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MATHEMATICS

CONTRIBUTIONS TO THE THEORY OF MODELS. I

BY

ALFRED TARSKI

(Communicated by Prof. A. HEYTING at the meeting of June 26, 1954)

Introduction

Within the last years a new branch of metamathematics has been developing. It is called the *theory of models* and can be regarded as a part of the semantics of formalized theories. The problems studied in the theory of models concern mutual relations between sentences of formalized theories and mathematical systems in which these sentences hold. Every set Σ of sentences determines uniquely a class $\mathbf{K}$ of mathematical systems; in fact, the class of all those mathematical systems in which every sentence of Σ holds. Σ is sometimes referred to as a *postulate system* for $\mathbf{K}$; mathematical systems which belong to $\mathbf{K}$ are called *models* of Σ. Among questions which naturally arise in the discussion of these notions, the following may be mentioned: Knowing some structural (formal) properties of a set Σ of sentences, what conclusions can we draw concerning mathematical properties of the correlated class $\mathbf{K}$ of models? Conversely, knowing some mathematical properties of a class $\mathbf{K}$ of mathematical systems, what can we say about structural properties of a set Σ which constitutes a postulate system for $\mathbf{K}$? Among publications in this field we may point out the articles and monographs BIRKHOFF [1], HENKIN [8], ROBINSON [16], and TARSKI [23] listed below in the Bibliography [1]).

In this paper we present some new results from the theory of models [2]). In § 1 we consider sets Σ consisting exclusively of first-order universal sentences; the correlated classes of models are called *universal classes of relational systems*. We give a simple mathematical characterization of

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1) The well-known Löwenheim–Skolem theorem, which was first proved in 1915, may be regarded as the earliest result in the theory of models. In the author's paper [20], part II, pp. 298–301, the development of the theory of models as a separate field of research was clearly anticipated, some general notions in this field were introduced, and some results were established; as indicated l. cit., p. 301, these results originated in 1926–1932. Cf. [20] also for bibliographical references to earlier publications. (The numbers in square brackets refer to the bibliography which will follow § 2 of this paper.)

2) Most results of this paper were obtained in 1949–1950 and discussed in a seminar conducted by the author in the University of California, Berkeley, in Fall 1950; they have been summarized in [21] and [24]. The last two theorems of § 2 have been found recently. The paper was prepared for publication while the author was working on a research project in the foundations of mathematics supported by the Office of Ordnance Research, U.S. Army.

universal classes, i.e., a set of simple mathematical conditions which are necessary and sufficient for a class **K** to be universal, and we establish some related results concerning these classes. In § 2 we are dealing with sets Σ consisting exclusively of algebraic equations (identities); the correlated classes of models are referred to as *equational classes of algebras.* A mathematical characterization of equational classes was first given in [1]. Using results of § 1, we establish a new set of necessary and sufficient conditions for a class of algebras to be equational. In the last two theorems of § 2 a special class of algebras, in fact, the class **L** of *representable relation algebras*, is discussed. We show that the class **L** is equational and we state some purely algebraic properties of **L** implied by its equational character (e.g., the fact that a homomorphic image of a representable relation algebra is itself a representable relation algebra). What seems especially interesting in connection with these last results is that no set of equations which could serve as a postulate system for **L** has yet been actually exhibited, and that no purely algebraic proof of those properties of **L** implied by its equational character is known at present.

§ 1. *Universal classes of relational systems*

In the subsequent discussion various set-theoretical notions will be involved, in particular those of an *ordinal* and a *cardinal.* We assume that ordinals have been defined in such a way that every ordinal coincides with the set of all smaller ordinals. A cardinal can be understood as an ordinal which has a larger power than every smaller ordinal. Given an ordinal α, an *α-termed sequence* (i.e., a sequence of type α) with consecutive terms $x_0, x_1, \ldots, x_\xi, \ldots$ is represented by $\langle x_\xi \rangle_{\xi<\alpha}$, sometimes also by $\langle x_0, \ldots, x_{\alpha-1} \rangle$ (if α is not a limit ordinal) or by $\langle x_0, \ldots, x_\xi, \ldots \rangle$ (without indicating the type of the sequence). A similar notation $\langle x_i \rangle_{i\in I}$ is sometimes used to represent a system of elements x_i indexed by an arbitrary set I (i.e., strictly speaking, a function, with domain I, which correlates an element x_i with every element i of I). Given a set B and an ordinal α, the set of all α-termed sequences $\langle x_\xi \rangle_{\xi<\alpha}$ in which all terms belong to B ($x_\xi \in B$ for $\xi < \alpha$) is denoted by B^α. Thus, α and β being two ordinals, β^α denotes here the set of all α-termed sequences in which every term is an ordinal smaller than β.

We identify finite ordinals with natural numbers; given a natural number ν, we do not distinguish between ν-termed sequences and *ordered ν-tuples* $\langle x_0, \ldots, x_{\nu-1} \rangle$. By a *$\nu$-ary relation* we understand an arbitrary set R of ordered ν-tuples; R is called a *finitary relation* or, simply, a *relation* if it is a ν-ary relation for some natural ν. In case the relation R is not empty, the number ν is uniquely determined by R and is called the *rank* of R; the empty set is regarded as a relation of any given rank ν.

By a *relational system* we understand an arbitrary system (sequence) $\mathfrak{R} = \langle A, R_0, \ldots, R_\xi, \ldots \rangle$ in which A is a non-empty set, $R_0, \ldots, R_\xi, \ldots$ are finitary relations, and each relation R_ξ is included in A^{ν_ξ} ($R_\xi \subseteq A^{\nu_\xi}$) where ν_ξ is the rank of R_ξ; the type of the sequence $\langle R_0, \ldots, R_\xi, \ldots \rangle$ is called the *order* of $\mathfrak{R}$[3]). Elements of the set A are sometimes referred to as elements of the system $\mathfrak{R}$; we speak of the power of $\mathfrak{R}$ meaning the power of A; $\mathfrak{R}$ is called finite if A is finite, etc. Two relational systems

(i) $$\mathfrak{R}=\langle A, R_0, \ldots, R_\xi, \ldots\rangle$$

and

(ii) $$\mathfrak{S}=\langle B, S_0, \ldots, S_\xi, \ldots\rangle$$

are called *similar* if they are of the same order α and if, for every $\xi<\alpha$, the corresponding relations R_ξ and S_ξ are of the same rank. A class consisting of all relational systems which are similar to a certain system $\mathfrak{R}$ is called a *species* or a *similarity class* (cf. [1], p. 439, and [10], part I, p. 896; since the notion of rank when applied to the empty relation is ambiguous, we may assume here that no relation R_ξ constituting $\mathfrak{R}$ is empty). When discussing a class $\boldsymbol{K}$ of relational systems, we shall always tacitly assume that any two systems of $\boldsymbol{K}$ are similar, i.e., that $\boldsymbol{K}$ is included in a similarity class.

$\mathfrak{R}$ and $\mathfrak{S}$ being two similar systems of order α defined by formulas (i) and (ii), we say that $\mathfrak{S}$ is a *subsystem* of $\mathfrak{R}$ if $B\subseteq A$ and if, for every $\xi<\alpha$, $S_\xi=R_\xi\cap B^{\nu_\xi}$ (i.e., S_ξ is the intersection of R_ξ and B^{ν_ξ}) where ν_ξ is the rank of R_ξ. Given a class $\boldsymbol{K}$ of relational systems, $\boldsymbol{S}(\boldsymbol{K})$ denotes the class of all subsystems of systems in $\boldsymbol{K}$. In case $\boldsymbol{K}$ consists of one system $\mathfrak{R}$, i.e., $\boldsymbol{K}=\{\mathfrak{R}\}$, we write $\boldsymbol{S}(\mathfrak{R})$ instead of $\boldsymbol{S}(\boldsymbol{K})$; the same applies to all symbols introduced below which involve a variable class $\boldsymbol{K}$. $\boldsymbol{S}_\gamma(\boldsymbol{K})$ (where γ is a cardinal) is the class of all systems of $\boldsymbol{S}(\boldsymbol{K})$ with power smaller than γ; thus, in particular, $\boldsymbol{S}_\omega(\boldsymbol{K})$is the class of all finite systems of $\boldsymbol{S}(\boldsymbol{K})$. The class $\boldsymbol{K}$ is called a *chain* if $\boldsymbol{K}$ is not empty and if, given any two systems in $\boldsymbol{K}$, one of them is always a subsystem of the other (i.e., we have $\mathfrak{R}\in\boldsymbol{S}(\mathfrak{S})$ or $\mathfrak{S}\in\boldsymbol{S}(\mathfrak{R})$ for any $\mathfrak{R}, \mathfrak{S}\in\boldsymbol{K}$).

