrow \exists x \alpha)$ — cannot be
correctly derived from the presupposed equivalence $(\forall x \alpha \leftrightarrow \neg \exists x \neg \alpha)$ plus
the law of double negation, $(\neg\neg\alpha \leftrightarrow \alpha)$ in the way attempted by Leibniz.
For 'not something is not', i.e. $\neg \exists x \neg \alpha$, is not the same as 'something is not
not',i.e. $\exists x \neg\neg\alpha$!

It cannot be overlooked, however, that the truth conditions quoted from
the early *De Conditionibus*, even when combined with Leibniz's later views
on possible worlds, fail to come up to the standards of modern possible
worlds semantics, since in Leibniz's work nothing corresponds to the access-
ability relation among worlds. Therefore it is almost impossible to decide
which of the diverse modern systems like T, S4, S5, etc. best conforms
with Leibniz's views. According to Poser [1969], Leibniz's modal logic is
tantamount to S5. This means in particular that Leibniz acknowledged the
characteristic axiom of S4:

(NEC 5) $\square\alpha \rightarrow \square\square\alpha$.

---

[64]Cf. **A** VI, 1, 469: "[...] necessarium non fieri et impossibile, coincidunt. Nam
etiam Nullus et omnis non coincidunt. Cur ita? quia nullus est non quidam. Omnis est
non quidam non. Ergo omnis non, est non quidam non non. Abjiciant se mutuò duo
posteriora non, superest non quidam."

[65]In so far as, again and again, Leibniz had serious problems in distinguishing ‚non est'
and ‚est non'; cf. [Lenzen, 1986].

[66]Cf. **A** VI, 1, 469: "[...] omne necessarium est possible. Nam semper, si omnis est,
etiam quidam est. Si enim Omnis est, non quidam non est seu quidam non non est. Ergo
quidam est".

Poser pointed out to the following passage in "De Affectibus": "For what can impossibly be actually the case, that can impossibly be possible"[67] which rather convincingly shows that, in Leibniz's view, any impossible proposition is impossibly possible:

(Nec 6) $\neg \Diamond \alpha \rightarrow \neg \Diamond \Diamond \alpha.$

However, Poser failed to give any quotation (or any other compelling reason) to show that Leibniz would also have accepted the stronger S5-principle $\Diamond \alpha \rightarrow \Box \Diamond \alpha$, according to which any possible proposition would be necessarily possible. Moreover, as was argued by Adams [1982], the latter principle appears to be incompatible with Leibniz's philosophical view of necessity as expressed, e.g., in the **GI**:

> (133) A true necessary proposition can be proved by reduction to identical propositions, or by reduction of its opposite to contradictory propositions; hence its opposite is called, impossible'.

> (134) A true contingent proposition canot be reduced to identical propositions, but is proved by showing that if the analysis is continued further and further, it constantly approaches identical propositions, but never reaches them. (**P**, 77).

If a *necessary* proposition α can be reduced in finitely many steps to an "identity", this means that a proposition α is *possible* if and only if it is not refutable in finitely many steps (i.e. its negation cannot be reduced in finitely many steps to an "identity"). But on this understanding of possibility and necessity, the S5 principle $\Diamond \alpha \rightarrow \Box \Diamond \alpha$ appears to be blatantly false.

6.3   Leibniz's Deontic logic

Leibniz saw very clearly that the logical relations between the "Modalia Iuris" *obligatory*, *permitted* and *forbidden* exactly mirror the corresponding relations between the alethic modal operators *necessary*, *possible* and *impossible* and that therefore all laws and rules of alethic modal logic may be applied to deontic logic as well:

> Just like 'necessary', 'contingent', 'possible' and 'impossible' are related to each other, so also 'obligatory', 'not obligatory', 'permitted', and 'forbidden'.[68]

---

[67]Cf. **Grua**, 534: "Nam quod impossibile est esse actu, id impossibile est esse possibile".

[68]Cf. **A** VI, 4, 2762: "Uti se habent inter se necessarium, contingens, possibile, impossibile; ita se habent debitum, indebitum, licitum, illicitum".

This structural analogy rests on the important discovery that the deontic notions can be defined by means of the alethic notions plus the additional "logical" constant of a morally perfect man ["vir bonus"]. Such a "virtuous man", b, is characterized by the requirements that (1) b strictly obeys all laws, (2) b always acts in such a way that he does no harm to anybody, and (3) b loves or is benevolent to all other people.[69] Given this understanding of the "vir bonus", b, Leibniz explains:

Obligatory is	what is	necessary	for the virtuous man as such
not obligatory is	what is	contingent	for the virtuous man as such
permitted is	what is	possible	for the virtuous man as such
forbidden is	what is	impossible	for the virtuous man as such.[70]

If we express the restriction of the modal operators $\Box$ and $\Diamond$ to the virtuous man by means of a subscript 'b', these definitions can be formalized as follows:

(Deon 1) $\quad O(\alpha) \leftrightarrow \Box_b(\alpha)$

(Deon 2) $\quad \mathbf{E}(\alpha) \leftrightarrow \Diamond_b(\alpha)$[71]

(Deon 3) $\quad \mathbf{F}(\alpha) \leftrightarrow \neg\Diamond_b(\alpha)$

Now, as Leibniz mentioned in passing, all that is unconditionally necessary will also be necessary for the virtuous man as such:[72]

(Nec 7) $\quad \Box(\alpha) \rightarrow \Box_b(\alpha)$.

Hence the fundamental laws for the deontic operators can be derived from corresponding laws of the alethic modal operators in much the same way as Anderson [1958] reduced deontic logic to alethic modal logic. As Leibniz pointed out, two different classes of theorems may be distinguished. First

---

[69]Cf. **A** VI, 1, 466: "Vir bonus est quisquis amat omnes"; **A** VI, 4, 2851: "Vir bonus est qui benevolus est erga omnes" and **A** VI, 4, 2856: "Vir bonus censetur, qui hoc agit ut prosit omnibus noceat[que] nulli." It is interesting to note that Leibniz denotes the entire discipline of jurisprudence as the "science of the virtuous man" ("scientia viri boni") and justice as the "voluntas viri boni".

[70]Cf. **A** VI, 4, 2758:

"Debitum	est, quod viro bono qua tali	necessarium
Indebitum	est, quod viro bono qua tali	contingens
Licitum	est, quod viro bono qua tali	possibile
Illicitum	est, quod viro bono qua tali	impossibile."

In the former edition in **Grua** 605 'debitum' was mistakenly associated with 'contingens'. Cf. also **A** VI, 4, 2863: "quod Viro bono possibile, impossibile, necessarium est, si nomen suum tueri velit, id justum sive licitum, injustum, ac denique debitum esse."

[71]We here use the letter 'E' (reminding of the German 'erlaubt') instead of 'P' for ‚permitted' in order to avoid any confusions with the operator for the possibility (or self-consistency) of concepts!

[72]Cf. **A** VI, 4, 2759: "Nam omne necessarium est necessarium viro bono".

we have some "Theorems in which the juridic modalities are combined by themselves", i.e. theorems describing the logical relations among the deontic operators, e.g.:

> Everything which is obligatory is permitted [. . .] Everything which is forbidden is not obligatory [. . .] Nothing which is obligatory is forbidden [. . .] Nothing which is forbidden is obligatory [. . .] Everything that is forbidden is obligatory to omit. And everything that is obligatory to omit is forbidden. [. . .] Everything that is forbidden to omit is obligatory and everything which is obligatory is forbidden to omit [. . .] Everything which is not obligatory is permitted to omit and everything that is permitted to omit is not obligatory.[73]

(DEON 4a) $O(\alpha) \rightarrow E(\alpha)$

(DEON 4b) $\neg E(\alpha) \rightarrow \neg O(\alpha)$

(DEON 5a) $O(\alpha) \rightarrow \neg F(\alpha)$

(DEON 5b) $F(\alpha) \rightarrow \neg O(\alpha)$

(DEON 6) $F(\alpha) \leftrightarrow O(\neg \alpha)$

(DEON 7) $O(\alpha) \leftrightarrow F(\neg \alpha)$

(DEON 8) $\neg O(\alpha) \leftrightarrow E(\neg \alpha)$

As Leibniz "demonstrates" (or, at least, makes it plausible to suppose), these laws are immediate counterparts of the well-known logical relations between the alethic modalities. E.g., concerning DEON 6 he remarks:

> Everything which is forbidden is obligatory to omit. And everything that is obligatory to omit is forbidden, i.e. 'forbidden' and 'obligatory to omit' coincide. Because 'necessarily not happen' and 'impossible' coincide. For also 'none' and 'everything not' coincide.[74] (Cf. **A** VI, 1, 469).

As a second class of theorems one obtains certain "Theorems in which the juridic modalities are combined with the logical modalities" [Theoremata

---

[73]Cf. **A** VI, 1, 468/9: "Omne debitum est justum" [...] "Omne injustum est indebitum" [...] "Nullum debitum est injustum" [... or equivalently] "Nullum injustum est debitum" [...] "Omne injustum est debitum omitti. Et omne debitum omitti est injustum" [...] "Omne injustum omitti est debitum et Omne debitum est injustum omitti" [... and] "Omne indebitum juste omittitur et omne quod juste omittitur est indebitum".

[74]Cf. **A** VI, 1, 469: "Omne injustum est debitum omitti. Et omne debitum omitti est injustum, seu injustum et debitum non fieri coincidunt. Quia necessarium non fieri et impossibile, coincidunt. Nam etiam Nullus et omnis non coincidunt".

quibus combinantur Iuris Modalia Modalibus Logicis seu justum cum possibili]. Thus in the "Elementa Juris Naturalis" Leibniz mentions the following principles concerning the relations between the alethic concepts 'necessary', 'possible' and 'impossible' on the one hand and the deontic notions 'obligatory, 'permitted' and 'forbidden' on the other hand: "Everything which is necessary is obligatory" [Omne necessarium debitum est], or, by contraposition: "Everything that is not obligatory is not necessary but contingent" [Cf. **A** VI, 1, 470: "Omne indebitum nec necessarium est, sed contingens"]:

(DEON 9a) $\Box(\alpha) \to O(\alpha)$

(DEON 9b) $\neg O(\alpha) \to \neg\Box(\alpha)$

Furthermore: "Everything that is necessary is permitted" [Omne necessarium justum est], or, again by contraposition, "Everything that is forbidden is not necessary but contingent" ["Quicquid injustum est, id nec necessarium est, sed contingens", *ibid.*]:

(DEON 10a) $\Box(\alpha) \to E(\alpha)$

(DEON 10b) $\neg E(\alpha) \to \neg\Box(\alpha)$

Next, "Everything that is permitted is possible" [Omne justum possibile est], or "Everything that is impossible is not permitted" ["Quicquid est impossibile, id injustum est", *ibid.*]:

(DEON 11a) $E(\alpha) \to \Diamond(\alpha)$

(DEON 11b) $\neg\Diamond(\alpha) \to \neg E(\alpha)$

Finally, "Everything which is obligatory is possible" [Omne debitum possibile est], or "Everything which is impossible is not obligatory, i.e. may be omitted by the good man" ["Omne impossibile indebitum seu omissibile est viro bono", *ibid.*]:

(DEON 12a) $O(\alpha) \to \Diamond(\alpha)$

(DEON 12b) $\neg\Diamond(\alpha) \to \neg O(\alpha)$

To illustrate Leibniz's way of demonstrating these laws in "Modalia et Elementa Juris Naturalis" let us consider DEON 10a which is formulated there with the word ,licitum' instead of ,justum' for 'permitted':

> Everything which is necessary is permitted, i.e. necessity has no law.

> For everything which is necessary is necessary for the good man. If something is necessary for the good man, its opposite is impossible for the good man. What is impossible for the good

man is anyway not possible for the good man as such, i.e. it is
not permitted. Therefore the opposite of something necessary
is not permitted. However, if the opposite of something is not
permitted, then itself is permitted.[75]

By means of the "bridge principle", Nec 7, $\Box(\alpha)$ is first shown to entail
$\Box_b(\alpha)$. Next Leibniz makes use of the following law Nec 8 which relativizes
the usual equivalence Nec 1 to the "virtuous man":

(Nec 8) $\Box_b(\alpha) \leftrightarrow \neg\Diamond_b(\neg\alpha)$.

According to Deon 2, the resulting formula $\neg\Diamond_b(\neg\alpha)$ is equivalent to $\neg E(\neg\alpha)$
which in turn entails the desired conclusion $E(\alpha)$ by way of the further the-
orem:

(Deon 13) $\neg E(\neg\alpha) \to E(\alpha)$.

Note, incidentally, that in an earlier proof which was later deleted by Leib-
niz, the conclusion $\Diamond_b(\alpha)$ or $E(\alpha)$ had been obtained more directly by in-
ferring $\Box_b(\alpha)$ from the premiss $\Box(\alpha)$ and then making use of the following
law which relativizes Nec 3 to the person b:

(Nec 9) $\Box_b(\alpha) \to \Diamond_b(\alpha)$

For, as Leibniz remarks: "Everything which is necessary for the good man is
anyway possible for the good man as such, i.e. it is permitted"[76]. Similarly
Leibniz proves Deon 12b as follows:

> Nothing which is impossible is obligatory, i.e. there is no obli-
> gation for impossibles.

> For everything which is imposible is impossible for the good
> man. Nothing which is impossible for the good man is anyway
> possible for the good man as such. What is not possible for the
> good man as such is not necessary for the good man as such, i.e.
> it is not obligatory.[77]

Here again by means of the "bridge principle" Nec 7, $\neg\Diamond_b(\alpha)$ is first shown
to follow from $\Box(\neg\alpha)$ or $\neg\Diamond(\alpha)$; second, Nec 9 in its contraposited form
$\neg\Diamond_b(\alpha) \to \neg\Box_b(\alpha)$ is used to derive $\neg\Box_b(\alpha)$ which, thirdly, according to
Deon 1, gives the desired conclusion $\neg O(\alpha)$.

---

[75]Cf. **A** VI, 4, 2759/60: "Omne necessarium est licitum, seu necessitas non habet
legem.Nam omne necessarium est necessarium viro bono. Quod est necessarium viro
bono, ejus oppositum est impossibile viro bono. Quod impossibile viro bono utcunque
non est possibile viro bono qua tali seu licitum. Ergo necessarii oppositum non est
licitum. Cujus autem oppositum non est licitum, id ipsum est licitum. "

[76]Cf. **A** VI, 4, 2759: "Omne necessarium viro bono utcunque est possibile viro bono
qua tali; hoc est licitum".

[77]Cf. **A** VI, 4, 2759: "Nullum impossibile est debitum, seu impossibilium nulla est
obligatio. Nam omne impossibile est impossibile viro bono. Nullum impossibile viro
bono utcunque est possibile viro bono qua tali. Quod non est possibile viro bono qua
tali non est necessarium viro bono qua tali, seu non est debitum. "

7 "INDEFINITE CONCEPTS" (QUANTIFIER LOGIC L2)

In many logical fragments Leibniz uses letters from the end of the alphabet $(x, y, \ldots, X, Y, Z, \ldots)$ and occasionally also from the mid of the alphabet $(Q, L, \ldots)$ for the representation of "indefinite concepts", while the "normal" concepts are symbolized by letters from the beginning of the alphabet $(A, B, C, \ldots, a, b, \ldots)$[78]. Below it will be shown

1. that indefinite concepts primarily function as (existential and universal) *quantifiers ranging over concepts*;

2. that Leibniz somehow "felt" the difference between an indefinite concept's functioning as an existential quantifier and as a universal quantifier, but that his elliptic formalization fails to bring out this difference with sufficient clarity and precision;

3. that Leibniz nevertheless anticipated some fundamental laws of quantifier logic and may thus be considered at least as a forerunner of modern quantification theory.

The bare essentials of his theory of indefinite concepts — as developed mainly in the **GI** — shall be outlined in this section (7), while some more details will be presented in the subsequent sections devoted to the theory of "quantification of the predicate" (8) and to Leibniz's view of possible individuals and possible worlds (9).

7.1 *The Existential Quantifier*

By the time around 1679 Leibniz became aware of the possibility to represent the universal affirmative (U.A.) proposition 'Every A is B' by the formula $A = BY$. The origin of this formalization appears to be due to the semantics of so-called "characteristic numbers", i.e. a numerical *model* for the theory of the syllogism which (1) assigns to the concepts $A, B, \ldots$ certain numbers $a, b, \ldots$[79] where (2) the ‚est'-relation among concepts is semantically interpreted by the condition of divisability of the corresponding numbers.

> *A categorical universal affirmative proposition* as 'Man is animal' will be expressed as follows: $\frac{b}{a} = y$, or $b = ya$. For it signifies that the number by which 'man' is expressed can be

---

[78]Cf. **GI**, §21: "Deinde definitas a me significari prioribus Alphabeti literis, indefinitas posterioribus, nisi aliud significetur." Similarly in **C**. 274-6: "Literae posteriores ut V, W, X, Y, Z, etc. significabunt indefinitum" or also in **C**. 264-70, ## (7,8): "A significat determinatum, Y vel Z vel alia litera posterior significat indeterminatum."

[79]In a later, more sophisticated approach Leibniz assigns a *pair* of such numbers to each concept. For details cf., e.g., Lukasiewicz [1957].

> divided by the number by which 'animal' is expressed, although
> the result of the division, namely y, is not considered here.[80]

Here y represents an "indefinite number" which is implicitly bound by an
existential quantifier. In §16 **GI** the "*Affirmative Proposition* A is B" is
similarly analyzed (without specific reference to characteristic numbers) as
follows:

> [...] That is, if we substitute a value for A, 'A coincides with
> BY' will appear [...] For by the sign Y I mean something
> undetermined, so that BY is the same as some B [...] So 'A is
> B' is the same as 'A is coincident with some B', or $A = BY$.[81]

This principle, according to which $A \in B$ is equivalent to $A = BY$, has to
be interpreted more exactly as the existentially quantified proposition that
A contains B if and only if *there exists some Y such that $A = BY$*:

(Cont 4) $A \in B \leftrightarrow \exists Y (A = BY)$.

This explicit introduction of the existential quantifier not only accords with
Leibniz's own intentions but it was also anticipated by him in some other
fragments. Thus in §10 of "The Primary Bases of a Logical Calculus"
(**C**. 235–7) he used the expression "there can be assumed a Y such that
$A = YB$" (**P**, 90). And in fragment **C**. 259–61 Leibniz starts by putting
forward the law

(Neg 6*) $A \notin B \leftrightarrow \exists Y (YA \in \overline{B})$

elliptically as "A is not B is the same as QA is non B" (§9), but when he
later offers a proof of this principle in §18, he uses the unambiguous and
explicit formulation "there exists a Q such that QA is $\overline{B}$" [datur Q tale ut
QA sit non B].

   Now, there is a minor problem connected with Neg 6*. In view of Conj
2, the concept $\overline{B}A$ contains $\overline{B}$; hence, trivially, there always exists at least
one Y such that $YA \in \overline{B}$, namely $Y = \overline{B}$. Therefore one should improve
Neg 6* by saying more exactly that the negation of the U.A., 'Some A is
not B', is true if and only if for some Y which is *compatible with A: YA*
contains $\overline{B}$:

---

[80]Cf. **C**., 57: "*Propositio categorica universalis affirmativa*, ut homo est animal, sic
exprimetur: $\frac{b}{a} = yaequ.$ y, vel b aequ. ya. significat enim numerum quo exprimitur
homo, divisibilem esse per numerum quo exprimitur animal, tametsi is quod dividendo
prodit nempe y hic non consideretur".

[81]**P**, 56; cf. also §§17, 158, 189 and 198 **GI** or **C**. 301. In the fragments **C**. 259-61
and **C**. 261-4, Leibniz used the letter 'L 'as an "indeterminate concept": "A *est* B, sic
exponitur literaliter $A \infty LB$, ubi L idem quod indefinitum quoddam" (**C**. 259); cf. also
C. 262/3: "cum A est B dici potest $A \infty LB$ [...] per L intelligi Ens vel aliud quiddam
quod jam in A continetur".

(Neg 6) $A \notin B \leftrightarrow \exists Y (\mathbf{P}(YA) \wedge YA \in \overline{B}).$

As a matter of fact, Leibniz himself hit upon the necessity of postulating that QA is self-consistent when he proved Neg 6 by means of the former principle Poss 1 as follows:

> 'A is not B' and 'QA is non B' coincide, i.e. to say 'A isn't B'
> is the same as to say 'there exists a Q such that QA is non B'.
> If 'A is B' is false, then 'A non B' is possible by [Poss 1]. 'Non
> B' shall be called 'Q'. Therefore QA is possible.[82]

In other places, however, Leibniz often overlooked this requirement or he simply took the self-consistency of the corresponding concept for granted. Thus in §§47, 48 **GI** after stating that "'A contains B' is a universal affirmative in respect of A" he suggests the following formalization for the P.N.: " 'AY contains B' is a particular affirmative in respect of A". Since $AY \in B$, i.e. more explicitly $\exists Y (AY \in B)$, follows from the trivial law $AB \in B$, this condition cannot, however, adequately express the content of the PN which rather has to be formalized by $\exists Y (\mathbf{P}(AY) \wedge AY \in B)$.

The basic inference of existential generalization,

(Exis 1) $\phi(A) \vdash \exists Y \phi(Y),$

according to which any proposition asserting that a certain concept A has the property ϕ entails that for some indefinite concept $\phi(Y)$, was formulated in §23 **GI** as follows:

> For any definite letter there can be substituted an indefinite
> letter not yet used [. . .] i.e. one can put $A = Y$.

Furthermore Leibniz provided several special instances or applications of this rule, e.g.:

(Exis 1.1) $A = AA \vdash \exists Y (A = AY)$

(Exis 1.2) $AB \in C \vdash \exists Y (AY \in C)$

(Exis 1.3) $A = AB \vdash \exists Y (A = YB).$

Thus in §24 **GI** he derives $\exists Y (A = AY)$ from the principle of idempotence, Conj 4, by noting:

> To any letter a new indefinite one can be added; e.g., for A we
> can put AY. For $A = AA$ (by 18 [i.e. Conj 4]), and A is Y (or,
> for A one can put Y, by 23 [i.e. by Exis 1]); therefore $A = AY$.
> (**P**, 57).

---

[82]Cf. **C.** 261: "A non est B et QA est non B coincidere seu dicere A non est B, idem esse ac dicere: datur Q tale ut QA sit non B. Si falsum est A est B, *possibile est A non B* per [Poss 1]. Non B vocetur Q. Ergo *possibile est QA*" (my emphasis).

In §49 **GI** he proves Exis 1.2 as follows: "If AB is C, it follows that AY is C; or, it follows that some A is C. For it can be assumed by 23 [i.e. by Exis 1] that $B = Y$" (**P**, 59). Furthermore, the validity of Exis 1.3 (that had already been maintained in §117 **GI**)[83] was proved, e.g., in a fragment of August 1st, 1690 as follows:

> If $A = AB$, there can be assumed a Y such that $A = YB$. This is a postulate but it can also be proved, for A itself at any rate can be designated by Y. (**P**, 90).

In # 13 of the same fragment Leibniz also shows the converse implication:

> If $A = YB$, it follows that $A = AB$. I prove this as follows. $A = YB$ (by hypothesis), therefore $AB = YBB$ (by [11]) $= YB$ (by 6 [i.e. Conj 4] $= A$ (by hypothesis).

Note, incidentally, that the inference from $A = YB$ to $AB = YBB$ is licensed by principle # 11 of the same essay ("If $A = C$, $AC = BC$") and not, as the editions of Couturat and Parkinson have it, by # 10. It is true that the manuscript contains "per (10)", but this slip is owing to the fact that Leibniz originally numbered the quoted principle as # (10), and when he later renumbered it as # 11, he forgot to change the reference accordingly.

Anyway, these examples show that Leibniz had a fairly good understanding of the rule for *introducing* an existential quantifier, Exis 1. Moreover, one may also ascribe to him at least a partial insight into the validity of the converse rule for *eliminating* existential quantifiers. In modern systems of natural deduction this rule says that from an existential proposition of the form $\exists Y \alpha[Y]$ one may deduce a corresponding singular proposition $\alpha[A]$ provided that the singular term A is a "new" one, which does not yet occur in the corresponding context:

(Exis 2) $\exists Y \phi(Y) \vdash \phi(A)$, for some "new" constant A.

In this vein also Leibniz notes in **GI** §27:

> Some $B = YB$, and therefore some $A = ZA$ [...] *but a new indefinite letter, namely Z*, is to be assumed for the latter equation just as Y had been assumed a little earlier. (**P**, 57; my emphasis).

This passage may be interpreted as saying that from a proposition, e.g., of the form 'Some A is C', i.e. $\exists Y (AY \in C)$, one may deduce that $AZ[\in C]$, provided that the indefinite concept Z is "new". In Lenzen [1984a] various other examples were discussed which show that Leibniz often applied the rule of inference, Exis 2, is just this sense.

---

[83] "$A = BY$ is the same as that $A = BA$". Cf. also §8 of fragment **C.**, 261–4.

7.2 The Universal Quantifier

Leibniz did not always recognize that the *negation* of a formula containing an indefinite concept as an *existential* quantifier gives rise to a *universally* quantified proposition. Thus in "De Formae Logicae Comprobatione" (**C**, 292–321) he tried to prove the syllogisms of the first figure within the quantifier system $L2$ as follows:

Barbara:	Every C is B	Every D is C	Therefore Every D is B.
	$C = BX$	$D = CY$	Therefore $D = BXY$.
Celarent:	No C is B	Every D is C	Therefore No D is B
	$C = X$ non-B	$D = CY$	Therefore $D = YX$ Non-B
Darii:	Every C is B	Some D is C	Therefore Some D is B
	$C = BX$	$D \neq Y$ non-C	Therefore $D \neq Y$ non-BX.

> But the desired $D \neq YX$ non-B does not follow from this [representation]. Hence there is still another difficulty in this calculus. Let's take an example: Every man is an animal. Some wise [being] is a man. Therefore Some wise [being] is an animal. According to the calculus: 'Man' is the same as 'rational animal'; 'wise' is not the same as 'Y not-man'. Therefore 'wise' is not the same as 'Y not-(rational animal)'.[84]

The proof of *Barbara* rests on the formalization of the universal affirmative proposition according to CONT 4. Thus 'Every C is B' is represented by '$C = BX$', i.e. more explicitly $\exists X(C = BX)$; similarly 'Every D is C' is represented by the corresponding formula $[\exists Y](D = CY)$; now substitution of BX for C in the latter equation yields $[\exists Y \exists X](D = BXY)$ which can easily be transformed into $\exists Z(D = BZ)$, i.e. 'Every D is B'. The latter inference, though not mentioned explicitly in the above quoted passage, had been stated, e.g., in the **GI** as follows:

> (19) [...] So when $A = BY$ and $B = CZ, A = CYZ$; or, A contains C.
> (20) It must be noted [...] that one letter can be put for any number of letters together: e.g. $YZ = X$. (**P**, 56/7).

---

[84]Cf. C., 301:

"*Barbara*:	Omne C est B.	Omne D est C.	Ergo Omne D est B.
	$C = BX$.	$D = CY$.	Ergo $D = BXY$.
Celarent:	Nullum C est B.	Omne D est C.	Ergo Null. D est B.
	$C = X$ non-B.	$D = CY$.	Ergo $D = YX$ Non-B.
Darii:	Omne C est B.	Qu. D est C.	Ergo Qu. D est B.
	$C = BX$.	D non $= Y$non$C$.	Ergo D non $= Y$non BX.

Sed hinc non sequitur: D non $= YX$non$B$ quod desideratur. Unde est alia adhuc in tali calculo difficultas. Exemplum sumamus: Omnis homo est animal. Quidam sapiens est homo. E. quidam sapiens est animal. Secundum calculum: Homo idem est quod animal rationale; sapiens non idem est quod Y non homo. Ergo sapiens non idem est quod Y non animal-rationale".

Next *Celarent* is proved in quite the same way as *Barbara* by making use of the traditional principle of obversion according to which the universal negative proposition (U.N.) 'No C is B' is equivalent to a U.A. with the negated predicate 'Every C is not-B'. Hence $C\overline{B}$, i.e., according to CONT 4, $\exists X(C = \overline{B}X)$, plus the second premiss $[\exists Y](D = CY)$ yields by substitution $[\exists Y \exists X](D = \overline{B}XY)$, which may be simplified to $\exists Z(D = \overline{B}Z)$, i.e. 'Every D is $\overline{B}$' or 'No D is B'.