Let again $\mathfrak{R}$ and $\mathfrak{S}$ be two systems of order α defined by (i) and (ii). A function f is said to *map* $\mathfrak{R}$ *homomorphically* onto $\mathfrak{S}$ if (1) the domain of f includes A; (2) B is the set of all elements $f(a)$ with $a\in A$; (3) for every $\xi<\alpha$, S_ξ consists of all ordered ν-tuples $\langle f(a_0), \ldots, f(a_{\nu-1})\rangle$ where ν is the rank of R_ξ and $\langle a_0, \ldots, a_{\nu-1}\rangle$ belongs to R_ξ. If, in addition, f is a biunique (univalent) function, it is said to *map* $\mathfrak{R}$ *isomorphically* onto $\mathfrak{S}$. In case there is a function f which maps $\mathfrak{R}$ homomorphically, or isomorphically, onto $\mathfrak{S}$, $\mathfrak{R}$ is said to be *homomorphic*, or *isomorphic*, to $\mathfrak{S}$, and $\mathfrak{S}$ is called a *homomorphic*, or *isomorphic*, *image* of $\mathfrak{R}$. (The relation of isomorphism, as opposed to that of homomorphism, is of course symmetric.) $\boldsymbol{H}(\boldsymbol{K})$ is the class of all homomorphic images and $\boldsymbol{I}(\boldsymbol{K})$ that of all isomorphic images of systems belonging to a given class $\boldsymbol{K}$; hence, e.g., $\boldsymbol{SI}(\boldsymbol{K})$ is the class of all subsystems of isomorphic images of systems in $\boldsymbol{K}$. A system $\mathfrak{S}$ is said to be *isomorphically embeddable* in a system $\mathfrak{R}$ if it is isomorphic to a subsystem of $\mathfrak{R}$, i.e.,

$$\mathfrak{S}\in\boldsymbol{IS}(\mathfrak{R}).$$

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[3]) In [10] (cf. part I, p. 897) the relational systems are referred to as *algebras in the wider sense*; in [7] the relational systems of finite order are called *polyrelations*.

The following formulas are easily seen to hold for every class $\mathbf{K}$ of relational systems:

$$\mathbf{K} \subseteq \mathbf{S}(\mathbf{K}) = \mathbf{SS}(\mathbf{K}),$$
$$\mathbf{K} \subseteq \mathbf{I}(\mathbf{K}) = \mathbf{II}(\mathbf{K}) \subseteq \mathbf{H}(\mathbf{K}) = \mathbf{HH}(\mathbf{K}),$$
$$\mathbf{IH}(\mathbf{K}) = \mathbf{HI}(\mathbf{K}) = \mathbf{H}(\mathbf{K}),$$
$$\mathbf{SH}(\mathbf{K}) \subseteq \mathbf{HS}(\mathbf{K}),$$
$$\mathbf{SI}(\mathbf{K}) = \mathbf{IS}(\mathbf{K}).$$

We also have for any two classes $\mathbf{K}$ and $\mathbf{L}$ of systems

$$\mathbf{S}(\mathbf{K} \cup \mathbf{L}) = \mathbf{S}(\mathbf{K}) \cup \mathbf{S}(\mathbf{L}), \quad \mathbf{H}(\mathbf{K} \cup \mathbf{L}) = \mathbf{H}(\mathbf{K}) \cup \mathbf{H}(\mathbf{L}), \quad \mathbf{I}(\mathbf{K} \cup \mathbf{L}) = \mathbf{I}(\mathbf{K}) \cup \mathbf{I}(\mathbf{L}),$$

and similarly for arbitrary many classes; hence

$$\mathbf{K} \subseteq \mathbf{L}$$

always implies

$$\mathbf{S}(\mathbf{K}) \subseteq \mathbf{S}(\mathbf{L}), \quad \mathbf{H}(\mathbf{K}) \subseteq \mathbf{H}(\mathbf{L}), \text{ and } \mathbf{I}(\mathbf{K}) \subseteq \mathbf{I}(\mathbf{L}).$$

All this is obvious, except perhaps for the inclusion $\mathbf{IS}(\mathbf{K}) \subseteq \mathbf{SI}(\mathbf{K})$, which constitutes a "half" of one of the formulas listed above and which can be established by means of the familiar "exchange procedure" (cf., e.g., [28], p. 42).

The *union* of two systems $\mathfrak{R}$ and $\mathfrak{S}$ defined by formulas (i) and (ii) is the system

(iii) $$\mathfrak{T} = \langle C, T_0, \ldots, T_\xi, \ldots \rangle$$

of the same order α in which $C = A \cup B$ (i.e., C is the set union of A and B) and similarly $T_\xi = R_\xi \cup S_\xi$ for every $\xi < \alpha$. Analogously we define the union of a class $\mathbf{K}$ of systems. As is easily seen, if $\mathbf{K}$ is a chain and $\mathfrak{R}$ is the union of $\mathbf{K}$, then $\mathbf{K} \subseteq \mathbf{S}(\mathfrak{R})$. By $\mathbf{U}(\mathbf{K})$ we denote the class of all unions of chains which are subclasses of $\mathbf{K}$. It may be noticed that all statements involving $\mathbf{U}(\mathbf{K})$ in the subsequent discussion remain valid if we agree to denote by $\mathbf{U}(\mathbf{K})$, not the class of all unions of chains, but either the less comprehensive class of all unions of *well-ordered chains* included in $\mathbf{K}$ or the more comprehensive class of all unions of *directed* subclasses of $\mathbf{K}$. (A class $\mathbf{L}$ is called directed if, for any systems $\mathfrak{R}, \mathfrak{S} \in \mathbf{L}$, there is a system $\mathfrak{T} \in \mathbf{L}$ such that $\mathfrak{R}, \mathfrak{S} \in \mathbf{S}(\mathfrak{T})$.)

The *cardinal* (*direct*) *product* $\mathfrak{R} \times \mathfrak{S}$ of two systems $\mathfrak{R}$ and $\mathfrak{S}$ defined by (i) and (ii) is the system $\mathfrak{T}$ defined by (iii) such that (1) C consists of all ordered couples $\langle a, b \rangle$ with $a \in A$ and $b \in B$, and (2), for every $\xi < \alpha$, T_ξ consists of all ordered ν-tuples $\langle \langle a_0, b_0 \rangle, \ldots, \langle a_{\nu-1}, b_{\nu-1} \rangle \rangle$ where ν is the common rank of R_ξ and S_ξ, $\langle a_0, \ldots, a_{\nu-1} \rangle \in R_\xi$, and $\langle b_0, \ldots, b_{\nu-1} \rangle \in S_\xi$. Analogously, we define the cardinal product $\mathfrak{P}_{i \in I}(R^{(i)})$ of a system $\langle \mathfrak{R}^{(i)} \rangle_{i \in I}$ of relational systems

$$\mathfrak{R}^{(i)} = \langle A^{(i)}, R_0^{(i)}, \ldots, R_\xi^{(i)}, \ldots \rangle$$

indexed by an arbitrary set I; instead of couples $\langle a, b \rangle$ with $a \in A$ and $b \in B$, and ordered ν-tuples of such couples, we use systems $\langle a^{(i)} \rangle_{i \in I}$

with $a^{(i)} \in A^{(i)}$ for every $i \in I$, and ordered ν-tuples of such systems. $\mathbf{K}$ being a class of relational systems, $\mathbf{P}(\mathbf{K})$ is the class of all cardinal products $\mathfrak{P}_{i\in I}(\mathfrak{R}^{(i)})$ where I is a non-empty set and $\mathfrak{R}^{(i)} \in \mathbf{K}$ for every $i \in I$; given a cardinal γ, $\mathbf{P}_\gamma(\mathbf{K})$ is the class of all such products in which, in addition, the power of I is smaller than γ. Thus, if we agree not to distinguish ordered 1-tuples from the only terms of these 1-tuples, then $\mathbf{P}_2(\mathbf{K})$ simply coincides with $\mathbf{K}$ while $\mathbf{P}_3(\mathbf{K})$ consists of all systems of $\mathbf{K}$ and all products $\mathfrak{R} \times \mathfrak{S}$ with $\mathfrak{R}, \mathfrak{S} \in \mathbf{K}$ (together with some systems isomorphic to these products).

For any given similarity class $\mathbf{R}$ we construct a formalized theory $T(\mathbf{R})$ within which relational systems of $\mathbf{R}$ can be discussed. This theory is simply the first-order predicate logic (with the identity symbol, without variable predicates) enriched by some non-logical symbols [4]). There are denumerably many distinct *individual variables* in $T(\mathbf{R})$, which are arranged in a simple infinite sequence $\langle v_0, \ldots, v_\nu, \ldots\rangle_{\nu<\omega}$. The set of *logical constants* of $T(\mathbf{R})$ consists of the *sentential connectives* $\rightarrow$, $\vee$, $\wedge$, $\sim$, *the universal quantifier* $\bigwedge$, *the existential quantifier* $\bigvee$, and the *identity symbol* $=$. The non-logical constants are *predicates*. They are arranged in a sequence $\langle P_0, \ldots, P_\xi, \ldots\rangle_{\xi<\alpha}$ where α is the common order of all relational systems $\langle A, R_0, \ldots, R_\xi, \ldots\rangle$ belonging to $\mathbf{R}$. Moreover, if R_ξ ($\xi < \alpha$) is a ν-ary relation, then P_ξ is a *ν-placed predicate*. The identity symbol is also a predicate and, in fact, a two-placed (binary) predicate; it is the only predicate which is a logical constant.