However, during his attempt to give a similar proof for *Darii* Leibniz faces another difficulty in his calculus [**C**. 301: "Unde est aliqua adhuc in tali calculo difficultas"] which is due, among others, to the fact that in '$D \neq Y$ not-C' the indefinite concept Y functions as a *universal* quantifier. The difficulty can be analyzed as follows. From 'Every C is B', i.e. $[\exists X](C = BX)$, plus 'Some D is C' which, as the negation of $D \in \overline{C}$, would have to be formalized explicitly as $\neg \exists Y(D = Y\overline{C})$, or $\forall Y(D \neq Y\overline{C})$, one obtains by way of substitution $\forall Y(D \neq \overline{BX})$. Leibniz formalizes this elliptically as $D \neq Y\overline{BX}$) and does not see how one might get from this the desired conclusion $D \neq YX\overline{B}$. As a matter of fact, the inference from $\forall Y(D \neq Y\overline{C})$ and $\exists X(C = BX)$ to $\forall Z(D \neq Z\overline{B})$ is not at all obvious, in particular for someone like Leibniz who never developed any laws that would allow him to transform a negated conjunction like $\overline{BX}$ into, say, a disjunction of $\overline{B}$ and $\overline{X}$. However, Leibniz might have solved this difficulty by observing that according to the law of contraposition, NEG 3, the premiss $C \in B$ entails $\overline{B} \in \overline{C}$, i.e. by CONT 4 $\exists X(\overline{B} = X\overline{C})$. Using *this* equation, $\forall Y(D \neq Y\overline{C})$ is easily shown to entail $\forall Z(D \neq Z\overline{B})$, because if there would exist some Z such that $D = Z\overline{B}$, the substitution $\overline{B} = X\overline{C}$ would yield $D = ZX\overline{C}$ which contradicts the premiss $\forall Y(D \neq Y\overline{C})$.

In view of the other difficulties that Leibniz encountered during his attempt to prove the syllogistic laws in "De Formae Logicae Comprobatione", it may be understandable that he did not fully realize the difference between the use of indefinite concepts functioning as existential and as universal quantifiers, respectively. In other fragments, however, he became more or less aware of this distinction . Thus in a somewhat confused passage of §112 **GI**[85] he said:

> It must be seen whether, when it is said that AY is B (i.e. that some A is B), Y is not taken in some other sense than when it is denied that any A is B, in such a way that not only is it denied that some A is B — i.e. that this indeterminate A is B — but also that any A out of a number of indeterminates is B, so that when it is said that no A is B, the sense is that it is denied that

---

[85]In order to avoid confusion with our formalization of conceptual negation, the symbol $\overline{Y}$ which Leibniz here uses for the "universal" indeterminate concept was replaced by '$\hat{Y}$'. Cf. also §§80–82 GI where Leibniz similarly uses two different symbols for indefinite concepts.

$A\hat{Y}$ is B; for $\hat{Y}$ is Y, i.e. any Y will contain this Y. So when I say that some A is B, I say that this some [hoc quoddam] A is B; if I deny that some A is B, or that this some A is B, I seem only to state a particular negative. But when I deny that any A is B, i.e. that not only this, but also this and this A is B, then I deny that $\hat{Y}$ is B. ($\mathbf{P}$, 72).

While the P.A. shall be formalized, according to Leibniz, by '$AY \in B$' with Y functioning as an *existential* quantifier, its negation shall not be represented as $AY \notin B$, but rather by means of a new symbol $\hat{Y}$ as $A\hat{Y} \notin B$, where this new type of indefinite concept $\hat{Y}$ denotes "any Y" [quodcunque Y] and thus represents a *universal* quantifier. To put it less elliptically: whereas 'Some A is B' may be formalized in $L2$ as $[\exists Y](AY \in B)^{86}$, the negation takes the form $[\forall \hat{Y}](A\hat{Y} \notin B)$ in accordance with the well-known law

(Univ 1) $\neg \exists Y \alpha[Y] \leftrightarrow \forall Y \neg \alpha[Y]$,

or its special instance

(Univ 1.1) $\neg \exists Y (AY \in B) \leftrightarrow \forall Y (AY \notin B)$.

In view of this explanation, Leibniz's incidental remark "$\hat{Y}$ is Y, i.e. any Y will contain this Y" [$\hat{Y}$ est Y, seu quodcunque Y continebit hoc Y] expresses another important law of the logic of quantifiers, namely: Each proposition of the form $\alpha[\hat{Y}]$ entails the corresponding proposition $\alpha[Y]$, or less elliptically:

(Univ 2) $\forall Y \alpha[Y] \rightarrow \exists Y \alpha[Y]$.

This principle was anticipated also in fragment $\mathbf{C}$. 270–3 where Leibniz had similarly used two types of indefinite concepts, Y and $\tilde{Y}$:[87]

Let us see in which way Y and $\hat{Y}$ differ from each other, namely like 'something' and 'whatsoever' but this happens by accident, and I want it to be Y simpliciter. This must be examined more carefully.[88]

Unfortunately, Leibniz never carried out the closer examination of this topic. Nevertheless it should be clear that Y as 'something' represents the

---

[86] More exactly, in view of the trivial law $AB \in B$, the P.A. should be formalized by $\exists Y (\mathbf{P}(AY) \wedge AY \in B)$ — cf. the discussion of principles Neg 6* and Neg 6 in section 7.1; this complication can, however, be ignored here.

[87] Here for typographical reasons 'X' has been replaced by 'D,Y' because my word processor only generates '$\tilde{Y}$' but not Leibniz's sign composed of an 'X' and '$\sim$'.

[88] Cf. C., 271: "Videndum quomodo Y et ? differant, scilicet ut aliquod et quodcunque sed id contingit per accidens, et velim qui sit Y simpliciter. Haec melius examinanda".

existential quantifier $\exists Y$ while $\tilde{Y}$ as 'whatsoever' corresponds to the universal quantifier $\forall Y$, and the remark that $\tilde{Y}$ should be "Y simpliciter" means that a universal proposition of the type $\forall \tilde{Y} \alpha[\tilde{Y}]$ *entails* the corresponding existential proposition $\exists Y \alpha[Y]$.

There are various other logical laws where Leibniz used indefinite concepts as universal quantifiers. Thus in **C.** 259–61 he formulates: "(15) A is B is the same as to say: If L is A, it follows L is also B" [A est B, idem est ac dicere si L est A sequitur quod et L est B]. Couturat [1901, p. 347, fn 2] thought that this principle would represent only a variant of the "principe du syllogisme", i.e. the law of transitivity of the $\in$-relation. But this interpretation is incompatible with the fact that Cont 2 has the form $A \in B \wedge L \in A \rightarrow L \in B$, or, equivalently, $A \in B \rightarrow (L \in A \rightarrow L \in B)$, where the first implication must never be strengthened into a biconditional. Furthermore Leibniz's explanation "L is to be understood as any term of which 'L is A' can be said" [Intelligitur autem L quicunque terminus de quo dici potest L est A] makes clear that here L is not a definite but an indefinite concept, i.e. a *variable* functioning as a universal quantifier. Therefore the principle has to be formalized more explicitly as follows:

(Univ 3) $(A \in B) \leftrightarrow \forall L(L \in A \rightarrow L \in B)$.

Leibniz's proof contains an anticipation of the contemporary rules for eliminating and introducing universal quantifiers:

> Let us assume the proposition 'A is B'. I say that it entails 'If L is A, it follows that L is B', which I prove as follows: Since A is B, hence $A = AB[\dots]$. But if L is A, then $L = LA$. Whereby (substituting for A the value AB) one obtains $L = LAB$. Therefore L is AB, hence L is B $[\dots]$. Now let us conversely prove that 'If L is A, it follos that L is B' entails 'A is B'. L however is to be understood as any term of which 'L is A' can be said. So assume the one $[\forall L(L \in A \rightarrow L \in B)]$ to be true and yet the other $[A \in B]$ to be false. $[\dots]$ Therefore the following proposition will be stated: QA is non-B. $[\dots]$ But QA is A. Therefore QA is B (*because QA is subsumed under* L). Hence QA is B non-B what is absurd.[89]

---

[89]Cf. **C.** 260: "Assumamus hanc propositionem A *est* B. dico hinc inferri si L est A, sequitur quod L est B. Hoc ita demonstro: Quia A est B, ergo $A \infty AB$ [...]. Jam si L est A, erit $L \infty LA$. Ubi (pro A substituendo valorem AB) fit $L \infty LAB$. Ergo L est AB. Ergo L est B [...]. Nunc inverse demonstremus, ex hac: Si L est A sequitur quod L est B, vicisssim inferri A est B. Intelligitur autem L quicunque terminus de quo dici potest L est A. Ponamus illud $[\forall L(L \in A \rightarrow L \in B)]$ esse verum, et tamen hoc $[A \in B]$ esse falsum. [...] Statuatur ergo haec enuntiatio: QA est non B. [...] Jam QA est A. Ergo QA est B (*quia QA comprehenditur sub* L) Ergo QA est B non B quod est abs." (my emphasis).

In the first part Leibniz derives $[\forall L](L \in A \to L \in B)$ from the premiss $A \in B$ by showing that, for any L, $L \in A$ (in conjunction with $A \in B$) entails $L \in B$. This follows the basic idea of the rule of $\forall$-introduction according to which $\forall Y \alpha[Y]$ may be established by showing that, for any arbitrary constant A, $\alpha[A]$. In the second part Leibniz proves indirectly that $A \notin B$ is incompatible with the premiss $[\forall L](L \in A \to L \in B)$, because if $A \in B$ was false, then according to Neg 6 there would exist some Q such that $QA \in \overline{B}$ (and $\mathbf{P}(QA)$); now, trivially, according to Conj 3 $QA \in A$; thus $[\forall L](L \in A \to L \in B)$ would allow us to conclude that $QA \in B$ ("because QA is subsumed under [the variable] L"); hence (by Conj 1) we would obtain $QA \in B\overline{B}$ which is "absurd" or, more correctly, which contradicts $\mathbf{P}(QA)$. This kind of proof follows the basic idea of $\forall$-elimination according to which $\forall Y \alpha[Y]$ entails, for any arbitrary constant A, $\alpha[A]$.

Another interesting law implicitly containing a universal quantifier may be found in a marginal note to §18 **GI**, where Leibniz first notes that AC=ABD does not generally entail $C = BD$; and where he adds that the following special case of this inference is valid:

> For it to be inferred from $AC = ABD$ that $C = BD$, it must be presupposed that nothing which is contained in A is contained in C unless it is also contained in BD, and conversely. (**P**, 56, Note 2).

If, for the sake of simplicity, we substitute 'E' for 'BD', this principle says that $AC = AE$ entails $C = E$ provided that each concept Y which is contained in A will be contained in C if and only if it is also contained in $E : \forall Y(A \in Y \to (B \in Y \leftrightarrow C \in Y)) \to (AB = AC \to B = C)$. Some further laws are discussed in [Lenzen, 1984a].

8   THE "QUANTIFICATION OF THE PREDICATE"

Leibniz's theory of "Quantification of the predicate" (TQP, for short) was developed mainly in the fragment "Mathesis rationis" which had first been edited in 1903 by Couturat (**C**, 193–206; cf. **P**, 95–104).[90] However, Couturat published not much more than the final version of the essay (sheets 1 and 2 of the manuscript **LH** IV, 6, 14),[91] while a preliminary draft and some related studies (sheets 3–5) were edited only in a very abridged form (cf. **C**, 203–206). Even the main text is far from complete since, among others, three important paragraphs that Leibniz decided to omit[92] did not

---

[90] The most important logical works are abbreviated as follows: *Comprobatione* = "De formae logicae comprobatione per linearum ductus" (**C**, 292–321); *Dissertatio* = *Dissertatio de Arte Combinatoria* (**A** VI, 1, 168–230).

[91] The classification of Leibniz's manuscripts (**LH**) follows the catalogue of E. Bodemann (**LH**).

[92] Cf. **LH** IV, 6, 14, 1 recto: "Omitti possunt 48, 49, 50".

find entrance into Couturat's edition. As will be shown below, the additional material of these §§provides the key for a proper understanding of §24 which - together with the related §§3-6 — forms the core of the whole essay.

Perhaps due to the lack of a complete and critical text, the real meaning of this fragment seems not to have been recognized so far. Most scholars agreed to Couturat's verdict that Leibniz sketched TQP, *only in order to refute it*.[93] Couturat [1901, p. 24] maintained this view although he was aware of the fact that Leibniz had stressed at several places the importance of TQP for a "foundation of all rules of the figures and moods of syllogistic theory". Couturat thought it necessary to close an apparent gap in Leibniz's syllogistic studies by providing a "Précis of classical logic" which basically consisted in a derivation of the theory of the syllogism from TQP. However, a closer analysis of the *Mathesis* reveals that Leibniz was in no need of such help since he not only developed TQP all by himself but also used it in much the same way as Couturat as a tool for deriving the basic laws of the syllogism.

8.1 Theory of the syllogism and universal calculus

Leibniz's great aim in logic was to construct a general calculus of concept logic that would enable him to strictly verify the traditional theory of the syllogism. It is not easy to chronologize this enterprise but the following can be claimed with some degree of certainty. On the one hand, Leibniz dealt with issues in the traditional theory of the syllogism practically throughout his (adult) life, namely from 1665 when he composed the *Dissertatio* until 1715 when the "Schedae de novis formis et figuris syllogisticis" (**C**, 206–210) were written. The various drafts of a general calculus, on the other hand, date from a much shorter period between 1680 and 1690, approximately. The validation of the theory of the syllogism by means of the "Calculus universalis" involves two tasks which can be referred to as 'soundness' and 'completeness', respectively. The *proof of soundness* amounts to showing that both the simple inferences of subalternation, opposition, and conversion and the 24 moods that were generally regarded as valid[94] can be derived as theorems of $L1$ or $L2$. If, as usual, **A**, **E**, **I**, and **O** symbolize the

---

[93]Cf. **C**, 194, fn.1: "Ici Leibniz conçoit nettement la quantification du prédicat, et la rejette."

[94]In many places Leibniz defended the view that there are exactly 6 valid moods in each of the 4 figures. He put forward this claim already in the *Dissertatio* (**A** VI, 1, 184: "Ita ignota hactenus figurarum harmonia detegitur, singulae enim modis sunt aequales"), but one may doubt whether at that time he was entitled to do so. On the one hand the table of the valid moods contained a 25th syllogism named Frisesmo which "[. . .] ex regulis modorum non sit inutilis" (**A** VI, 1, 185/6). On the other hand Leibniz mistakenly listed a syllogism Colanto among the valid moods of the IVth figure while in fact it had to be replaced by Calerent.

categorical forms of a universal affirmative, universal negative, particular affirmative, and particular negative proposition, the simple consequences may be formalized as follows:

(OPP 1) $\neg \mathbf{A}(B,C) \leftrightarrow \mathbf{O}(B,C)$

(OPP 2) $\neg E(B,C) \leftrightarrow \mathbf{I}(B,C)$

(SUB 1) $\mathbf{A}(B,C) \rightarrow \mathbf{I}(B,C)$

(SUB 2) $\mathbf{E}(B,C) \rightarrow \mathbf{O}(B,C)$

(CONV 1) $\mathbf{E}(B,C) \leftrightarrow \mathbf{E}(C,B)$

(CONV 2) $\mathbf{E}(B,C) \rightarrow \mathbf{O}(C,B)$

(CONV 3) $\mathbf{A}(B,C) \rightarrow \mathbf{I}(C,B)$

(CONV 4) $\mathbf{I}(B,C) \leftrightarrow \mathbf{I}(C,B)$.

The perfect moods of the Ist figure accordingly take the shape:

(BARBARA) $\mathbf{A}(C,D) \wedge \mathbf{A}(B,C) \rightarrow \mathbf{A}(B,D)$

(CELARENT) $\mathbf{E}(C,D) \wedge \mathbf{A}(B,C) \rightarrow \mathbf{E}(B,D)$

(DARII) $\mathbf{A}(C,D) \wedge \mathbf{I}(B,C) \rightarrow \mathbf{I}(B,D)$

(FERIO) $\mathbf{E}(C,D) \wedge \mathbf{I}(B,C) \rightarrow \mathbf{O}(B,D)$.

Actually, the proof of soundness could be simplified to demonstrating these 4 moods only plus the laws of opposition. For Leibniz had shown in "De formis syllogismorum Mathematice definiendis" (**C**, 410–416) that:

1. the laws of subalternation, SUB 1, 2, follow from DARII and FERIO;

2. by means of SUB 1 and 2 the remaining two moods of the Ist figure, BARBARI and CELARO, can be proved;

3. the moods of figures II and III can be reduced to those of the Ist by means of a primitive inference called '*regressus*'; and

4. the laws of conversion can be derived from moods of the IInd and IIIrd figure.

Finally in *Mathesis* Leibniz also proved that

5. the moods of the IVth figure follow from the previous ones by means of the rules of conversion.[95]

---

[95]Cf. **LH** IV, 6, 14, 3 recto - 3 verso. Another proof of the IVth figure is given in **C**, 209.

Hence {BARBARA, CELARENT, DARII, FERIO, OPP 1,2} constitutes an axiomatic basis of the theory of the syllogism.

Leibniz who already in 1679 had developed a *semantical* method for validating these principles by means of characteristic numbers[96] started a series of *syntactic* derivations in *Comprobatione* which was probably written around 1686. At that time, however, the various attempts to derive the basic principles of the theory of syllogism from the "universal calculus" remained without success. As was shown in Lenzen [1988], it was not before 1690 that Leibniz found a satisfactory proof of the soundness of syllogistic theory[97]. The *proof of completeness*, on the other hand, should have

- to demonstrate the traditional canon of general rules including the so-called rules of quantity and quality;

- to derive from them some more specific rules for the single figures; and

- to show that the latter suffice to invalidate all but those syllogisms already proven to be sound.

Before investigating how Leibniz tackled this threefold task in *Mathesis*, let us take a closer look at the traditional version of this syllogistic doctrine as described, e.g., in the famous *Port-Royal Logic*.

8.2   Axioms and rules of traditional syllogistics

The first axiom of Arnauld/Nicole [1683] is nothing but the above mentioned law of subalternation. Three further axioms contain the *theory of quantity and quality*, that is:

(QUAN)   The subject of a universal proposition is universal. The subject of a particular proposition is particular.

(QUAL)   The predicate of an affirmative proposition is particular. The predicate of a negative proposition is universal.

These axioms are said to be the basis for the subsequent *general rules of the syllogism*, although Arnauld/Nicole fail to show how the latter might be derived from the former.

---

[96]Cf. the series of essays of April 1679 (**C.** 42-92 + 245-247) where Leibniz maintains "Ex hoc calculo omnes modi et figurae derivari possunt per solas regulas Numerorum" (**C.** 247). For a possible extension of Leibniz's method to a language containing negation cf. [Sotirov, 1999].

[97]Cf. the marginal note: "Hic demonstrantur Modi primae figurae, et regulae oppositionum. Quarum ope (ut alibi jam ostendimus) demonstrantur deinde conversiones et modi reliquarum figurarum." (**C**, 229).

(GR 1)   The middle term may not be particular in both premisses.

(GR 2)   If a term is universal in the conclusion then it must also be universal in the premiss.

(GR 3)   At least one of the premisses must be affirmative.

(GR 4)   If the conclusion is negative, one of the premisses also has to be negative.

Next: "The conclusion always follows the weaker part, i.e. if one of the two propositions is negative, the conclusion must be negative, and if one is particular, it must be particular"[98]. It will be convenient to split this rule up into

(GR 5.1)   If one of the premisses is particular, then the conclusion must be particular;

(GR 5.2)   If one of the premisses is negative, then the conclusion must be negative.

Finally one has:

(GR 6)   At least one of the premisses must be universal.

These general rules in turn are supposed to entail the following *special rules* for the single figures, although, again, Arnauld/Nicole fail to indicate how the latter might be obtained from the former. The *first figure* is defined by the fact that the middle term, C, is the subject in the minor-premiss, i.e. the premiss containing the minor-term, B, while C is the predicate in the major-premiss (which contains the major-term D). Here the following restrictions obtain:

(SR I.1)   In the first figure the minor-premiss must be affirmative

(SR I.2)   In the first figure the major-premiss must be universal.

In the *second figure*, which is defined by having the middle term both times as a predicate, the corresponding restrictions run as follows:

(SR II.1)   In the second figure one of the premisses must be negative

(SR II.2)   In the second figure the major-premiss must be universal.

---

[98]Cf. Arnauld/Nicole [1683, p. 186]: "La conclusion suit toûjours la plus foible partie, c'est-à-dire, que s'il y a une des deux propositions negatives, elle doit être negative; & s'il y en a une particuliere, elle doit être particuliere".

The *third figure* is characterized by having the middle term both times as subject. Here the following conditions apply:

(SR III.1)   In the third figure the minor-premiss must be affirmative

(SR III.2)   In the third figure the conclusion must be particular.

Finally, with regard to the *fourth figure* where the middle term is predicate in the major-premiss and subject in the minor-premiss, [Arnauld and Nicole, 1683, p. 200] mention three conditions:

> If the major is affirmative, the minor is always universal [...] If the minor is affirmative, the conclusion is always particular [...] In all negative moods the major must be general.

In view of the general rules GR 4 and GR 5.2, a mood is negative if and only if it has a negative conclusion. Hence we can paraphrase the above conditional restrictions as follows:

(SR IV.1)   In the fourth figure, if the major-premiss is affirmative, the minor-premiss must be universal

(SR IV.2)   In the fourth figure, if the minor-premiss is affirmative, the conclusion must be particular

(SR IV.3)   In the fourth figure, if the conclusion is negative, the major-premiss must be universal.

8.3   *Leibniz's early attempts at a proof of completeness*

Leibniz appears to have been acquainted with this traditional doctrine already as a youth. In the *Dissertatio* he does not state the axioms QUAN and QUAL, though, but he mentions in passing the general rules GR 2, 3, 5, 6[99], and he also formulates the special rules in a very condensed way[100]. Only Leibniz's conditions for the IVth figure differ quite considerably from the traditional restrictions: "In the IVth the conclusion is never a UA. The major never PN. And if the minor is N, the major is UA".[101] In *Comprobatione*, probably written 2 decades after the *Dissertatio*, Leibniz gives a riper version of the laws of the syllogism, and he makes some first steps towards a proof of completeness. First he mentions (although he does not prove yet) the proper rules of quantity and quality when he points out that

---

[99]"Ex puris particularibus nihil sequitur [...] Conclusio nullam ex praemissis quantitate vincit [...] Ex puris negativis nihil sequitur [...] Conclusio sequitur partem in qualitate deteriorem" (**A** VI, 1, 181).

[100]Cf. **A** VI, 1, 184: "Imae autem et 2dae figurae semper major propositio est U[niversalis ...] Imae et IIItiae semper minor A[ffirmativa ...] In IIda semper Conclusio N[egativa ...] In IIItia Conclusio semper est P[articularis]".

[101]Cf. **A** VI, 1, 184: "In IV^{ta} Conclusio nunquam est UA. Major nunquam PN. Et si Minor N, Major UA".

A *distributed term* is the same as a total or universal one; a non-*distributed* is one which is particular or partial. The subject has the same quantity as the proposition. [...] But the predicate in each affirmative proposition is partial or non-distributed, and in each negative proposition it is total or distributed.[102]

Second he is now able to demonstrate the validity of the general rules (omitting only GR 4) as follows. As regards GR 1:

The middle [term] must be distributed or total in at least one of the premisses, otherwise no coincidence can be established; if something of the minor term coincides or fails to coincide with a part of the middle term, and something of the major term in turn coincides or fails to coincide with a part of the middle term, different parts of the middle term might be concerned.[103]

Similarly, we read with respect to GR 2:

[...] it can generally be said that a term cannot be more ample in the conclusion than it is in the premisses, otherwise that which would not enter into the logical consideration, namely that part of the term which is not concerned in the premisses, would enter into the conclusion [...]. And this is what is ordinarily stated as '*A term which is not distributed* [...] *in a premiss cannot be distributed in the conclusion*.[104]

Concerning GR 3 Leibniz explains:

It is also evident that nothing can be inferred from merely negative propositions. For if you only exclude that which is in an extreme [minor or major] term from that which is in the middle [term] you cannot infer any coincidence, indeed you cannot even infer the exclusion of that what is in one of the extremes from that which is in the other.[105]

---

[102]Cf. **C.**, 312: "*Terminus distributivus* est idem qui totalis seu universalis; *non distributus*, qui particularis seu partialis. Subjectum est ejusdem quantitatis cujus propositio. [...] Sed praedicatum in omni propositione affirmativa est partiale seu non distributum, et in omni propositione negativa est totale seu distributum".

[103]Cf. **C.**, 317: "*Medius debet esse in alterutra praemissarum distributus seu totalis*; alioqui nulla potest effici coincidentia, si minoris termini aliquid parti medii coincidit aut non coincidit, et majoris termini aliquid rursus parti medii coincidit aut non coincidit, diversae partes medii affici poterunt".

[104]Cf. **C.**, 316: "[...] generaliter dici potest terminum non posse [esse] ampliorem in conclusione quam in praemissa, alioqui id quod non venisset in ratiocinationem, ea nempe pars termini, quae in praemissa non afficitur, veniret in conclusionem [...] Atque hoc est quod vulgo dicitur *Terminum non distributum [...] in praemissa nec posse esse distributum in conclusione*".

[105]Cf. **C.**, 318: "Manifestum etiam est ex meris negativis propositionibus nil sequi.

The proof of the remaining rules GR 5, 6 is somewhat less satisfactory because Leibniz restricts it to the case of affirmative propositions noting that "all negative syllogisms can be transformed into affirmative ones by changing a negative [proposition] into an affirmative with an indefinite [i.e. negative predicate]".[106]

The special rules for the single figures, however, are not derived very systematically by Leibniz. He just mentions some restrictions that happen to come to his mind as immediate consequences of the general rules. Thus, as a corollary of GR 1, he notes: "Therefore in the figures [?] where the middle term is always the predicate [i.e., only in the IInd figure] the conclusion must be negative" [Hinc in figuris ubi medius terminus semper est praedicatum conclusio debet esse negativa], i.e. SR II.1, and "where [the middle term] always is the subject [i.e., in the IIIrd figure], the conclusion must be particular" [ubi semper est subjectum conclusio debet esse particularis], i.e. SR III.2. Furthermore Leibniz infers from GR 2 some *conditional* restrictions which, however, are much weaker than the traditional rules.[107] Finally, Leibniz promises to derive further rules for the Ist and IVth figure once GR 6 and GR 5 were proven, but he fails to make this announcement true.

8.4   *Proving the special rules*

By the time of the *Mathesis,* probably around 1705[108], Leibniz has gained a clear knowledge of the logical foundations of the general rules. In what I consider as a preliminary version of the essay, he gives the following summary of the "fundaments of all theorems of the figures and the moods":

> (1) The middle term must be universal in at least one premiss [...]
>
> (2) At least one premiss must be affirmative [...]
>
> (3) A particular term in a premiss is also particular in the conclusion [...]

Nam sola exclusio ejus quod est in termino extremo ab eo quod est in medio non infert utique ullam coincidentiam, sed ne quidem inferre potest exclusionem ejus quod in uno extremo ab eo quod est in alio extremo."

[106]Cf. **C**, 319: "omnes syllogismos negativos posse mutari in affirmativos, ex negativa faciendo affirmativam indefiniti [praedicati]".

[107]Cf., e.g., **C**, 316: "[...] si conclusio est universalis, Minorem propositionem esse universalem in figuris ubi terminus minor est praemissae suae subjectum, scilicet prima et secunda". This condition and three similar ones reappear in *Mathesis* as §§34 - 36.

[108]According to a communication of Prof. Schepers from the Leibniz-Forschungsstelle Münster, the water-sign of the manuscript indicates that *Mathesis* was written at about that time. The present investigation also suggests that *Mathesis* is a rather late fragment, at any rate later than *Comprobatione* because the TQP-version of the categorical forms given there (cf. **C**, 311) is clearly inferior to the one presented in *Mathesis.*

(4) If one premiss is negative, also the conclusion is negative [...]

(5) The subject of a universal proposition is universal, that of a particular is particular [...]

(6) Because of the logical form, the predicate of an affirmative proposition is particular, that of a negative is universal.

From these [six] fundamentals all theorems concerning the figures and moods can be proved."[109]

It is not without interest to note that Leibniz sees no need to distinguish the traditional axioms QUAL and QUAN from the theorems GR 1–6; he rather considers them all alike as fundamentals. Actually, the above list contains only a part of the traditional rules, *viz.* GR 1, 2, 3, and 5.1. Leibniz evidently forgot to state also GR 4, but in the final version of *Mathesis* he recognizes this slip when he inserts into his formulation of GR 5.2 "Nor is it less evident that if one of the premisses is negative, the conclusion also must be negative"[110] the remark "and vice versa". In contrast, the fact that also GR 5.1 and GR 6 no longer range among the fundamentals should not be taken as another slip of Leibniz but rather as the result of his insight that both principles follow from the remaining ones. Corresponding proofs are provided in §§32 and 33 of the main text.

In an admirably clear and strictly deductive way Leibniz shows in §§37, 38, 39, 42, 43 that the fundamental principles (in conjunction with the definition of the figures as stated in §22) entail the following special rules for the first 3 figures:

- SR II.1: "[...] in the second figure, the conclusion must be negative";

- SR II.2: "In the same figure, the major proposition is always universal";

- SR III.2: "[...] in the third figure, the conclusion must be particular";

- SR III.1 + SR I.1: "In the first and the third figure the minor proposition is affirmative";

---

[109] Cf. **LH** IV, 6, 14, 4 verso: "(1) Medius terminus debet esse universalis in alterutra praemissa [...]

(2) Alterutra praemissa debet esse affirmativa [...]

(3) Terminus particularis in praemissa est particularis in conclusione [...]

(4) Si una praemissa sit negativa, etiam conclusio est negativa [...]

(5) Subjectum propositionis universalis est universale, particularis particulare

(6) Praedicatum propositionis affirmativae vi formae est particulare, negativae universale.