An expression $P_\xi(v_{\varkappa_0}, \ldots, v_{\varkappa_{\nu-1}})$ where P_ξ is a ν-placed predicate and $v_{\varkappa_0}, \ldots, v_{\varkappa_{\nu-1}}$ are variables is called an *atomic formula*; appropriate combinations of atomic formulas by means of sentential connectives and *quantifier expressions* (i.e., quantifiers followed by variables) are referred to as *formulas*. A formula without free variables is called a *sentence*; it is called a *universal sentence* if it is of the form

$$\bigwedge v_{\varkappa_0} \ldots \bigwedge v_{\varkappa_\nu} (\phi)$$

where ϕ is a formula containing no quantifiers.

We assume it to be clear under what conditions a sentence σ of $T(\mathbf{R})$ is *satisfied* in a system $\mathfrak{R} = \langle A, R_0, \ldots, R_\xi, \ldots\rangle$ of $\mathbf{R}$. Roughly speaking, this means that σ proves to be true if all the variables of $\mathbf{R}$ are assumed to range over the set A, the logical constants are interpreted in the usual way, and the predicates $P_0, \ldots, P_\xi, \ldots$ are understood to denote the relations $R_0, \ldots, R_\xi, \ldots$, respectively. A system $\mathfrak{R}$ is called a *model* of a sentence σ if σ is satisfied in $\mathfrak{R}$; it is called a model of a set Σ of sentences if it is a model of every sentence of this set. A class $\mathbf{K}$ of systems (included in $\mathbf{R}$) is called an *arithmetical class*, in symbols $\mathbf{K} \in \mathbf{AC}$, if it consists of all models of a certain sentence σ of $T(\mathbf{R})$. $\mathbf{K}$ is called an *arithmetical class in*

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[4]) For a more detailed discussion of such theories (the so-called *theories with standard formalization*) see [25], pp. 5 ff.

the wider sense, in symbols $\mathbf{K} \in \mathbf{AC}_\Delta$, if it consists of all models of a certain set Σ of sentences; this is equivalent to saying that $\mathbf{K}$ is an intersection of arbitrarily many arithmetical classes [5] [6]). Two systems $\mathfrak{R}$ and $\mathfrak{S}$ are called *arithmetically equivalent* if every sentence of $T(\mathbf{R})$ which holds in one of these systems holds also in the other; or, what amounts to the same, if every class $\mathbf{K} \in \mathbf{AC}$ which contains one of these systems contains also the other. If, instead of considering arbitrary sentences of $T(\mathbf{R})$, we restrict ourselves to universal sentences, we arrive at the notions of a *universal* (*arithmetical*) *class*, *a universal class in the wider sense*, and *universally equivalent* systems [7]). The formula $\mathbf{K} \in \mathbf{UC}$, or $\mathbf{K} \in \mathbf{UC}_\Delta$, expresses the fact that $\mathbf{K}$ is a universal class, or a universal class in the wider sense, respectively. For instance, among systems $\langle A, R\rangle$ in which R is a binary relation, the class of all partially ordered systems and that of all simply ordered systems belong to $\mathbf{UC}$. On the other hand, the class of all reflexive systems without cycles belongs to $\mathbf{UC}_\Delta$, but not to $\mathbf{UC}$. ($\langle A, R\rangle$ is said to be a reflexive system without cycles if $\langle x, x\rangle \in R$ for every $x \in A$ and if, for every natural number $\nu \neq 0$ and every sequence $\langle x_0, \ldots, x_\nu\rangle \in A^{\nu+1}$, the formulas $\langle x_0, x_1\rangle \in R, \ldots, \langle x_{\nu-1}, x_\nu\rangle \in R$, and $\langle x_\nu, x_0\rangle \in R$ imply $x_0 = x_\nu$.)

Various elementary properties of $\mathbf{UC}$ and $\mathbf{UC}_\Delta$ directly follow from the definitions of these notions. Thus, e.g., unions and intersections of finitely many classes in $\mathbf{UC}$ are again classes in $\mathbf{UC}$; similarly, unions of finitely many and intersections of arbitrarily many classes in $\mathbf{UC}_\Delta$ are again classes in $\mathbf{UC}_\Delta$.

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[5]) The notions of arithmetical classes and arithmetical equivalence were first introduced in [20], part II, pp. 298 ff.; they were applied there only to relations, and not to relational systems, and instead of "arithmetical class" and "arithmetical equivalence" the terms "*elementary property*" and "*elementary equivalence*" were used. A purely mathematical definition of arithmetical classes (and a discussion of their fundamental mathematical properties) can be found in [23].

[6]) In [8], p. 418, the notion of a *quasi-arithmetical class* is introduced; a class (of relational systems) is called quasi-arithmetical if it consists of all models of a set of first-order sentences. It could thus seem that quasi-arithmetical classes coincide with arithmetical classes in the wider sense. Actually this is not the case since the terms "sentence" and "model" are not used in [8] exactly in the same sense with which they appear in the present article. To define quasi-arithmetical classes in our terminology, we assume that the formalized theory $T(\mathbf{R})$ is provided with arbitrarily (not only denumerably) many variables and we use the notion of simultaneous satisfiability of an arbitrary set of formulas (not necessarily sentences); a class is quasi-arithmetical if it consists of all relational systems in which a certain set of formulas is simultaneously satisfiable. From this definition it is easily seen that every arithmetical class in the wider sense is quasi-arithmetical. The converse in general fails. For instance, given any relational system $\mathfrak{R}$, the class $\mathbf{K}$ of all systems $\mathbf{S}$ such that $\mathbf{IS}(\mathfrak{S})$ does not contain $\mathfrak{R}$ is always quasi-arithmetical (for an idea of a proof see [8], pp. 414 f.). In general, however, $\mathbf{K}$ is not in $\mathbf{AC}_\Delta$; it is certainly not in $\mathbf{AC}_\Delta$ if $\mathfrak{R}$ is a non-denumerable system of finite or denumerable order.

[7]) Universal classes were first discussed in [14], p. 190, where they were referred to as *universally definable classes*. By means of the method developed in [23], pp. 707 ff., the notion of a universal class can be defined in purely mathematical terms.

We shall give below simple and purely mathematical criteria, formulated in terms of subsystem and isomorphism, for a class of relational systems to be universal in the wider sense and for two relational systems to be universally equivalent; cf. Ths. 1.2, 1.3, 1.2′, 1.3′, and 1.4. An even simpler criterion makes it possible to single out universal classes from among all arithmetical classes; cf. Ths. 1.7 and 1.8.

Theorem 1.1. *Let* $\mathfrak{S}=\langle B, S_0, \ldots, S_{\nu-1}\rangle$ *be a finite relational system of finite order* ν, *and let* $\mathbf{K}$ *be the class of all systems* $\mathfrak{R}$ (*similar to* $\mathfrak{S}$) *in which* $\mathfrak{S}$ *is not isomorphically embeddable. Then* $\mathbf{K} \in \mathbf{UC}$.

Proof: The elements of B are assumed to be arranged in a finite sequence $\langle b_0, \ldots, b_{\pi-1}\rangle$, $\pi>0$, without repeating terms. Let $\mathbf{R}$ be the similarity class containing $\mathfrak{S}$. For every $\mu<\nu$ let ϱ_μ be the rank of the relation S_μ and hence also the place number of the predicate P_μ in $T(\mathbf{R})$. Let Φ be the set consisting of the following formulas of $T(\mathbf{R})$: (i) all the formulas $v_\varkappa = v_\lambda$ with $\varkappa<\lambda<\pi$; (ii) all the formulas $P_\mu(v_{\varkappa_0}, \ldots, v_{\varkappa_\lambda}, \ldots)$ such that $\mu<\nu$, $\langle \varkappa_0, \ldots, \varkappa_\lambda, \ldots\rangle \in \pi^{\varrho_\mu}$, and the ordered ϱ_μ-tuple $\langle b_{\varkappa_0}, \ldots, b_{\varkappa_\lambda}, \ldots\rangle$ does not belong to S_μ; (iii) all the formulas $\sim P_\mu(v_{\varkappa_0}, \ldots, v_{\varkappa_\lambda}, \ldots)$ such that $\mu<\nu$, $\langle \varkappa_0, \ldots, \varkappa_\lambda, \ldots\rangle \in \pi^{\varrho_\mu}$, and $\langle b_{\varkappa_0}, \ldots, b_{\varkappa_\lambda}, \ldots\rangle \in S_\mu$. Clearly, the set Φ of formulas is finite, so that its elements can be arranged in a finite sequence $\langle \varphi_0, \ldots, \varphi_{\sigma-1}\rangle$. It is easily seen that the universal sentence

(1) $$\wedge v_0 \ldots \wedge v_{\pi-1} (\varphi_0 \vee \ldots \vee \varphi_{\sigma-1})$$

holds in a system $\mathfrak{R}$ if and only if $\mathfrak{R}$ has no subsystem isomorphic to $\mathfrak{S}$. Hence $\mathbf{K}$ is the class of all models of (1) and consequently $\mathbf{K} \in \mathbf{UC}$.