Ex his [sex] fundamentis omnia Theoremata de Figuris et modis demonstrari possunt."

[110] Cf. **C.**, 196: "Nec minus manifestum est, una praemissa existente negativa, etiam conclusionem esse negativam".

- SR I.2: "In the first figure, the major proposition is universal."

Moreover, the number of special rules for the IVth figure also can be reduced to two. The former SR IV.1 is stated in §46 as follows: "In the fourth figure, the minor proposition is not particular at the same time as the major proposition is affirmative"; and instead of SR IV.2 + IV.3 Leibniz now formulates: "In the fourth figure, the major proposition is not particular at the same time as the minor proposition is negative." (§45). Hence Leibniz who in general was fond of symmetries and harmonies happily concludes: "Any figure, therefore, has two limitations" (§47).

A careful analysis of the Leibnitian proof of the special rules reveals that each of the six fundamentals (and no other principle) is used as a premiss. As will be shown in section 8.6 below, the special rules in turn are necessary and sufficient to carry out the final step in the proof of completeness by proving "[...] that there are not more [than the 24 valid moods], and this must be done, not by an enumeration of illegitimate moods, but from the laws of those which are legitimate" (**P**, 104). First, however, we will have to describe Leibniz's version of TQP which is the basis for the first step of the completeness proof, *viz.* for validating the six fundamentals.

8.5   *The Quantification of the Predicate*

In order to discuss Leibniz's TPQ let us consider, e.g., the universal affirmative proposition:

> (3) When I say 'Every A is B', I understand that any of those which are called A is the same as some one of those which are called B.

What kind of entitites are the informal quantifier-expressions 'any' and 'some' assumed to refer to, and how is the relation of 'being called' A (or B) to be understood? For a contemporary logician it may be most natural to interpret the quantifiers as referring to *individuals* which are elements of the *set* A (or individuals to which the *predicate* A applies). In this case one arrives at the following version of TQP. The universal affirmative proposition 'Every A is B' will be paraphrased as: 'Every individual x which is an element of A is identical with some individual y which is an element of B'. Since the symbol '$\in$' is here used to designate the containment relation between concepts, we now better chose another symbol, say ε, for expressing the set-theoretical relation between a certain object x and a set A. Furthermore, in distinction to Leibniz's quantifiers, $\forall$ and $\exists$, ranging over *concepts*, let us introduce another pair of quantifiers, Λ and V, which range over *objects*. Leibniz's extensional characterization of the U.A. then takes the following form:

(UA 1) $\Lambda x(x\varepsilon A \rightarrow V y(y\varepsilon B \wedge y = x))$.

The particular affirmative proposition 'Some A are B' in the sense of "(4) [...] some one of those which are called A [are] the same as some one of those which are called B" accordingly can be formalized as follows:

(PA 1) $Vx(x\varepsilon A \wedge Vy(y\varepsilon B \wedge y = x))$.

The universal negative proposition, 'No A are B', in the sense of "(5) [...] any one of those which are called A is different from any one of those which are called B" amounts to:

(UN 1) $\Lambda x(x\varepsilon A \rightarrow \Lambda y(y\varepsilon B \rightarrow y \neq x))$.

Finally, the particular negative proposition, 'Some A are not B', in the sense of "(6) [...] some one of those which are called A [are] different from any one of those which are called B" can be rendered as:

(PN 1) $Vx(x\varepsilon A \wedge \Lambda y(y\varepsilon B \rightarrow y \neq x))$.

Under the present interpretation the additional propositions mentioned in §7 make a clear sense, although they are "superfluous" [inutile] and "not in accordance with our linguistic usage" [non est in usu in nostris linguis]. To say that "every A is every B" means that "all those which are called A are the same as all those which are called B" (**P**, 95; cf. **C**., 193: "omnes qui dicuntur A esse eosdem cum omnibus qui dicuntur B"). This can be formalized as follows:

(NC 1) $\Lambda x(x\varepsilon A \rightarrow \Lambda y(y\varepsilon B \rightarrow y = x))$.

But this will never be the case unless the sets A and B are singletons which contain exactly one and the same element.

In the same way the corresponding proposition "Some As are the same as all Bs" (**P**., 95, cf. **C**., 194: "quosdam A esse eosdem cum omnibus B") has to be formalised as

(NC 2) $Vx(x\varepsilon A \wedge \Lambda y(y\varepsilon B \rightarrow y = x))$.

Again this can't be true unless the set B is a singleton.[111]

The other two propositions which Leibniz obtained by negating NC 1 and NC 2: "[...] any one of those which are called A is different from some one of those which are called B" and "[...] some one of those which are called A is different from some one of those which are called B" (**P**., 95), i.e.

---

[111] Note, incidentally, that Leibniz commits a fallacy when he says that NC 2 might equivalently be expressed by saying "Omnes B esse A". According to UA 1, the latter amounts to the condition $\Lambda x(x\varepsilon B \rightarrow Vy(y\varepsilon A \wedge y = x))$. However, one may not at all interchange the two quantifiers within that formula.

(NC 3) $\Lambda x(x \varepsilon A \to V y(y \varepsilon B \wedge y \neq x))$

(NC 4) $V x(x \varepsilon A \wedge V y(y \varepsilon B \wedge y \neq x))$,

will in general be tautological statements the truth of which is self-evident
["per se patet"] unless, again, "B is unique" (**P.**, 95, cf. **C**. 194: "nisi B sit
unicum").

It strikes me as somewhat incomprehensible that not only Couturat but
also modern commentators regarded this as a *rejection* of TQP[112]. Even if
Leibniz's remarks about the artificiality ("non est in usu in nostris linguis")
and the redundancy ("inutilis") of the non-categorical propositions NC 1–4
(which exhaust all possibilities of a quantification of the predicate) might
be interpreted as a rejection of this particular *part* of TQP, still it could
hardly be denied that Leibniz advocated the other, more relevant part of
TQP which relates to the categorical forms UA 1, PA 1, UN 1, and
PN 1. Furthermore, it cannot be overlooked that Leibniz took this very
(semi)-formalization of the categorical forms as a conclusive proof of the
traditional rules of quantity and quality:

> (9) So [. . .] it is evident that every affirmative proposition (and
> only such a proposition) has a particular predicate, by art. 3 et
> 4.,
>
> (10) and that every negative proposition (and only such a propo-
> sition) has a universal predicate, by art. 5 et 6.
>
> (11) Further, the proposition itself is called 'universal' or 'partic-
> ular' by virtue of the universality or particularity of its subject.
> (**P**, 96)

As a matter of fact, these counterparts of Qual and Quan follow imme-
diately from the quantification both of the subject and of the predicate as
illustrated in UA 1, PA 1, UN 1, and PN 1, provided that the terms A,
B are taken to be universal or particular just in case they are modified by
a universal or by a particular (i.e., existential) quantifier.

Before discussing a second version of TQP presented in §§24, 48–50, let
me briefly touch upon Leibniz's proofs of the remaining fundamentals. They
basically follow the lines of the corresponding demonstrations in *Comproba-*

---

[112]Parkinson remarked in the same vein as Couturat that: "[. . .] Leibniz conceives
the idea of the quantification of the predicate, only to reject it." (**P**, liii). [Kauppi,
1960, p. 199] says that "[. . .] die Quantifikation des Prädikats [wird] als unnötig verwor-
fen". Burkhardt [1980, p. 44] shares Couturat's opinion that "[Leibniz hatte] die Quan-
tifizierung des Prädikates [. . .] noch im arithmetischen Kalkül von 1679 abgelehnt". He
correctly recognizes, however, that in §24 "Leibniz noch ein Zeichensystem zur Darstel-
lung der vier kategorischen Satzformen entwickelt [hat], mit dessen Hilfe es möglich ist,
Subjekt und Prädikat zu quantifizieren" (o.c., 45).

tione. Thus Leibniz immediately infers the fundamental principles GR 3, GR 4 + GR 5.2 from the logical laws for identity stated in §§12 and 13[113]:

> (15) It is at once inferred from this that a syllogism cannot be made out of two negative propositions; for in this way it would be stated that L is different from M, and that M is different from N. [**P**, 96 ...]

> (21) It is none the less evident that if one premiss is negative, the conclusion also is negative, and conversely; for the reasoning used here is just the same as that whose principle was stated in article 13 [...] (**P**, 97).

The proof of the other fundamentals GR 1, 2 resorts in addition to the following definition of a categorical syllogism:

> (12) What are called, simple categorical syllogisms' elicit a third proposition from two others [...]

> (16) It is also evident that in the simple categorical syllogism there are three terms, as we are using some third term, and while we compare this equally with the one and the other of the extremes we are seeking a method of comparing these extremes with each other. (**P**, 96)

This third term, the medius, must be universal in at least one premiss, as Leibniz argues in §:

> (19) [...] For [...] if the middle term in each premiss is particular, it is not certain that the contents of the middle term which are used in one premiss are the same as the contents of the middle term which are used in the other premiss, and therefore nothing can be inferred from this about the identity and difference of the extremes. (**P**, 97)

And in the subsequent §he shows that if a term is particular in a premiss, it will also be particular in the conclusion:

> (20) It can also be seen easily that a particular term in the premiss does not imply a universal term in the conclusion, for it is not known to be the same or different in the conclusion unless it is known that it is the same as or different from the middle term in the premiss.

---

[113] "[...] thus if L is the same as M and M is the same as N, L and N are the same"; "[...] Thus, if L is the same as M, and M is different from N, L and N are also different."

8.6 The $\Psi B \Psi D$-formalism

Another version of the TQP is developed in §24 which is difficult to read in several places since the text is written in very small letters on the margin. The main differences between the text-critical edition given in [Lenzen, 1990b] and the previous edition in **C** (or in **P**) are the following. Leibniz inserted the last sentence of §24 'propositionis quaecunque [...]' on top of the sentence 'S significabit [...]'. That's why a certain word which Couturat somewhat diffidently interpreted as 'unurarem' seemed to belong to the former sentence while in fact it reads as 'terminum' and belongs to the latter sentence. Accordingly, the passage:

> S signifies the universal, P the particular, V, Y, Ψ the indetermined. [cf. **C**., 196: "S significabit universalem, P particularem, V, Y, Ψ incertam"]

has to be corrected to "S significabit *terminum* universalem, P particularem, V, Y, Ψ *incertum*." This is quite important since it conclusively establishes that the symbols 'S' and 'P' characterize the universality and particularity of a *term* and not, as, e.g., Parkinson assumed[114], the corresponding property of a *proposition*. Accordingly 'Ψ' symbolizes that it is undetermined whether the subsequent term is universal or particular; it does not, however, as Burkhardt [1980, p. 47] has maintained, constitute itself an indefinite term. The resulting formalisation of the categorical forms is read by Couturat as

> Therefore the sign SBSD is the universal negative proposition, SBPD the universal saffirmative. IBSD the particular negative. IBID the particular affirmative.

> Signum itaque SBSD est propositio universalis negativa. SBPD universalis affirmativa. IBSD particularis negativa. IBID, particularis affirmativa. (**C**., 196)

The opening word, however, actually belongs to the preceding sentence: "The quantity of the proposition will be designated by the universal sign of the subject, the quality [of the proposition] by the sign of the predicate". [Propositionis quantitas designabitur per subjecti signum universale, qualitas per praedicati signum]. Furthermore, the text of the manuscript does not necessarily speak in favor of a letter 'I' within the formulae 'IBSD' and 'IBID', but allows one to read this letter instead as a very slim 'P' where what at first glance to be a point above 'I' really is a tiny crook of a 'P'. That Leibniz at any rate *meant to write* 'P' instead of 'I' is evident from the deleted §§48 where one can read very clearly:

---

[114]Cf. **P**, 98: "S will stand for a universal, P for a particular, V, Y, Ψ for an indefinite proposition".

> If we do not take care about what are the premisses, the terms
> will be F, G, and similar ones. In general the universal proposi-
> tion $SF\Psi G$, the particular proposition $PF\Psi G$, the affirmative
> proposition ΨFPG, the negative proposition ΨFSG. In partic-
> ular, the universal affirmative proposition $SFPG$, the particular
> affirmative $PFPG$, the universal negative $SFSG$, the particular
> negative PFSG.[115]

This unambiguous statement also confirms that the concluding sentence
of § 24 ends with the words "is generally expressed by $\Psi F\Psi G$" [generaliter
exprimitur $\Psi F\Psi G$] and not, as **C** has it, with "generaliter exprimitur un-
urarem $\Psi F.\Psi S$."[116]

Let us now consider in which way Leibniz used this symbolism to complete
his proof of completeness. In §45 he proved the special rule IV.1 indirectly
as follows. If one would have at the same time that "the major proposition is
not particular [. . . and] the minor proposition is negative", one could argue:

> [. . .]   Let the particular major proposition in this figure (by
> 24) be $PD\Psi C$, and the negative minor proposition be [ΨCSB];
> then the negative conclusion will be $PBSD$. But this is absurd,
> since (art. 20 [i.e. GR 2]) there cannot be PD in the major
> proposition and SD in the conclusion. (**P**, 103)[117]

In §46 it is similarly shown that:

> [. . .]   the minor proposition is not particular at the same time
> as the major proposition is affirmative. For suppose that they
> are: then the major proposition will be ΨDPC, and the minor
> proposition $PC\Psi B$. But in this way the middle term, C, is
> particular in each, which is contrary to art. 19 [i.e. contrary to
> GR 1]. (**P**, 103/104).

Systematically much more important, however, is the sketch of a proof that
Leibniz gives at the very end of *Mathesis* to show that there are not more
valid moods than the 24 ones proven elsewhere:

---

[115] Cf. **LH** IV, 6, 14, 2v., margin: "(48) [. . .] Ubi nullus respectus ad praemissas, ter-
mini erunt F, G, vel tales. In genere propositio universalis SFΨG propositio particularis
$PF\Psi G$ propositio Affirmativa ΨFPG propositio negativa ΨFSG. In specie Universalis
Affirmativa $SFPG$, Particularis affirmativa $PFPG$, Universalis negativa SFSG, particu-
laris negativa $PFSG$."

[116] Even more misleading is the interpretation of this formula by Parkinson who suggests
"$\Psi P.\Psi S$" — cf. **P**, 98, fn. 1.

[117] Couturat pointed out in **C**, 202, fn. 1 and 2, that the formula for the negative minor-
premiss has to be ΨCSB instead of Leibniz's $SC\Psi B$, and that the "in minore" of the
manuscript must be read as "in conclusione". Leibniz's third inaccuracy of symbolizing
the "conclusio negativa" as $PBSD$ instead of ΨBSD is harmless, since under the given
premisses the conclusion also has to be particular, hence PBSD.

"It must be maintained that there are no more moods, and this must be done, not by an enumeration of illegitimate moods, but from the laws of those which are legitimate. For example, in the first figure the premisses $SC.\Psi D, \Psi B.PD$ give:

			A Barbara	1
	SBPD	AA		
SCPD			I Barbari	2
	PBPD	AI	I Darii	3
			E Celarent	4
	SBPD	EA		
SCSD			O Celaro	5
	PBPD	EI	O Ferio	6". (**P**, 104).

In its present form, however, this schema is incomplete and incorrect. As was stated in §22, the position of the terms in the Ist figure is: "Fig.1. $CD.BC.BD$." The special rule I.1, according to which the minor-premiss is affirmative, therefore has to be formalized as 'ΨBPC', whereas Leibniz erroneously has 'ΨBPD' which would symbolize an affirmative *conclusion*. Hence only the following combination of premisses (obtained by substituting 'S' and 'P' successively in the place of 'Ψ') is legitimate:

	SBPC
SCPD	
	PBPC
	SBPC
SCSD	
	PBPC.

In the first two cases, in view of GR 4, the conclusion must itself be affirmative: ΨBPD; moreover, in the second subcase it has to be particular according to GR 3: PBPD. In the last two cases, in contrast, the conclusion has to be negative on account of GR 4: ΨBSD; in the second subcase, again, it also must be particular: PBSD. Hence Leibniz's schema for the only valid moods of the Ist figure has to be modified as follows:

		SBPD	BARBARA	1
	SBPC			
SCPD		PBPD	BARBARI	2
	PBPC	PBPD	DARII	3
		SBSD	CELARENT	4
	SBPC			
SCSD		PBSD	CELARO	5
	PBPC	PBSD	FERIO	6

As was shown at length in Lenzen [1990b], this formal method of eliminating the invalid moods "ex legibus legitimorum" can be applied to the other

figures as well. To round off the present discussion of the *Mathesis*, I want to delineate in the following section in which respect the $\Psi B \Psi C$-formalism may be considered as a second version of TQP.

8.7 Formalisations of the Categorical forms [118]

The most immediate way of expressing the universal affirmative proposition within the general calculus of a logic of concepts is simply to drop the informal quantifier-expression 'Every' in 'Every A is B', thus obtaining the formula 'A is B', or symbolically

(UA 2) $A \in B$.

According to CONT 3 and CONT 4 this formula can be reduced to one of the following identities:

(UA 3) $A = AB$

(UA 4) $\exists Y (A = BY)$.

Now in "A paper on 'some logical difficulties" (**P**, 115–121) Leibniz recognized that the UA can equivalently be expressed by the generalized statement that every A is B in the sense of $\forall X (XA \in B)$. Somewhat more exactly, Leibniz first defined the following formal criterion for the universality or non-universality, i.e. particularity, of a term B (within a certain proposition):

> In general we can tell if a term [. . .] B is universal if [. . .] YB can be substituted for [. . .] B, where Y can be anything which is compatible with B (**P**, 119).

Next he went on to prove that the term A is in fact universal within the proposition $A \in B$ by pointing out: "In the universal affirmative, $AB = A$, therefore [for every Y] $YAB = YA$".[119] Hence $A \in B$ entails $\forall Y (AY \in B)$. On the other hand, $\forall Y (AY \in B)$ entails, for arbitrary concepts Y, that $AY \in B$, especially for $Y = A : AA \in B$, i.e., because of the trival law CONJ 4, $A \in B$. Hence one obtains the further formalisation

(UA 5) $\forall X (XA \in B)$.

---

[118] Leibniz made enormous efforts to formalize the single categorical forms within his system(s) of concept logic, and he worked with enumerable "homogeneous" and inhomogeneous combinations of these formulas, not all of which turned out to be correct and useful. Here only the most important homogeneous schemata shall be considered. For more details cf. [Lenzen, 1988].

[119] **P** 119. In the same passage Leibniz also proves all the remaining theorems of quantity and quality.

The remaining '$\in$' can either be eliminated, as Leibniz did in the quoted passage, by means of CONT 3, or by means of CONT 4. In the latter case one obtains the following representation with *two* quantifiers:

(UA 6) $\forall X \exists Y (XA = YB)$.

The *PA* 'Some A is B', on the other hand, was formalized by Leibniz among others as 'XA est B' where the indefinite concept X now plays the rôle of an *existential* quantifier:

(PA 2) $\exists X (XA \in B)$.[120]

Eliminating, again, the '$\in$' by means of CONT 4, one obtains the doubly-quantified version

(PA 3) $\exists X \exists Y (XA = YB)$,

which Leibniz expressed somewhat elliptically as: "the *particular affirmative* Some C is B can be expressed thus: $XB = YC$" [cf. **C.**, 302: "*particularis affirmativa* Qu. C est B sic exprimetur: $XB = YC$"]

In view of the laws of opposition, the universal negative proposition can accordingly be formalized as: "No C is B, i.e. $XC \neq YB$" [cf. **C.**, 303: "Nullum C est B id est XC non $= YB$"], where both indefinite concepts X, Y now function as *universal* quantifiers:

(UN 2) $\forall X \forall Y (XA \neq YB)$.

Finally, for the particular negative proposition one obtains as the negation of UA 6:

(PN 2) $\exists X \forall Y (XA \neq YB)$.

Putting these formal representations together into the schema:

(UA) $\forall X \exists Y (XA = YB)$ $\forall X \forall Y (XA \neq YB)$ (UN)
(PA) $\exists X \exists Y (XA = YB)$ $\exists X \forall Y (XA \neq YB)$ (PN)

one obtains *the real meaning of the $\Psi B \Psi C$-formalism*. All that has to be observed is that the original version of § 24:

(UA)   SA PB   SA SB   (UN)
(PA)   PA PB   PA SB   (PN)

implicitly contained corresponding '$=$' and '$\neq$'-symbols as Leibniz explained in the deleted §49:

---

[120]Cf., e.g., **GI**, §48: "AY contains B is [the] particular affirmative". However, in view of the trivial law CONJ 2 there always exists at least one Y such that $AY \in B$. Therefore Leibniz's formalisation of the P.A. should be modified by requiring that Y is compatible with A. Corresponding remarks apply to the subsequent formulas PA 3 and UN 2.

> We can also reduce everything by means of the calculus to identi-
> ties and non-identitites. [. . .] thus if I want to express a negative
> proposition [. . .] ΨFSG, it will be ΨF non $= SG$.[121]

Hence the intended meaning of the above schema is better formalized as
follows:

(UA)	SA = PB	SA $\neq$ SB	(UN)
(PA)	PA = PB	PA $\neq$ SB	(PN)

Here the "sign" [signum] S has to be interpreted as an indefinite con-
cept governed by a universal quantifer while P accordingly represents an
indefinite concept governed by a particular (or existential) quantifier.

8.8 Conclusion

To conclude, I want to show that the first, "extensional" version of TQP
discussed in section 8.6 is provably equivalent to the second, "intensional"
version elaborated in the preceding section, where this equivalence can be
established by means of principles of a genuinely Leibnizian logic. For rea-
sons of space, however, I can here only sketch how the two version of, e.g.,
the UA can be derived from each other. A more detailed account may be
found in [Lenzen, 1990b].

In section 8.7 several laws of $L2$ were quoted to show that the "inten-
sional" UA with quantified subject and quantified predicate, $\forall X \exists Y(XA =
YB)$, is equivalent to the simple formalization of the "*Affirmative Proposi-
tion A is B or A contains B*" (**GI**, §16). Now, as Leibniz observed in **C**,
260, the UA can also be expressed as a *universal conditional*: "A is B, is
the same as to say If L is A, it follows that L is B" [A est B, idem est ac
dicere si L est A, sequitur quod et L est B]. Hence another formalisation
of the UA is:

(UA 7) $\forall X(X \in A \rightarrow X \in B)$.

Next observe that Leibniz developed several logical criteria for a concept
A being a *complete* concept (of an individual substance) or, for short, *an
individual concept*, e.g.:

> [..] if two propositions *with exactly the singular subject* are pre-
> sented such that one of them has one of two contradictory terms
> as predicate while the other proposition has the other term as

---

[121] Cf. **LH** IV, 6, 14, 2v.: "Possumus etiam reducere omnia ad principium identitatis
et diversitatis per calculum. [. . .] ut si velim exprimere propositionem negativam fiet
ΨFSG, erit ΨF non $= SG$".

predicate, then necessarily one proposition is true and the other false".[122]

This can be formalized as follows:

(DEF 12) $\mathrm{Ind}(A) \leftrightarrow_{\mathrm{df}} \forall X (A \in \overline{X} \leftrightarrow A \notin X)$.

With the help of this definition, one can introduce new quantifiers ranging over individual(concept)s:

(DEF 13) $\Lambda X\alpha \leftrightarrow_{\mathrm{df}} \forall X(\mathrm{Ind}(X) \to \alpha)$

(DEF 14) $V X\alpha \leftrightarrow_{\mathrm{df}} \exists X(\mathrm{Ind}(X) \wedge \alpha)$.

These quantifiers allow us to represent the UA, alternatively to UA 7, also as

(UA 8) $\Lambda X(X \in A \to X \in B)$.

This formula captures the meaning of Leibniz's example:

> The universal affirmative proposition *Every b is c* can be reduced to this hypothetical proposition *If a is b, a will be c*, e.g.: *Every man is an animal*, i.e. *If someone is a man (b), he (a, or Titus) is c (animal)*.[123]

The last but one step in the proof of the equivalence between the "extensional" and the "intensional" approach consists in the trivial law according to which the condition $V y(y = x \wedge \alpha)$ is only a complicated version of $\alpha[x]$. Hence UA 1 may be simplified to

(UA 9) $\Lambda x(x\varepsilon A \to x\varepsilon B)$.

Now, the intension and the extension of a concept A in general are linked together by the so-called law of reciprocity which also applies to *individual*-concepts. As captured in DEF.1, their intension is maximal. The extension of an individual-concept, therefore, will be minimal, which means that it consists of exactly *one* (possible) individual only. In this sense individuals may properly be called the lowest species "whose name cannot be restricted to fewer"[124], or in other words: "The absolutely lowest species is the individuum" [Cf. **A** VI, 4, 32: "Species absoluta infima est *individuum*"].

---

[122]**LH** IV, 5, 8d, 17 verso; cf. **C**, 67: "[...] si duae exhibeantur propositiones *ejusdem praecise subjecti singularis* quarum unius unus terminorum contradictoriorum, alterius alter sit praedicatum, tunc necessario unam propositionem esse veram et alteram falsam". A discussion of this important passage may be found in [Lenzen, 1986], esp. pp. 23–24.

[123]Cf. **A** VI, 4, 126: "Propositio Universalis affirmativa *Omne b est c* reduci potest ad hanc hypotheticam *Si a est b, a erit c*, verbi gratia: *Omnis homo est animal* id est, *Si quis est homo (b) is (a vel Titius) est c (animal)*".

[124]Cf. **A** VI, 4, 31:"[...] cuius nomen ad pauciora restringi non potest"

To sum up: the individual concept X contains the concept A: '$X \in A$', iff X's extension, i.e. the unit-set $\{x\}$ containing exactly the individual x, is contained in the extension of A, i.e. iff x itself has the property A or is a member of the set of all As: $x \varepsilon A!$[125] In this sense the "extensional" formalisation UA 9 coincides with the "intensional" version UA 8.

9   POSSIBLE INDIVIDUALS AND POSSIBLE WORLDS

Since the publication of Russell [1900], a lot of books and articles have been written about Leibniz's logic on the one hand and about his metaphysics on the other. Most Leibniz scholars followed Russell in recognizing the intimate relationship between these two areas of Leibniz's philosophy. After all, Leibniz himself had repeatedly pointed out the close connection between his metaphysical and his logical ideas. Thus in a famous letter to Duchess Sophie he declared that "[. . .]the true Metaphysics is hardly different from the true Logic" (**GP** 4, 292). However, modern commentators consider this statement as an absolutely unfounded exaggeration. They are conficdent that Leibniz's logic of concepts is much too weak to serve as a basis either for defining the central notions of his ontology or even for deriving certain metaphysical propositions which Leibniz had referred to as "logical" propositions. Thus in their standard exposition of *The Development of Logic*, W. and M. Kneale [1962, p. 337] summarize their evaluation of Leibniz's logical achievements as follows:

> When he began, he intended, no doubt, to produce something wider than traditional logic. [. . .] But although he worked on the subject in 1679, in 168[6], and in 1690, he never succeeded in producing a calculus which covered even the whole theory of the syllogism.

If this were correct, then it would be absurd to expect that any interesting element of Leibniz's "true metaphysics" might be derived from his "true logic". In particular, it would be silly to believe that the core of Leibniz's proof of the existence of God, namely the statement "If the necessary being is possible, then it exists" might turn out as a *logical* truth. But this is at any rate what Leibniz himself claimed to be the case when he characterized this statement as "a modal proposition, perhaps one of the best fruits of the entire logic".[126]

---

[125] As the formalizations UA 8 and UA 9 make clear, there is always a logical relation between the individual(-concept) x (or X) and the general concept A whether the latter is taken extensionally as a set or intensionally as an idea. Modern predicate logic, however, misleadingly veils this relation behind the functional brackets of '$A(x)$'. For a more detailed discussion of this point cf. [Lenzen, 1989b].

[126] Cf. **GP** 4, 406: "On pourrait encore faire à ce sujet une proposition modale qui seroit un des meilleurs fruits de toute la logique, scavoir que *si L'Estre necessaire est*

Hopefully the present exposition has convincingly shown that 20th century scepticism concerning the strength of traditional logic in general and concerning Leibniz's achievements in particular is rather unfounded. Anyway, in Lenzen [1990a] a self-consistent reconstruction of the "Universal Calculus" has been provided which actually allows one to derive the quoted thesis about the existence of the necessary being as a logical theorem! For reasons of space I will here confine myself to giving a logical reconstruction of the main elements of Leibnitian *ontology*, to wit the notions of a possible individual and of a possible world. Let us begin by considering §§71–72 **GI** where Leibniz presents his views on *existence* and on *individuals*:

> (71) What is to be said about the proposition '*A* is an existent' or '*A* exists'? Thus, if I say about an existing thing, '*A* is *B*', it is the same as if I were to say '*AB* is an existent'; e.g. 'Peter is a denier', i.e. 'Peter denying is an existent'. The question here is how one is to proceed in analysing this; i.e. whether the term 'Peter denying' involves existence, or whether 'Peter existent' involves denial — or whether 'Peter' involves both existence and denial, as if you were to say 'Peter is an actual denier', i.e. is an existent denier; which is certainly true. Undoubtedly, one must speak in this way; and this is the difference between an individual or complete term and another. For if I say 'Some man is a denier', 'man' does not contain 'denial', as it is an incomplete term, nor does 'man' contain all that can be said of that of which it can itself be said.