Theorem 1.2. *Let* $\mathbf{K}$ *be a class of relational systems of finite order. For* $\mathbf{K} \in \mathbf{UC}_\Delta$ *it is necessary and sufficient that* $\mathbf{K}$ *satisfy the following three conditions:*

(i) $\mathbf{S}(\mathbf{K}) \subseteq \mathbf{K}$;

(ii) $\mathbf{I}(\mathbf{K}) \subseteq \mathbf{K}$;

(iii) *for every* $\mathfrak{R}$, *if* $\mathbf{S}_\omega(\mathfrak{R}) \subseteq \mathbf{K}$, *then* $\mathfrak{R} \in \mathbf{K}$.

Condition (iii) *can be replaced by*

(iii′) $\mathbf{U}(\mathbf{K}) \subseteq \mathbf{K}$;

and the three conditions (i)—(iii) *can be replaced by one:*

(iv) *for every* $\mathfrak{R}$, *if* $\mathbf{S}_\omega(\mathfrak{R}) \subseteq \mathbf{IS}(\mathbf{K})$, *then* $\mathfrak{R} \in \mathbf{K}$.

Proof: If $\mathbf{K} \in \mathbf{UC}_\Delta$, then $\mathbf{K}$ consists of all models of a certain set Σ of universal sentences. It is obvious that, if a universal sentence σ holds in a system $\mathfrak{R}$, it also holds in every subsystem of $\mathfrak{R}$ and every isomorphic image of $\mathfrak{R}$. Also, it can easily be shown (e.g., by contradiction) that if σ holds in every finite subsystem of $\mathfrak{R}$, then it holds in the system $\mathfrak{R}$ itself. Hence, conditions (i)—(iii) are necessary for $\mathbf{K} \in \mathbf{UC}_\Delta$.

Condition (iv) clearly follows from (i)—(iii). Assume now that the class

$\boldsymbol{K}$ satisfies (iv). Let $\boldsymbol{K}'$ be the intersection of all classes $\boldsymbol{L} \in \mathbf{UC}$ which include $\boldsymbol{K}$. Obviously, $\boldsymbol{K}' \in \mathbf{UC}_\Delta$ and $\boldsymbol{K} \subseteq \boldsymbol{K}'$. Suppose there is a system $\mathfrak{R}$ which belongs to $\boldsymbol{K}'$ but not to $\boldsymbol{K}$. By (iv), a certain finite subsystem $\mathfrak{S}$ of $\mathfrak{R}$ is not isomorphically embeddable in any system of $\boldsymbol{K}$. Let $\boldsymbol{M}$ be the class of all systems in which $\mathfrak{S}$ is not isomorphically embeddable. Thus $\boldsymbol{K} \subseteq \boldsymbol{M}$ and, by 1.1, $\boldsymbol{M} \in \mathbf{UC}$. Hence, by the definition of $\boldsymbol{K}'$, $\boldsymbol{K}' \subseteq \boldsymbol{M}$. Therefore $\mathfrak{R} \in \boldsymbol{M}$; in other words, the subsystem $\mathfrak{S}$ of $\mathfrak{R}$ is not isomorphically embeddable in $\mathfrak{R}$, which is obviously absurd. In view of this contradiction, the classes $\boldsymbol{K}$ and $\boldsymbol{K}'$ must coincide, and $\boldsymbol{K} \in \mathbf{UC}_\Delta$. Thus condition (iv) is sufficient for $\boldsymbol{K} \in \mathbf{UC}_\Delta$.

Finally, as regards the relation between (iii) and (iii′), the first of these conditions implies the second for every class $\boldsymbol{K}$ of relational systems satisfying (i). For, if $\boldsymbol{L}$ is a chain included in $\boldsymbol{K}$ and $\mathfrak{R}$ is the union of $\boldsymbol{L}$, then, as is easily seen, $\boldsymbol{S}_\omega(\mathfrak{R}) \subseteq \boldsymbol{S}(\boldsymbol{K})$; hence $\boldsymbol{S}_\omega(\mathfrak{R}) \subseteq \boldsymbol{K}$ by (i), and $\mathfrak{R} \in \boldsymbol{K}$ by (iii). Conversely (iii′) implies (iii) for any class $\boldsymbol{K}$ of systems whatsoever. In fact, assume to the contrary that (iii′) holds while (iii) fails. Thus there are systems $\mathfrak{R}$ for which $\boldsymbol{S}_\omega(\mathfrak{R}) \subseteq \boldsymbol{K}$ and which do not belong to $\boldsymbol{K}$; let $\mathfrak{R}_0$ be a system with these properties and with the smallest possible power. Obviously $\mathfrak{R}_0$ is infinite and hence, by the well-ordering principle, it can be represented as the union of a chain $\boldsymbol{L}$ of systems each of which has a smaller power than $\mathfrak{R}_0$. Clearly, for every $\mathfrak{S} \in \boldsymbol{L}$ we have $\boldsymbol{S}_\omega(\mathfrak{S}) \subseteq \boldsymbol{S}_\omega(\mathfrak{R}_0) \subseteq \boldsymbol{K}$ and therefore $\mathfrak{S} \in \boldsymbol{K}$. Consequently, $\boldsymbol{L} \subseteq \boldsymbol{K}$ and, by (iii′), $\mathfrak{R}_0 \in \boldsymbol{K}$, which contradicts our assumption. Thus, (iii) can be replaced by (iii′), and the proof has been completed.

It has been observed by Vaught in [27] that, by slightly modifying 1.2, we obtain a mathematical criterion for $\boldsymbol{K} \in \mathbf{UC}$; for this purpose it suffices to strengthen condition (iii) in the following way:

(iii*) *There is a natural number* ν *such that, for every* $\mathfrak{R}$, *if* $\boldsymbol{S}_\nu(\mathfrak{R}) \subseteq \boldsymbol{K}$, *then* $\mathfrak{R} \in \boldsymbol{K}$.

Instead of modifying 1.2 (iii), we can modify 1.2 (iv) analogously.

Theorem 1.3. *Let* $\mathfrak{R}$ *and* $\mathfrak{S}$ *be two similar relational systems of finite order. For* $\mathfrak{R}$ *and* $\mathfrak{S}$ *to be universally equivalent it is necessary and sufficient that*

$$\boldsymbol{S}_\omega(\mathfrak{R}) \subseteq \boldsymbol{IS}(\mathfrak{S}) \text{ and } \boldsymbol{S}_\omega(\mathfrak{S}) \subseteq \boldsymbol{IS}(\mathfrak{R}).$$

Proof: Assume first that $\mathfrak{R}$ and $\mathfrak{S}$ are universally equivalent. Let $\mathfrak{R}' \in \boldsymbol{S}_\omega(\mathfrak{R})$ and let $\boldsymbol{K}$ be the class of all systems $\mathfrak{T}$ (similar to $\mathfrak{R}$) such that $\mathfrak{R}'$ does not belong to $\boldsymbol{IS}(\mathfrak{T})$; in other words, $\mathfrak{R}'$ is a finite subsystem of $\mathfrak{R}$ and $\boldsymbol{K}$ is the class of all systems in which $\mathfrak{R}'$ is not isomorphically embeddable. By 1.1, $\boldsymbol{K} \in \mathbf{UC}$. Therefore, since $\boldsymbol{K}$ obviously does not contain $\mathfrak{R}$, it cannot contain $\mathfrak{S}$, so that $\mathfrak{R}' \in \boldsymbol{IS}(\mathfrak{S})$. Hence

$$\boldsymbol{S}_\omega(\mathfrak{R}) \subseteq \boldsymbol{IS}(\mathfrak{S}). \tag{1}$$

For the same reasons

(2) $$\mathbf{S}_\omega(\mathfrak{S}) \subseteq \mathbf{IS}(\mathfrak{R}).$$

Thus formulas (1) and (2) are necessary for $\mathfrak{R}$ and $\mathfrak{S}$ to be universally equivalent.

Assume now, conversely, that formulas (1) and (2) hold. Consider a class $\mathbf{K} \in \mathbf{UC}$ which contains one of the systems $\mathfrak{R}$ and $\mathfrak{S}$, say $\mathfrak{S}$. (1) then implies

(3) $$\mathbf{S}_\omega(\mathfrak{R}) \subseteq \mathbf{IS}(\mathbf{K}).$$

Since $\mathbf{K} \in \mathbf{UC}$, we have *a fortiori* $\mathbf{K} \in \mathbf{UC}_\Delta$ and therefore $\mathbf{K}$ satisfies 1.2 (iv). Hence, by (3), $\mathbf{K}$ contains $\mathfrak{R}$. Similarly, if $\mathbf{K}$ contains $\mathfrak{R}$, it must also contain $\mathfrak{S}$. Thus formulas (1) and (2) are jointly sufficient for $\mathfrak{R}$ and $\mathfrak{S}$ to be universally equivalent, and the proof is complete.

It is easily seen that the two inclusions formulated in 1.3 can be replaced by one equality:

$$\mathbf{IS}_\omega(\mathfrak{R}) = \mathbf{IS}_\omega(\mathfrak{S}).$$

As has been noticed by Vaught in [27], Ths. 1.1—1.3 do not directly extend to relational systems of infinite order. To obtain analogous results for systems of arbitrary order, we introduce the following notation.