> (72) So if we have *BY*, and the indefinite term *Y* is superfluous (i.e., in the way that 'a certain Alexander the Great' and 'Alexander the Great' are the same), then *B* is an individual. If there is a term *BA* and *B* is an individual, *A* will be superfluous; or if *BA* = *C*, then *B* = *C*.

First we have to clarify the central notions 'existing', 'individual', and 'individual-term'. Leibniz has often been blamed for not carefully distinguishing between terms and their denotations. The quoted passage certainly justifies such a criticism, but Leibniz's rather careless use of the word 'individual' to refer alternatively either to individual-terms or to individuals does not give rise to serious misunderstandings. One may assume that there is a 1-to-1-correspondence between individuals and individual-terms, and the context makes perfectly clear what Leibniz is talking about. What has to be kept in mind, however, is that an individual-term for Leibniz nevertheless is a concept, i.e. an "intensional" entity which may contain (or be contained in) other concepts. Hence its extension must be conceived of as a *subset* — and not as an *element* — of the universe of discourse. E.g., the extension

---

possible, il existe."

of the individual-concept 'Peter' is not the individual Peter but the *unit-set* containing exactly that individual.

As regards the notion of existence, Leibniz is treating it on a par with the other concepts by forming corresponding conjunctions 'Petrus existens', 'ab-negans existens' which enter into the fundamental relation of containment, '∈'. Therefore 'existens' may be abbreviated by a distinguished concept letter, say $E*$, which has to be interpreted extensionally, like any other concept letter, as a certain subset of the universe of discourse.[127]

Now, generalizing from the above examples, Leibniz is maintaining that whenever A is the complete term of an existing individual, then the statement 'A is B' is equivalent both to i) 'AB is Existing' and to ii) 'A Existing is B', and also to iii) 'A is Existing B'. These principles may easily be shown to be theorems of the algebra of concepts regardless of whether the subject-term A is a "normal" concept or an individual-concept. What, then, had Leibniz in mind when he went on to explain: "Undoubtedly, one must speak in this way; and this is the difference between an individual or complete term and another."

At first sight the answer may be surprising. The difference between an individual concept and an ordinary one is that the proposition 'A exists' or 'A is existing' may only in the former but not in the latter case be regarded as a relation of conceptual containment and hence be formalized as '$A \in E*$'. Why this is the case will be explained below in connection with §§144–150 **GI**. First, however, I want to deal with some other criteria for distinguishing individual-concepts from ordinary concepts.

A first difference is vaguely outlined by Leibniz's remark that from the truth of the particular proposition 'Some man is a denier' it does not follow that the universal proposition 'Every man is a denier' or, for short 'Man is denier' be true as well: "'man' does not contain 'denial'". Here one evidently has to add the unspoken claim that the corresponding inference from a particular to a universal proposition does hold if the subject term is an individual-concept. This stands in close connection with the parenthetical remark of §72: "'a certain Alexander the Great' and 'Alexander the Great' are the same", and also with the following passage from "A paper on 'some logical difficulties'":

> How is it that opposition is valid in the case of singular propo-
> sitions - e.g. 'The Apostle Peter is a soldier' and 'The Apostle
> Peter is not a soldier' - since elsewhere a universal affirmative
> and a particular negative are opposed? Should we say that a sin-
> gular proposition is equivalent to a particular and to a universal

---

[127] According to Leibniz, the extreme cases that this set is either empty or universal should be excluded. For he not only believed it to be "impossible that nothing exists" (**A** VI 4, 17), but he also held the view that not all of the possible individuals are compossible and that therefore some individuals will not be created by God but will remain mere possibles.

> proposition? Yes, we should. [...] For 'some Apostle Peter'
> and 'every Apostle Peter' coincide, since the term is singular.
> (**P** 115; cf. **GP** 7, 214)

Let us see how this claim, which has been dubbed by Englebretsen [1988] the
"Wild Quantity Thesis", can be verified within Leibniz's calculus. Observe,
first, that the UA 'Every A is B', i.e. $A \in B$, can in general (for arbitrary
subject-terms A) be represented, in $L2$, in the form of $\forall Y(YA \in B)$.[128]
In the case of a singular proposition - i.e. a proposition with an individual
term such as 'Apostle Peter' as subject - this means that, e.g., Apostle Peter
is a denier if and only if every Apostle Peter is a denier, or, in short, that
the subject term 'Apostle Peter' is equivalent to the universally quantified
term 'every Apostle Peter'. Thus the first part the "Wild Quantity Thesis"
is already verified.

As regards the second part, observe that according to NEG 6 the particu-
lar affirmative proposition 'Some A is B', i.e. the negation of the UN $A \in \overline{B}$,
can in general be formalized as $\exists Y(\mathbf{P}(AY) \wedge AY \in B)$. Now if the subject-
term A is an individual concept — formally $\mathrm{Ind}(A)$ — then the predication
'A is B' turns out to be *equivalent* to the formula $\exists Y(\mathbf{P}(AY) \wedge AY \in B)$:

(IND 1) $\mathrm{Ind}(A) \to (A \in B \leftrightarrow \exists Y(\mathbf{P}(AY) \wedge AY \in B))$.

In other words: the singular predication 'A is B' is tantamount to the
particular proposition 'Some A are B', or — in our previous example —
Apostle Peter is a denier iff some Apostle Peter is a denier.

The validity of IND 1 is based on the completeness-condition for individ-
ual concepts which Leibniz mentions in the concluding sentence of §72 **GI**.
There he calls a concept A "superfluous" (with respect to concept B) iff (for
every C) $BA = C$ entails that $B = C$. This condition may be simplified
by just requiring that A is already contained in B.[129] Now, when Leibniz
goes on to maintain "If there is a term BA and B is an individual, A will
be superfluous" (**P**., 65, fn. 1), he seems to maintain that any term A is
superfluous with respect to any individual term B. But this is absurd since
otherwise an individual-concept B would be "completely complete" in the
sense of containing every concept A, in particular besides A also Non-A,
and hence B would be inconsistent.

To resolve this difficulty, observe that Leibniz begins the sentence in ques-
tion by saying "Si sit terminus BA" which Parkinson translated as "If there

---

[128]On the one hand, $\forall Y(YA \in B)$ immediately entails $AA \in B$ and thus, because of
the trivial law $AA = A$ also $A \in B$; conversely $YA \in B$ follows, for arbitrary Y, from the
premiss $A \in B$ and from the trivial conjunction law $YA \in A$ by means of the transitivity
of '$\in$'.

[129]For, on the one hand, substituting 'BA' for 'C' yields that $BA = BA$ entails $B = BA$;
conversely, if $BA = B$ then (for any C) $BA = C$ entails that $B = C$. Hence A is
superfluous with respect to B just in case that $B = BA$, i.e. $B \in A$.

is a term BA". In other contexts, this translation surely would be appropriate to express the sense of a mere stipulation: "Let there be a term BA [...]". In the present context, however, Leibniz meant to say: "Let the term BA be", i.e. let BA be a consistent term, or, let us suppose that $\mathbf{P}(BA)$! There are several passages within and without the **GI** where Leibniz paraphrases the condition of self-consistency of a concept A just by saying 'A is'. Therefore the interpretation of "Si sit terminus BA" as meaning 'Let BA be a possible term' is very plausible, and it entails the necessary condition: B is an individual-concept only if — unlike other concepts — B is complete in the precise sense of already containing any concept A with which it is compatible (i.e. for which $\mathbf{P}(BA)$ holds). Since A here stands for any arbitrary concept, it may be replaced by an indefinite concept Y and then be bound by a universal quantifier:

(IND 2) $\mathrm{Ind}(B) \to \forall Y (\mathbf{P}(BY) \to B \in Y)$.

That this is what Leibniz had in mind is evidenced by the fact that the converse implication

(IND 3) $\forall Y (\mathbf{P}(BY) \to B \in Y) \to \mathrm{Ind}(B)$

is recognized by him as a sufficient condition for B to be an individual-concept when he says: "So if BY is [possible], and the arbitrary indefinite term Y is superfluous, then B is an individual". We thus obtain the following Leibnizian definition of individual-concepts:

(IND 4) $\mathrm{Ind}(A) \leftrightarrow \mathbf{P}(A) \wedge \forall Y (\mathbf{P}(AY) \to A \in Y)$,

where the trivial condition $\mathbf{P}(A)$ not mentioned by Leibniz has been added. This definition is semantically adequate and it enables us to prove the open part of the "Wild Quantity Thesis", IND 1, as follows: If $\mathrm{Ind}(A)$ and $A \in B$, then, trivially, $AA \in B$ and $\mathbf{P}(AA)$, from which $\exists Y (\mathbf{P}(AY) \wedge AY \in B)$ follows by existential generalization; conversely, let there be some Y such that $\mathbf{P}(AY) \wedge AY \in B$; since A is presupposed to be an individual-concept, $\mathbf{P}(AY)$ according to IND 4 implies that $A \in Y$, i.e. $A = AY$, so that $AY \in B$ yields the desired $A \in B$.

So far I have been concerned with the truth-conditions for attributing existence to individuals. Let us now consider §§144–150 **GI** where Leibniz investigates the truth-conditions for corresponding non-singular categorical propositions.

> (144) Propositions are either essential or existential, and both are either *secundi adjecti* or *tertii adjecti*. [...] An existential proposition *tertii adjecti* is 'Every man exists liable to sin'. [...] An existential proposition *secundi adjecti* is 'A man liable to sin exists, i.e. is actually an entity' ["existit seu est ens actu"].

(145) From every proposition *tertii adjecti* a proposition *secundi adjecti* can be made, if the predicate is compounded with the subject into one term and this is said to [be or to] exist ["esse vel existere"], i.e. is said to be a thing, whether in any way whatsoever, or actually existing ["esse res sive utcunque, sive actu existens"].

(146) The particular affirmative proposition, 'Some A is B', transformed into a proposition *secundi adjecti*, will be 'AB exists' ["AB est"], i.e. 'AB is a thing' - either possible or actual ["AB est res, nempe vel possibilis vel actualis"], depending on whether the proposition is essential or existential. [...]

(148) The particular negative proposition, 'Some A is not B', will be transformed into a proposition *secundi adjecti* as follows: 'A, not-B exists'. That is, A which is not B is a certain thing - possible or actual, depending on whether the proposition is essential or existential.

(149) The universal negative is transformed into a proposition *secundi adjecti* by the negation of the particular affirmative. So, for example, 'No A is B', i.e. 'AB does not exist' ["AB non est"], i.e. 'AB is not a thing' [...]

(150) The universal affirmative is transformed into a proposition *secundi adjecti* by the negation of the particular negative, so that 'Every A is B' is the same as 'A not-B does not exist, i.e. is not a thing' ["A non B non est, seu non est res"] (**P**, 80-81; cf. **C**., 392).

These ideas may be summarized and formalized in the following diagram:

Categorical form	Formalization "secundi adjecti"	Formalization "tertii adjecti"
U.A. "Every A is B"	$\neg P(A\overline{B})$	$A \in B$
U.N. "No A is B"	$\neg P(AB)$	$A \in \overline{B}$
P.A. "Some A is B"	$P(AB)$	$A \notin \overline{B}$
P.N. "Some A is not B"	$P(A, \overline{B})$	$A \notin B$

Figure 1. "Essential" propositions

Leibniz's thesis of the reducibility of the categorical forms *tertii adjecti* to propositions *secundi adjecti* amounts to the claim that the corresponding

formulae are provably equivalent. This, however, easily follows from our former axiom Poss 1.

Let us now turn to the "existential" interpretation of the categorical forms. Just as the truth of the "essential" P.A. "AB is a possible [...] thing" according to our semantics requires that there is at least one possible individual $x \in U$ such that x is an AB, i.e. x is both an A and a B, so the stronger "existential" P.A. "AB is an actual [...] thing" should be considered as true if and only if there is an actually existing individual x which is both an A and a B. How can this be expressed, however, within the logic of concepts? The answer to this question may be found in an untitled fragment where Leibniz is wondering whether:

> [...] the way of transforming logical propositions into terms by adding just 'ens' or 'non ens' also works in the case of existential propositions. [...] For example: 'Some pious is poor', i.e. 'Pious poor is existing'. [...] Let us see whether 'existing' can also be moved into the term so that only 'ens' or 'non Ens' remains. Such that 'Pious poor is existing' yields 'Pious poor existing is Ens'. (cf. **C**, 271).

Generalizing from this example, an "existential" P.A. "AB is existing" shall be reduced to a proposition *secundi adjecti* by maintaining that the conjunction ABE(xistens), or $ABE*$, is "Ens", i.e. is self-consistent: $\mathbf{P}(ABE*)$! Similarly, an "existential" P.N. "A Not-B is existing" will have to be represented by $P(A\overline{B}E*)$, as Leibniz illustrates when he transforms "Some pious [man] is not poor, i.e. 'Pious not poor' is existing" ["quidam pius non est pauper, seu Pius non pauper est existens"] into "'Pious not poor existing' is Ens or possible" ["Pius existens non pauper est Ens seu possible", *ibid.*]. Since "existential" versions of the universal propositions can be obtained by negating P.A. and P.N., respectively, one arrives at the following schema:

Categorical form	Formalization "secundi adjecti"	Formalization "tertii adjecti"
U.A.* "Every existing A is B"	$\neg P(A\overline{B}E*)$	$A \in *B$
U.N. "No existing A is B"	$\neg P(ABE*)$	$A \in *\overline{B}$
P.A. " Some existing A is B"	$P(ABE*)$	$A \notin *\overline{B}$
P.N. "Some existing A isn't B"	$P(A, \overline{B}E*)$	$A \notin *B$

Figure 2. "Essential" propositions

Here, of course, the new operator of existential containment, $\in *$, must be interpreted extensionally as saying that each actually existing individual which falls under concept A also falls under concept B. This operator might be defined in terms of ordinary containment plus the concept of existence as follows:

(DEF 15) $A \in *B \leftrightarrow AE* \in B$.

To conclude the discussion of §§144–150 GI, let me explain and prove the former claim that only in the case of individual concepts A, the statement 'A exists' may be represented by the formula '$A \in E*$'. If A is an ordinary concept, say that of a horse, then a statement of the form '$A \in B$' always has to be understood as a *universal* affirmative proposition saying that every individual which is an A also is a B, say, every horse is an animal. Hence substituting the concept '$E*$' in the place of the predicate 'B' one obtains that 'Horse $\in$ E(xistence)*' is true if and only if *every* horse actually exists. Existential propositions of the type 'Horses exist', however, only maintain that *some* horses exist. Hence, where A is a normal concept, 'As exist' will have to be represented in $L2$ by the formula '$A \notin \overline{E*}$' which expresses an particular affirmative proposition.