With every relational system

(i) $$\mathfrak{R} = \langle A, R_0, \ldots, R_\xi, \ldots \rangle$$

of order α and with every sequence

(ii) $$\langle \zeta_0, \ldots, \zeta_\xi, \ldots \rangle \in \alpha^\beta$$

we correlate the system

(iii) $$\mathfrak{R}_{\zeta_0, \ldots, \zeta_\xi, \ldots} = \langle A, R_{\zeta_0}, \ldots, R_{\zeta_\xi}, \ldots \rangle$$

of order β, called the *reduct of* $\mathfrak{R}$ *indexed by the sequence* $\langle \zeta_0 \ldots, \zeta_\xi, \ldots \rangle$. If $\mathbf{K}$ is any class of systems (i) and $\langle \zeta_0, \ldots, \zeta_\xi, \ldots \rangle$ any sequence satisfying (ii), we denote by $\mathbf{K}_{\zeta_0, \ldots, \zeta_\xi, \ldots}$ the class of all reducts (iii) correlated with systems $\mathfrak{R} \in \mathbf{K}$.

For our immediate purposes we need exclusively *finite reducts*, i.e., reducts indexed by finite sequences. Using this notion, we shall now formulate and prove Ths. 1.1′—1.3′ which can be recognized as extensions of 1.1—1.3 to systems of arbitrary order.

Theorem 1.1′. *Let* $\mathfrak{S} = \langle B, S_0, \ldots, S_\xi, \ldots \rangle$ *be a finite relational system of order* α*; given a finite sequence* $\langle \zeta_0, \ldots, \zeta_{\nu-1} \rangle \in \alpha^\nu$, *let* $\mathbf{K}$ *be the class of all systems* $\mathfrak{R}$ *(similar to* $\mathfrak{S}$*) such that* $\mathfrak{S}_{\zeta_0, \ldots, \zeta_{\nu-1}}$ *is not isomorphically embeddable in* $\mathfrak{R}_{\zeta_0, \ldots, \zeta_{\nu-1}}$. *Then* $\mathbf{K} \in \mathbf{UC}$.

Proof: entirely analogous to that of 1.1.

Theorem 1.2′. *Let* $\mathbf{K}$ *be a class of relational systems of order* α. *For* $\mathbf{K} \in \mathbf{UC}_\Delta$ *it is necessary and sufficient that* $\mathbf{K}$ *satisfies conditions* (i), (ii) *of* 1.2 *and the following condition:*

(iii) *for every* $\mathfrak{R}$, *if* $\mathbf{S}_\omega(\mathfrak{R}_{\zeta_0,\ldots,\zeta_{\nu-1}}) \subseteq \mathbf{K}_{\zeta_0,\ldots,\zeta_{\nu-1}}$ *for every finite sequence* $\langle \zeta_0, \ldots, \zeta_{\nu-1} \rangle \in \alpha^\nu$, *then* $\mathfrak{R} \in \mathbf{K}$.

These three conditions can be replaced by one:

(iv) *for every* $\mathfrak{R}$, *if* $\mathbf{S}_\omega(\mathfrak{R}_{\zeta_0,\ldots,\zeta_{\nu-1}}) \subseteq \mathbf{IS}\,(\mathbf{K}_{\zeta_0,\ldots,\zeta_{\nu-1}})$ *for every finite sequence* $\langle \zeta_0, \ldots, \zeta_{\nu-1} \rangle \in \alpha^\nu$, *then* $\mathfrak{R} \in \mathbf{K}$.

Proof: With small changes, we argue as in the proof of 1.2, applying 1.1′ instead of 1.1.

Theorem 1.3′. *Let* $\mathfrak{R}$ *and* $\mathfrak{S}$ *be two relational systems of arbitrary order. For* $\mathfrak{R}$ *and* $\mathfrak{S}$ *to be universally equivalent it is necessary and sufficient that, for every finite sequence* $\langle \zeta_0, \ldots, \zeta_{\nu-1} \rangle \in \alpha^\nu$, *we have*

$$\mathbf{S}_\omega(\mathfrak{R}_{\zeta_0,\ldots,\zeta_{\nu-1}} \subseteq \mathbf{IS}(\mathfrak{S}_{\zeta_0,\ldots,\zeta_{\nu-1}}) \text{ and } \mathbf{S}_\omega(\mathfrak{S}_{\zeta_0,\ldots,\zeta_{\nu-1}}) \subseteq \mathbf{IS}(\mathfrak{R}_{\zeta_0,\ldots,\zeta_{\nu-1}})$$

(*and thus that the systems* $\mathfrak{R}_{\zeta_0,\ldots,\zeta_{\nu-1}}$ *and* $\mathfrak{S}_{\zeta_0,\ldots,\zeta_{\nu-1}}$ *be universally equivalent*).

Proof: analogous to that of 1.3, applying 1.1′ and 1.2′ instead of 1.1 and 1.2.

With the help of 1.1′—1.3′ we shall be able to establish most of the subsequent theorems of this section without restricting ourselves to systems of finite order.

MATHEMATICS

CONTRIBUTIONS TO THE THEORY OF MODELS. II

BY

ALFRED TARSKI

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Theorem 1.4. *Let $\mathfrak{R}$ and $\mathfrak{S}$ be two similar finite relational systems of arbitrary order α. For $\mathfrak{R}$ and $\mathfrak{S}$ to be universally equivalent it is necessary and sufficient that they be isomorphic.*

Proof: The sufficiency of the condition is obvious. To prove its necessity, we suppose that the systems $\mathfrak{R}$ and $\mathfrak{S}$ are universally equivalent but not isomorphic. We put

$$\mathfrak{R} = \langle A, R_0, \ldots, R_\xi, \ldots \rangle,$$
$$\mathfrak{S} = \langle B, S_0, \ldots, S_\xi, \ldots \rangle.$$

Since the sets A and B are finite, there are only finitely many functions f whose domain is A and which map A onto B; they can be arranged in a finite sequence $\langle f_0, \ldots, f_{\nu-1} \rangle$. None of these functions $f_\varkappa$, $\varkappa < \nu$, maps $\mathfrak{R}$ isomorphically onto $\mathfrak{S}$; hence, as is easily seen, we can correlate with each such function an ordinal $\zeta_\varkappa < \alpha$, such that $f_\varkappa$ does not map $\langle A, R_{\zeta_\varkappa} \rangle$ isomorphically onto $\langle B, S_{\zeta_\varkappa} \rangle$. Consequently, none of the functions $f_0, \ldots, f_{\nu-1}$ maps the system

$$\langle A, R_{\zeta_0}, \ldots, R_{\zeta_{\nu-1}} \rangle = \mathfrak{R}_{\zeta_0, \ldots, \zeta_{\nu-1}} \tag{1}$$

isomorphically onto the system

$$\langle B, S_{\zeta_0}, \ldots, S_{\zeta_{\nu-1}} \rangle = \mathfrak{S}_{\zeta_0, \ldots, \zeta_{\nu-1}}. \tag{2}$$

But since the isomorphism of these two systems cannot be established by any function different from $f_0, \ldots, f_{\nu-1}$, we arrive at the conclusion that the systems $\mathfrak{R}_{\zeta_0, \ldots, \zeta_{\nu-1}}$ and $\mathfrak{S}_{\zeta_0, \ldots, \zeta_{\nu-1}}$ are simply not isomorphic. On the other hand, the systems $\mathfrak{R}$ and $\mathfrak{S}$ being universally equivalent, we conclude by 1.3′ that $\mathfrak{R}_{\zeta_0, \ldots, \zeta_{\nu-1}}$ is isomorphic to a subsystem $\langle B', S'_{\zeta_0}, \ldots, S'_{\zeta_{\nu-1}} \rangle$ of $\mathfrak{S}_{\zeta_0, \ldots, \zeta_{\nu-1}}$, and similarly $\mathfrak{S}_{\zeta_0, \ldots, \zeta_{\nu-1}}$ is isomorphic to a subsystem $\langle A', R'_{\zeta_0}, \ldots, R'_{\zeta_{\nu-1}} \rangle$ of $\mathfrak{R}_{\zeta_0, \ldots, \zeta_{\nu-1}}$. Consequently, the set A has the same power as the subset B' of B, and the set B has the same power as the subset A' of A. Remembering that the sets A and B are finite, we obtain $B = B'$ and $\mathfrak{S}_{\zeta_0, \ldots, \zeta_{\nu-1}} = \langle B', S'_{\zeta_0}, \ldots, S'_{\zeta_{\nu-1}} \rangle$, so that the systems $\mathfrak{R}_{\zeta_0, \ldots, \zeta_{\nu-1}}$ and $\mathfrak{S}_{\zeta, \ldots, \zeta_{0\nu-1}}$ now turn out to be isomorphic. Thus our supposition leads to a contradiction. We conclude that, if $\mathfrak{R}$ and $\mathfrak{S}$ are universally equivalent, they are isomorphic, and the proof has been completed.

The following result of HENKIN will be involved in our discussion:

Theorem 1.5. *Let* $\mathbf{K}$ *be a class of relational systems of arbitrary order such that* $\mathbf{K} \in \mathbf{AC}_\Delta$. *Then, for every relational system* $\mathfrak{R}$ *(similar to systems in* $\mathbf{K}$*), we have* $\mathfrak{R} \in \mathbf{S}(\mathbf{K})$ *provided* $\mathbf{S}_\omega(\mathfrak{R}) \subseteq \mathbf{S}(\mathbf{K})$.

Proof: see [8], pp. 414 f.

It may be noticed that, for every class $\mathbf{K}$ of systems, the conclusion of 1.5 is equivalent to the following formula:

$$\mathbf{US}(\mathbf{K}) \subseteq \mathbf{S}(\mathbf{K}).$$

For the equivalence of the two formulations compare the proof of 1.2 (last part).

Actually we shall apply, not 1.5, but a somewhat stronger result:

Theorem 1.5′. *Let* $\mathbf{K}$ *be a class of relational systems of arbitrary order* α *such that* $\mathbf{K} \in \mathbf{AC}_\Delta$. *Then, for every relational system* $\mathfrak{R}$ *(similar to systems in* $\mathbf{K}$*), we have* $\mathfrak{R} \in \mathbf{S}(\mathbf{K})$ *provided* $\mathbf{S}_\omega(\mathfrak{R}_{\zeta_0,\ldots,\zeta_{\nu-1}}) \subseteq \mathbf{S}(\mathbf{K}_{\zeta_0,\ldots,\zeta_{\nu-1}})$ *holds for every finite sequence* $\langle \zeta_0, \ldots, \zeta_{\nu-1} \rangle \in \alpha^\nu$.

Proof: entirely analogous to that of 1.5.

It may be observed that in the discussion preceding 1.5 we have not applied the axiom of choice except when deriving 1.2(iii′) from 1.2(iii); if we omitted condition (iii′) (which is rather irrelevant for the subsequent discussion) in formulating 1.2, we could state that all the results preceding 1.5 are independent of the axiom of choice. On the other hand, this axiom is essentially involved in the proofs of 1.5 and 1.5′ [8]). In our further discussion we shall find results which are based upon 1.5 or 1.5′ and essentially depend on the axiom of choice (1.6 and 1.9) as well as results independent of this axiom (1.11, 2.1, and 2.2 except for 2.2(i)). In some cases the situation is less clear. For instance, the proofs of Ths. 1.7 and 1.8 which are given here are based upon 1.5′; however, as was pointed out by C. C. Chang, it is possible to modify these proofs so as to dispense with the use of the axiom of choice.

Theorem 1.6. *Let* $\mathbf{K}$ *be a class of relational systems of arbitrary order* α *such that* $\mathbf{K} \in \mathbf{AC}_\Delta$. *Then* $\mathbf{S}(\mathbf{K}) \in \mathbf{UC}_\Delta$.

Proof: Let

(1) $$\mathbf{L} = \mathbf{S}(\mathbf{K}).$$

We easily see that

(2) $$\mathbf{S}(\mathbf{L}) \subseteq \mathbf{L}$$

and also, since $\mathbf{I}(\mathbf{K}) \subseteq \mathbf{K}$,

(3) $$\mathbf{I}(\mathbf{L}) = \mathbf{IS}(\mathbf{K}) \subseteq \mathbf{SI}(\mathbf{K}) \subseteq \mathbf{S}(\mathbf{K}) = \mathbf{L}.$$

[8]) As was shown in [8], p. 415, Th. 1.5 (and hence *a fortiori* 1.5′) implies the representation theorem for Boolean algebras; on the other hand, it is known that the latter theorem cannot be established without the help of the axiom of choice. The problem remains open whether 1.5 (or 1.5′) is equivalent to the axiom of choice.

By 1.5′ and (1) we conclude:

(4) *for every system* $\mathfrak{R}$, *if* $\mathbf{S}_\omega(\mathfrak{R}_{\zeta_0,\dots,\zeta_{\nu-1}}) \subseteq \mathbf{S}(\mathbf{L}_{\zeta_0,\dots,\zeta_{\nu-1}})$ *holds for every finite sequence* $\langle\zeta_0, \dots, \zeta_{\nu-1}\rangle \in \alpha^\nu$, *then* $\mathfrak{R} \in \mathbf{L}$.

From (1)—(4), by applying 1.2′ (with $\mathbf{K}$ replaced by $\mathbf{L}$), we immediately obtain the conclusion.

We thus see that 1.6 can be derived in an elementary way from 1.5′ with the help of 1.2′. The derivation in the opposite direction—of 1.5′ from 1.6—is even simpler. In view of the rather elementary character of 1.2′ we can therefore regard Th. 1.6 as an equivalent formulation of Th. 1.5′. As a matter of fact, from 1.2′ it is seen that the conclusions of 1.5′ and 1.6 are equivalent for every class $\mathbf{K}$ of relational systems such that $\mathbf{I}(\mathbf{K}) \subseteq \mathbf{K}$ (and not only for every class $\mathbf{K} \in \mathbf{AC}_\Delta$). Similarly, by 1.2, the conclusions of 1.5 and 1.6 are equivalent for every class $\mathbf{K}$ of systems of finite order such that $\mathbf{I}(\mathbf{K}) \subseteq \mathbf{K}$. In both cases the equivalence is independent of the axiom of choice.

Theorem 1.7. *Let* $\mathbf{K}$ *be a class of relational systems of arbitrary order. For* $\mathbf{K} \in \mathbf{UC}_\Delta$ *it is necessary and sufficient that* $\mathbf{K} \in \mathbf{AC}_\Delta$ *and* $\mathbf{S}(\mathbf{K}) \subseteq \mathbf{K}$.

Proof: If $\mathbf{K} \in \mathbf{UC}_\Delta$, then obviously $\mathbf{K} \in \mathbf{AC}_\Delta$ and $\mathbf{S}(\mathbf{K}) \subseteq \mathbf{K}$ (cf. 1.2′). If, conversely, $\mathbf{K} \in \mathbf{AC}_\Delta$ and $\mathbf{S}(\mathbf{K}) \subseteq \mathbf{K}$, we clearly have $\mathbf{S}(\mathbf{K}) = \mathbf{K}$ and we obtain $\mathbf{S}(\mathbf{K}) \in \mathbf{UC}_\Delta$ by 1.6, so that finally $\mathbf{K} \in \mathbf{UC}_\Delta$.

Theorem 1.8. *Let* $\mathbf{K}$ *be a class of relational systems of arbitrary order. For* $\mathbf{K} \in \mathbf{UC}$ *it is necessary and sufficient that* $\mathbf{K} \in \mathbf{AC}$ *and* $\mathbf{S}(\mathbf{K}) \subseteq \mathbf{K}$.

Proof: The necessity of the conditions is obvious. To prove their sufficiency, assume that $\mathbf{K} \in \mathbf{AC}$ and $\mathbf{S}(\mathbf{K}) \subseteq \mathbf{K}$. Then, by 1.7, $\mathbf{K} \in \mathbf{UC}_\Delta$; i.e., there is a family $\mathbf{F}$ of classes $\mathbf{L} \in \mathbf{UC}$ such that $\mathbf{K}$ is the intersection of all these classes. Both all the classes $\mathbf{L} \in \mathbf{F}$ and their intersection $\mathbf{K}$ are arithmetical. Hence, by the compactness theorem for arithmetical classes (cf. [23], p. 710, and also [15]), there is a finite subfamily $\mathbf{G}$ of $\mathbf{F}$ such that $\mathbf{K}$ coincides with the intersection of all classes $\mathbf{L} \in \mathbf{G}$. Clearly, the intersection of finitely many universal classes is itself universal. Hence $\mathbf{K} \in \mathbf{UC}$, which completes the proof.

We thus see that 1.7 remains valid if we replace in it both $\mathbf{UC}_\Delta$ by $\mathbf{UC}$ and $\mathbf{AC}_\Delta$ by $\mathbf{AC}$. It can be shown that 1.6 loses its validity if modified in the same manner. In fact, the class $\mathbf{K}$ of all groups treated as systems with one binary operation is clearly in $\mathbf{AC}$ (cf. the beginning of § 2 of this paper); on the other hand, it easily follows from the results in [12] that $\mathbf{S}(\mathbf{K})$ is not in $\mathbf{UC}$.

A class $\mathbf{L}$ of relational systems of order β is called *pseudo-arithmetical*, or *pseudo-arithmetical in the wider sense*, in symbols $\mathbf{L} \in \mathbf{PC}$, or $\mathbf{L} \in \mathbf{PC}_\Delta$, if there is a class $\mathbf{K}$ of systems of some order α and a sequence $\langle\zeta_0, \dots, \zeta_\xi, \dots\rangle \in \alpha^\beta$ such that $\mathbf{K} \in \mathbf{AC}$, or $\mathbf{K} \in \mathbf{AC}_\Delta$, and $\mathbf{L} = \mathbf{K}_{\zeta_0,\dots,\zeta_\xi,\dots}$. By analyzing the proofs of 1.5 and 1.5′ we notice that these two theorems can be extended to arbitrary pseudo-arithmetical classes in the wider sense, i.e., that they remain valid if $\mathbf{AC}_\Delta$ is replaced by $\mathbf{PC}_\Delta$ [9]). Consequently 1.6 and 1.7 can also be extended in an analogous way. (However, 1.8 loses its validity if **AC** is replaced by **PC**. For instance, let K be the class of all systems $\langle A, R\rangle$ where R is a binary relation such that, for some relation S, $R \subseteq S$ and $\langle A, S\rangle$ is a simply ordered system. As is easily seen, $\mathsf{K} \in \mathbf{PC}$ and $\mathsf{S}(\mathsf{K}) \subseteq \mathsf{K}$; it can be shown, however, that K is not in **UC**. K proves to coincide with the class of reflexive systems without cycles, which was previously mentioned in this discussion.)