Now, as was shown in connection with the "Wild quantity thesis", the completeness of an individual-concept A entails that the particular proposition 'Some A are B' becomes equivalent to the universal proposition 'Every A is B'. Therefore the existence of an *individual* may well be expressed also in the form of the simple attribution '$A \in E*$'.

So far, only a very small portion of Leibnitian ontology has been dealt with. Let me conclude by sketching in bare outlines how a more complete logical reconstruction of Leibniz's metaphysics would have to proceed[130]. First, quantification over individuals should be modelled by restricting the quantifiers to *individual concepts* as in DEF 13, 14. With the help of these quantifiers, the following axiom can be formulated which reflects the basic idea of the set-theoretical semantics underlying concept-logic, namely the idea that a concept is possible if and only if, within the realm of all possible individuals, it has a non-empty extension:

(POSS 5) $\mathbf{P}(A) \leftrightarrow VX(X \in A)$.

The second step towards a logical reconstruction of Leibnizian ontology requires the introduction of the modal *propositional operators* of possibility and necessity. This involves a generalization of our former extensional semantics in the usual way: i.e. one has to take into account of a non-empty set W of possible worlds; relativize the truth of each propositions to the elements of W; and let the modalized propositions $\Diamond\alpha$ and $\Box\alpha$ be true if

---

[130]Cf. Lenzen [1991; 1992].

and only if the unmodalized proposition α is true in at least one/or in every world w.

Third, the former concept of actual existence, $E*$, has to be generalized or relativized in such a way that in every possible world w it refers to the set of all individuals which belong to w and in this sense "exist in w". Then the crucial relation of compossibility among individuals can be defined to obtain if and only if X and Y will co-exist in some possible world, i.e. if they possibly coexist:

(DEF 16) $\Lambda X \Lambda Y (\mathrm{Comp}(X, Y) \leftrightarrow_{df} \Diamond (X \in E \wedge Y \in E))$.

Fourth, *possible worlds* will be constructed as maximal sets of compossible individuals in roughly the following way:

(DEF 17) $W(A) \leftrightarrow_{df} \Lambda X (X \in A \leftrightarrow \Lambda Y (Y \in A \rightarrow \mathrm{Comp}(Y, X)))$.

Finally the *actual world* $w*$ may be singled out from the set of all possible worlds by the fact that it contains the greatest number of elements. Then our former notion of (actual) existence, $E*$, may be regarded as the extension of the world-bound concept of existence, E, in the real world $w*$. This chain of logical moves seems to stand behind Leibniz's insight that 'existens'

> can be defined as 'that which is compatible with more things than anything else which is incompatible with it'. (**P** 51).

BIBLIOGRAPHY

Editions of Leibniz's works and manuscripts

A Akademie-Ausgabe, i.e.: German Academy of Science, ed., G. W. Leibniz, *Sämtliche Schriften und Briefe*, esp. SeriesVI *Philosophische Schriften*, Darmstadt 1930, Berlin 1962 ff.

C Louis Couturat (ed.), *Opuscules et fragments inédits de Leibniz*, Paris 1903, reprint Hildesheim (Olms) 1960.

GI *Generales Inquisitiones de Analysi Notionum et Veritatum – Allgemeine Untersuchungen über die Analyse der Begriffe und Wahrheiten*, ed., by F. Schupp, Hamburg (Meiner), 1982.

GM C. I. Gerhardt (ed.), *Leibnizens mathematische Schriften*, 7 vol. Berlin 1875-90, reprint Hildesheim (Olms) 1960-1.

GP C. I. Gerhardt (ed.), *Die philosophischen Schriften von G. W. Leibniz*, 7 volumes Berlin/Halle 1849–63, reprint Hildesheim (Olms) 1962.

Grua G. Grua (ed.), *G. W. Leibniz – Textes inédits*, Paris 1948.

LBr E. Bodemann, *Der Briefwechsel des Gottfried Wilhelm Leibniz in der Königl. öffentl. Bibl. zu Hannover*, Hannover 1889, reprint Hildesheim (Olms) 1966.

LH E. Bodemann, *Die Leibniz-Handschriften der Königlichen öffentlichen Bibliothek zu Hannover*, Hannover & Leipzig 1895, reprint Hildesheim (Olms) 1966.

OP J. E. Erdmann (ed.): *God. Guil. Leibnitii Opera Philosophica quae extant* ... Berlin 1840, reprint Aalen 1959.

R Rud. Eric Raspe (ed.): *Oeuvres philosophiques de feu Mr. Leibniz*, Amsterdam/Leipzig 1765.

English Translations

L G. W. Leibniz, *Philosophical Papers and Letters*, ed. and translated by Leroy E. Loemker, 2nd ed. Dordrecht (Reidel) 1969.
P G. W. Leibniz, *Logical Papers*, ed. and translated by G. H. R. Parkinson, Oxford (Clarendon Press) 1966.

OTHER LITERATURE

[Adams, 1982] R. M. Adams. Leibniz's Theories of Contigency. In [Hooker, 1982, pp. 243–83].
[Adams, 1994] R. M. Adams. *Leibniz – Determinist, Theist, Idealist*, New York & Oxford (Oxford University Press), 1994.
[Anderson, 1958] A. R. Anderson. A Reduction of Deontic Logic to Alethic Modal Logic. *Mind* LXVII, 100–103, 1958.
[Arnauld and Nicole, 1683] A. Arnauld and P. Nicole. *La Logique ou L'Art de Penser*, 5th edition, 1683. Reprint 1965 Paris (Presses universitaires de France); the 1st edition appeared in 1662.
[Broad, 1975] C. D. Broad. *Leibniz: An Introduction*, Cambridge (Cambridge University Press), 1975.
[Burkhardt, 1980] H. Burkhardt. *Logik und Semiotik in der Philosophie von Leibniz*, München (Philosophia), 1980.
[Couturat, 1901] L. Couturat. *La Logique de Leibniz d'après des documents inédits*, Paris, 1901. Reprint Hildesheim (Olms) 1961.
[Dürr, 1930] K. Dürr. *Neue Beleuchtung einer Theorie von Leibniz*, Darmstadt, 1930. (Abhandlungen der Leibniz-Gesellschaft, ed. by P. Ritter, Vol. II.)
[Englebretsen, 1988] G. Englebretsen. A Note on Leibniz's Wild Quantity Thesis. *Studia Leibnitiana*, 20, 87–89, 1988.
[Frankfurt, 1972] H. Frankfurt. *Leibniz: A Collection of Critical Essays*, New York (Doubleday), 1972.
[Hooker, 1982] M. Hooker. *Leibniz: Critical and Interpretive Essays*, Minneapolis (University of Minnesota Press), 1982.
[Ishiguru, 1972] H. Ishiguru. *Leibniz's Philosophy of Logic and Language*, Ithaca (Cornell University Press), 1972.
[Kauppi, 1960] R. Kauppi. *Über die Leibnizsche Logik*, Helsinki (Acta Philosophica Fennica), 1960.
[Kneale and Kneale, 1962] W. Kneale and M. Kneale. *The Development of Logic*, Oxford (Oxford University Press), 1962.
[Knecht, 1981] H. H. Knecht. *La Logique chez Leibniz – Essai sur le Rationalisme Baroque*, Lausanne (Editions L'Age d'Homme), 1981.
[Knobloch, 1976] E. Knobloch, ed. *G. W. Leibniz – Ein Dialog zur Einführung in die Arithmetik und Algebra*, Stuttgart (Frommann & Holzboog), 1976.
[Kulstad, 1991] M. Kulstad. *Leibniz on Apperception, Consciousness and Reflection*, Munich (Philosophia), 1991.
[Kvet, 1857] F. Kvet. *Leinitz'ens Logik – Nach den Quellen dargestellt*, Prague (F. Tempsky), 1857.
[Lenzen, 1980] W. Lenzen. *Glauben, Wissen und Wahrscheinlichkeit*, Wien (Springer), 1980.
[Lenzen, 1983] W. Lenzen. Zur extensionalen und "intensionalen" Interpretation der Leibnizschen Logik. *Studia Leibnitiana* 15, 129-148, 1983.
[Lenzen, 1984a] W. Lenzen. 'Unbestimmte Begriffe' bei Leibniz. *Studia Leibnitiana* 16, 1–26, 1984.

[Lenzen, 1984b] W. Lenzen. Leibniz und die Boolesche Algebra. *Studia Leibnitiana* 16, 187–203, 1984.

[Lenzen, 1986] 'Non est' non est 'est non' - Zu Leibnizens Theorie der Negation. *Studia Leibnitiana* 18, 1–37, 1986.

[Lenzen, 1987] W. Lenzen. Leibniz's Calculus of Strict Implication. In J. Srzednicki (ed.) *Initiatives in Logic (Reason and Argument* 1), Dordrecht, 1–35, 1987.

[Lenzen, 1988] W. Lenzen. Zur Einbettung der Syllogistik in Leibnizens 'Allgemeinen Kalkül'. *Studia Leibnitiana Sonderheft* 15, 38–71, 1988.

[Lenzen, 1989a] W. Lenzen. Arithmetical vs. 'Real' Addition - A Case Study of the Relation between Logic, Mathematics, and Metaphysics in Leibniz. In [Rescher, 1989, pp. 149–157].

[Lenzen, 1989b] W. Lenzen. Concepts vs. Predicates - Leibniz's Challenge to Modern Logic. In *The Leibniz Renaissance* (ed. by Centro Fiorentino di Storia e Filosofia della Scienzia), Florenz, 153–172, 1989.

[Lenzen, 1990a] W. Lenzen. *Das System der Leibnizschen Logik*, Berlin (de Gruyter), 1990.

[Lenzen, 1990b] W. Lenzen. On Leibniz's Essay '*Mathesis rationis*' (Critical Edition and Commentary). *Topoi* 9, 29–59, 1990.

[Lenzen, 1991] W. Lenzen. Leibniz on Ens and Existence. In W. Spohn, B. C. van Fraassen & B. Skyrms (eds.): *Existence and Explanation.* Dordrecht (Kluwer) 1991, 59–75.

[Lenzen, 1992] W. Lenzen. Leibniz on Properties and Individuals", in K. Mulligan (ed.): *Language, Truth, and Ontology.* Dordrecht (Kluwer), 193–204, 1992.

[Lenzen, 2000] W. Lenzen. Guilielmi Pacidii Non plus ultra – oder – Eine Rekonstruktion des Leibnizschen Plus-Minus-Kalküls. *Philosophiegeschichte und logische Analyse* 3, 71–118, 2000.

[Lenzen, 2001] W. Lenzen. Zur Logik alethischer und deontischer Modalitäten bei Leibniz. In W. Stelzner (ed.), *Ursprünge und Entwürfe nichtklassischer logischer Ansätze im Übergang von traditioneller zu moderner Logik*, Paderborn (Mentis), 335–351, 2001.

[Lewis, 1918] C. Lewis. *A Survey of Symbolic Logic*, University of California Press, 1918. Here quoted according to the Dover edition, New York 1960.

[Lewis and Langford, 1932] C. I Lewis and C. H. Langford. *Symbolic Logic*, New York, 1932. 2nd edition, 1959 (Dover Publications).

[Lukasiewicz, 1957] J. Lukasiewicz. *Aristotle's Syllogistic*, Oxford (Oxford University Press), 1957.

[Mates, 1986] B. Mates. *The Philosophy of Leibniz: Metaphysics and Language*, Oxford (Oxford University Press), 1986.

[Mugnai, 1992] M. Mugnai. *Leibniz' Theory of Relations*, Stuttgart (F. Steiner), 1992.

[O'Briant, 1968] W. H. O'Briant. *Gottfried Wilhelm Leibniz's General Investigations Concerning the Analysis of Concepts and Truths*, Athens (University of Georgia Monographs no. 17.), 1968.

[Parkinson, 1965] G. H. R. Parkinson. *Logic and Reality in Leibniz's Metaphysics*, Oxford, 1965.

[Poser, 1969] H. Poser. *Zur Theorie der Modalbegriffe bei G. W. Leibniz*, Wiesbaden (Steiner), 1969.

[Quine, 1953] W. V. O. Quine. *From a Logical Point of View*, New York (Harper & Row), 1953.

[Ravier, 1937] E. Ravier. *Bibliographie des OEuvres de Leibniz*, Paris, 1937. Reprint Hildesheim 1966.

[Rescher, 1954] N. Rescher. Leibniz's interpretation of his logical calculus. *Journal of Symbolic Logic* 19, 1–13, 1954.

[Rescher, 1967] N. Rescher. *The Philosophy of Leibniz*, Englewood Cliffs (Rowman & Littlefield), 1967.

[Rescher, 1979] N. Rescher. *Leibniz - An Introduction to his Philosophy*, Oxford (Blackwell), 1979.

[Rescher, 1989] N. Rescher. *Leibnizian Inquiries - A Group of Essays* (Proceeding of the 5th Annual Conference in Philosophy of Science), Lanham (University of America Press), 1989.

[Russell, 1900] B. Russell. *A Critical Exposition of the Philosophy of Leibniz*, Cambridge (George Allen & Unwin), 1900.

[Rutherford, 1995] D. Rutherford. *Leibniz and the Rational Order of Nature*, Cambridge (Cambridge University Press), 1995.

[Sauer, 1946] H. Sauer. Über die logischen Forschungen von Leibniz. In *G. W. Leibniz. Vorträge der aus Anlaß seines 300. Geburtstages in Hamburg abgehaltenen wissenschaftlichen Tagung*, Hamburg, 46–78, 1946.

[Schepers, 1975] H. Schepers. Leibniz' Disputation ,De Conditionibus': Ansätze zu einer juristischen Aussagenlogik. In *Akten des II. Internationalen Leibniz-Kongresses*, Bd. V, 1–17, 1975.

[Schupp, 2000] F. Schupp, ed. G. W. Leibniz, *Die Grundlagen des logischen Kalküls*, Hamburg (Meiner), 2000.

[Sleigh, 1990] R. C. Sleigh, Jr. *Leibniz & Arnauld – A Commentary of Their Correspondence*, New Haven & London (Yale University Press), 1990.

[Sotirov, 1999] V. Sotirov. Arithmetizations of Syllogistic à la Leibniz. *Journal of Applied Non-Classical Logics* 9, 387–405, 1999.

[Swoyer, 1995] C. Swoyer. Leibniz on Intension and Extension. *Noûs* 29, 96–114, 1995.

[Thom, 1981] P. Thom. *The Syllogism*, München (Philosophia), 1981.

[Wilson, 1989] C. Wilson. *Leibniz's Metaphysics – The Concept of Substance in Seventeenth Century Metaphysics*, Princeton (Princeton University Press), 1989.

[Woolhouse, 1981] R. S. Woolhouse, ed. *Leibniz: Metaphysics and Philosophy of Science*, Oxford (Oxford University Press), 1981.