For further reference we state here explicitly (though without using the symbol $\mathbf{PC}_\Delta$) one of the extensions just mentioned, in fact, that of 1.6.

Theorem 1.9. *Let K be a class of relational systems of arbitrary order α such that $\mathsf{K} \in \mathbf{AC}_\Delta$; let β be any ordinal and $\langle \zeta_0, \ldots, \zeta_\xi, \ldots\rangle$ be any sequence in α^β. Then $\mathsf{S}(\mathsf{K}_{\zeta_0,\ldots,\zeta_\xi,\ldots}) \in \mathbf{UC}_\Delta$.*

Proof: We first extend 1.5′ to pseudo-arithmetical classes in the wider sense by repeating, with inessential changes, the original proof of 1.5 in [8], pp. 414 f. Hence, with the help of 1.2′, we derive our theorem by arguing as in the proof of 1.6.

Besides the class of reducts $\mathsf{K}_{\zeta_0,\ldots,\zeta_\xi,\ldots}$, a closely related class $\mathsf{K}^{\zeta_0,\ldots,\zeta_\xi,\ldots}$ deserves attention. By definition, $\mathsf{K}_{\zeta_0,\ldots,\zeta_\xi,\ldots}$ consists of all relational systems $\mathfrak{S}$ such that for *some* system $\mathfrak{R}$ with $\mathfrak{R}_{\zeta_0,\ldots,\zeta_\xi,\ldots} = \mathfrak{S}$ we have $\mathfrak{R} \in \mathsf{K}$. On the other hand, $\mathsf{K}^{\zeta_0,\ldots,\zeta_\xi,\ldots}$ is defined as the class of all systems $\mathfrak{S}$ such that for *every* system $\mathfrak{R}$ with $\mathfrak{R}_{\zeta_0,\ldots,\zeta_\xi,\ldots} = \mathfrak{S}$ we have $\mathfrak{R} \in \mathsf{K}$. $\mathsf{K}^{\zeta_0,\ldots,\zeta_\xi,\ldots}$ can also be defined by means of the formula

$$\mathsf{K}^{\zeta_0,\ldots,\zeta_\xi,\ldots} = \mathsf{R}_{\zeta_0,\ldots,\zeta_\xi,\ldots} - (\mathsf{R} - \mathsf{K})_{\zeta_0,\ldots,\zeta_\xi,\ldots}$$

where R is the similarity class including K (and $\mathsf{L}-\mathsf{K}$ denotes as usual the difference of the classes L and K). It turns out that 1.9 does not remain valid if $\mathsf{K}_{\zeta_0,\ldots,\zeta_\xi,\ldots}$ is replaced by $\mathsf{K}^{\zeta_0,\ldots,\zeta_\xi,\ldots}$. For instance, as is easily seen, the class L of all well-ordered systems $\langle A, R\rangle$ can be represented in the form $\mathsf{L} = \mathsf{K}^{\zeta_0,\ldots,\zeta_\xi,\ldots}$ where $\mathsf{K} \in \mathbf{AC}$; nevertheless, $\mathsf{S}(\mathsf{L})$ is not in $\mathbf{UC}_\Delta$ since clearly $\mathsf{S}(\mathsf{L}) = \mathsf{L}$ and 1.2′(iii) fails for $\mathsf{K} = \mathsf{L}$. On the other hand we have the following:

Theorem 1.10. *Let K be a class of relational systems of arbitrary*

[9]) Henkin points out in [8], pp. 418 f., that Th. 1.5 extends to quasi-arithmetical classes (cf. footnote 6 above). From some further remarks in [8], pp. 425 f., it follows that the conclusion of 1.5 applies to still other classes of relational systems; in fact to every class which consists of all models of a set of second-order sentences having the form

$$\bigvee R_0 \ldots \bigvee R_\nu\ (\varphi)$$

where $R_0,\ldots,R_\nu$ are variable predicates and φ is a first-order formula (cf. the remarks below following 1.10). As is easily seen, both the quasi-arithmetical classes and the classes just mentioned are special instances of $\mathbf{PC}_\Delta$.

order α such that $\mathbf{K} \in \mathbf{UC}_\Delta$; *let β be any ordinal and $\langle \zeta_0, \ldots, \zeta_\xi, \ldots \rangle$ be any sequence in α^β. Then*

(i) $$\mathbf{K}_{\zeta_0,\ldots,\zeta_\xi,\ldots} \in \mathbf{UC}_\Delta,$$

(ii) $$\mathbf{K}^{\zeta_0,\ldots,\zeta_\xi,\ldots} \in \mathbf{UC}_\Delta.$$

Proof: To obtain (i), notice that the premise $\mathbf{K} \in \mathbf{UC}_\Delta$ obviously implies $\mathbf{K} \in \mathbf{AC}_\Delta$ and hence, by 1.9,

(1) $$\mathbf{S}(\mathbf{K}_{\zeta_0,\ldots,\zeta_\xi,\ldots}) \in \mathbf{UC}_\Delta.$$

On the other hand, from $\mathbf{K} \in \mathbf{UC}_\Delta$ we conclude that $\mathbf{S}(\mathbf{K}) \subseteq \mathbf{K}$ (cf. 1.2′). Hence, as is easily seen, $\mathbf{S}(\mathbf{K}_{\zeta_0,\ldots,\zeta_\xi,\ldots}) \subseteq \mathbf{K}_{\zeta_0,\ldots,\zeta_\xi,\ldots}$ and therefore

(2) $$\mathbf{S}(\mathbf{K}_{\zeta_0,\ldots,\zeta_\xi,\ldots}) = \mathbf{K}_{\zeta_0,\ldots,\zeta_\xi,\ldots}.$$

Formulas (1) and (2) give at once (i).

Conclusion (ii) is a direct consequence of 1.2′; its proof does not involve 1.9 (and is independent of the axiom of choice). We simply check that, under the assumption $\mathbf{K} \in \mathbf{UC}_\Delta$, the class $\mathbf{L} = \mathbf{K}^{\zeta_0,\ldots,\zeta_\xi,\ldots}$ satisfies all the conditions mentioned in 1.2′, i.e., 1.2(i), 1.2(ii), and 1.2′(iii) (with $\mathbf{K}$ and α replaced by $\mathbf{L}$ and β). Details need not be given.

If we replace $\mathbf{UC}_\Delta$ by $\mathbf{UC}$ everywhere in 1.10, conclusion (ii) remains valid. However, (i) falls away; this is seen from the example constructed above to show that $\mathbf{UC}$ cannot be replaced by $\mathbf{PC}$ in 1.8.

Theorem 1.10 implies an interesting consequence which can be described in metamathematical terms as follows. Assume that the relational systems of a similarity class $\mathbf{R}$ are discussed, not within the theory $T(\mathbf{R})$, but within the second-order formalized theory $T''(\mathbf{R})$ obtained from $T(\mathbf{R})$ by adding variable predicates. Thus $T''(\mathbf{R})$ contains, in addition to logical constants, three kinds of symbols: individual variables $v_0, v_1, \ldots$, constant predicates $P_0, P_1, \ldots$, and variable predicates $R_0, R_1, \ldots$. As is easily seen, every sentence in $T''(\mathbf{R})$ can be equivalently transformed into a sentence σ consisting of three consecutive parts: (i) a succession of arbitrary (universal or existential) quantifiers, each followed by a variable predicate; (ii) a succession of arbitrary quantifiers, each followed by an individual variable; (iii) an arbitrary formula without quantifiers. In fact, the structure of part (ii) of σ can be further specified; it can be assumed that in this part all the universal quantifiers precede all the existential quantifiers (or conversely). Now consider those sentences σ of the form just described in which part (ii) contains no existential quantifiers. By an induction based upon both conclusions of 1.10 we easily show that every sentence σ of this kind is equivalent to a set of first-order universal sentences; in other words, if Σ is any set of sentences of this kind and $\mathbf{K}$ is the class of all models of Σ, then $\mathbf{K} \in \mathbf{UC}_\Delta$ [10]).

[10]) When speaking of models of second-order sentences, we have exclusively in mind the so-called *standard models;* cf., e.g., [8], p. 425. In other words, we assume that the notion of satisfaction underlying that of a model is the one which was discussed in a detailed way in [19], §§ 3 and 4.

From 1.9 and 1.10 we can derive various more special consequences in which the notion of a reduct is not involved. For instance, it turns out that $\mathbf{SH}(\mathbf{K}) \in \mathbf{UC}_\Delta$ and $\mathbf{SIP}_3(\mathbf{K}) \subseteq \mathbf{UC}_\Delta$ whenever $\mathbf{K} \in \mathbf{AC}_\Delta$, and that $\mathbf{H}(\mathbf{K}) \in \mathbf{UC}_\Delta$ whenever $\mathbf{K} \in \mathbf{UC}_\Delta$. The derivation of these results is simple, but not quite direct. E.g., we do not know whether, under our definitions of $\mathbf{AC}_\Delta$ and $\mathbf{PC}_\Delta$, the formula $\mathbf{K} \in \mathbf{AC}_\Delta$ implies $\mathbf{H}(\mathbf{K}) \in \mathbf{PC}_\Delta$, although with the help of 1.9 we can show that it implies $\mathbf{SH}(\mathbf{K}) \in \mathbf{UC}_\Delta$.

The next theorem is of a more special character. In a somewhat more general form (given in Theorem 1.14 below) it will be applied in § 2, in the proof of 2.2.

Theorem 1.11. *Let* $\mathbf{K}$ *be a class of relational systems of arbitrary order. If* $\mathbf{K} \in \mathbf{UC}_\Delta$ *and* $\mathbf{P}_3(\mathbf{K}) \subseteq \mathbf{K}$, *then* $\mathbf{P}(\mathbf{K}) \subseteq \mathbf{K}$.

Proof: By hypothesis, $\mathbf{K}$ is the class of all models of a certain set Σ of universal sentences (belonging to the theory $T(\mathbf{R})$ where $\mathbf{R}$ is the similarity class including $\mathbf{K}$). It is well known that every universal sentence is equivalent to a conjunction of sentences of the form

$$\bigwedge v_{\varkappa_0} \ldots \bigwedge v_{\varkappa_\nu} (\varphi_0 \vee \ldots \vee \varphi_\pi) \tag{1}$$

where each of the expressions $\varphi_0, \ldots, \varphi_\pi$ is either an atomic formula or the negation of an atomic formula. Hence, without loss of generality, we can assume that all sentences in Σ are of form (1). The hypothesis $\mathbf{P}_3(\mathbf{K}) \subseteq \mathbf{K}$ implies that $\mathfrak{R} \times \mathfrak{S} \in \mathbf{K}$ for any two systems $\mathfrak{R}$, $\mathfrak{S}$ of $\mathbf{K}$. Using this fact and applying an argument essentially due to McKinsey [13], pp. 66 f., we show that every sentence σ in Σ can be replaced by a sentence σ' which is also of form (1) and in which, in addition, at most one term of the disjunction $\varphi_0 \vee \ldots \vee \varphi_\pi$ is an atomic formula (while the remaining terms are negations of atomic formulas). We thus arrive at a set Σ' of sentences such that (i) $\mathbf{K}$ is the class of all models of Σ', and (ii) every sentence in Σ' is of form (1) and has the additional property just mentioned. From (ii) it is easily seen that a sentence of Σ' always holds in the cardinal product $\mathfrak{P}_{i \in I}(\mathfrak{R}_i)$ of systems $\mathfrak{R}_i$ whenever it holds in all the systems $\mathfrak{R}_i$ $(i \in I)$; cf., e.g., Horn [9], p. 17, Th. 4 [11]). Hence, by (i), we arrive at the desired conclusion: $\mathbf{P}(\mathbf{K}) \subseteq \mathbf{K}$.

As is easily seen, under the assumption that $\mathbf{K} \in \mathbf{UC}_\Delta$ (or $\mathbf{K} \in \mathbf{AC}_\Delta$ or, more generally, $\mathbf{I}(\mathbf{K}) \subseteq \mathbf{K}$), the formula $\mathbf{P}_3(\mathbf{K}) \subseteq \mathbf{K}$ occurring in the hypothesis of 1.11 is equivalent to the formula $\mathbf{P}_\omega(\mathbf{K}) \subseteq \mathbf{K}$.

It may be mentioned that Vaught has recently obtained the following result which presents a far reaching improvement of 1.11:

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[11]) McKinsey in [19] and Horn in [9] concern themselves exclusively with algebras (in the sense of § 2 of this paper). However, their results can easily be extended to arbitrary relational systems once the notion of a cardinal product has been properly defined for these systems.

Let $\mathbf{K}$ *and* $\mathbf{L}$ *be two classes of relational systems of arbitrary order. If* $\mathbf{K} \in \mathbf{AC}_\Delta$ *and* $\mathbf{P}_\omega(\mathbf{L}) \subseteq \mathbf{K}$, *then* $\mathbf{P}(\mathbf{L}) \subseteq \mathbf{K}$.

In addition to the notions discussed so far in this section, we can consider more general notions of a relative character. Let $\mathbf{M}$ be a class of relational systems included in a similarity class $\mathbf{R}$, and let $\mathbf{K}$ be another subclass of $\mathbf{R}$. $\mathbf{K}$ is said to be *arithmetical relative to* $\mathbf{M}$, in symbols $\mathbf{K} \in \mathbf{AC}(\mathbf{M})$, if $\mathbf{K} = \mathbf{L} \cap \mathbf{M}$ for some class $\mathbf{L} \in \mathbf{AC}$; in other words, if $\mathbf{K}$ consists of all those systems of $\mathbf{M}$ which are models of a certain sentence σ of the formalized theory $T(\mathbf{R})$. In an entirely analogous way we define the notions $\mathbf{AC}_\Delta(\mathbf{M})$, $\mathbf{UC}(\mathbf{M})$, and $\mathbf{UC}_\Delta(\mathbf{M})$. In case $\mathbf{M} = \mathbf{R}$, $\mathbf{AC}(\mathbf{M})$ obviously coincides with $\mathbf{AC}$; in case $\mathbf{M} \in \mathbf{AC}$, $\mathbf{AC}(\mathbf{M})$ is simply the family of all those subclasses of $\mathbf{M}$ which belong to $\mathbf{AC}$; similarly for $\mathbf{AC}_\Delta(\mathbf{M})$, $\mathbf{UC}(\mathbf{M})$, and $\mathbf{UC}_\Delta(\mathbf{M})$. Various results stated above and involving $\mathbf{UC}$ and $\mathbf{UC}_\Delta$ can easily be extended, with some minor changes in formulations, to the relativized notions $\mathbf{UC}(\mathbf{M})$ and $\mathbf{UC}_\Delta(\mathbf{M})$; in certain cases, however, this requires additional assumptions concerning $\mathbf{M}$. For further reference some extensions thus obtained will be stated here explicitly.

Theorem 1.12. *Let* $\mathbf{K}$ *and* $\mathbf{M}$ *be two classes of relational systems of finite order. For* $\mathbf{K} \in \mathbf{UC}_\Delta(\mathbf{M})$ *it is necessary and sufficient that the following conditions be satisfied:*

(i) $\mathbf{K} \subseteq \mathbf{M}$;

(ii) *for every* $\mathfrak{R} \in \mathbf{M}$, *if* $\mathbf{S}_\omega(\mathbf{R}) \subseteq \mathbf{IS}(\mathbf{K})$, *then* $\mathfrak{R} \in \mathbf{K}$.

Proof: The necessity of conditions (i) and (ii) obviously follows from the definition of $\mathbf{UC}_\Delta(\mathbf{M})$ and from Theorem 1.2. Assume now that these two conditions are satisfied, and let $\mathbf{K}'$ be the intersection of all classes $\mathbf{L} \in \mathbf{UC}$ which include $\mathbf{K}$. By arguing as in the proof of 1.2 (second part), we easily show with the help of 1.1 that $\mathbf{K} = \mathbf{K}' \cap \mathbf{M}$; since $\mathbf{K}' \in \mathbf{UC}_\Delta$, we conclude that $\mathbf{K} \in \mathbf{UC}_\Delta(\mathbf{M})$.

Theorem 1.13. *Let* $\mathbf{K}$ *and* $\mathbf{M}$ *be two classes of relational systems of arbitrary order. If* $\mathbf{M} \in \mathbf{AC}_\Delta$ *and* $\mathbf{K} \in \mathbf{AC}_\Delta(\mathbf{M})$, *then* $\mathbf{S}(\mathbf{K}) \cap \mathbf{M} \in \mathbf{UC}_\Delta(\mathbf{M})$.

Proof: By hypothesis, $\mathbf{K} = \mathbf{L} \cap \mathbf{M}$ for some $\mathbf{L} \in \mathbf{AC}_\Delta$; hence $\mathbf{K} \in \mathbf{AC}_\Delta$. The conclusion now easily follows by 1.6.

Theorem 1.14. *Let* $\mathbf{K}$ *and* $\mathbf{M}$ *be two classes of relational systems of arbitrary order. If* $\mathbf{P}(\mathbf{M}) \subseteq \mathbf{M}$, $\mathbf{K} \in \mathbf{UC}_\Delta(\mathbf{M})$, *and* $\mathbf{P}_3(\mathbf{K}) \subseteq \mathbf{K}$, *then* $\mathbf{P}(\mathbf{K}) \subseteq \mathbf{K}$.

Proof: Entirely analogous to that of 1.11.

By taking in 1.12—1.14 the whole similarity class including $\mathbf{K}$ for $\mathbf{M}$, we clearly obtain 1.2, 1.6, and 1.11.