Notes on Tarski's Definition of Satisfaction

Here is a statement of the usual syntax of FOL

A ***syntax for first-order language*** **FOL** consists of a series of sets ("parts of speech") $\mathbf{Vbls_{FOL}}$, $\mathbf{Cons_{FOL}}$, $\mathbf{Funcs_{FOL}}$, $\mathbf{Preds_{FOL}}$, $\mathbf{Trms_{FOL}}$, $\mathbf{AF_{FOL}}$ and $\mathbf{F_{FOL}}$, and the rule set $\mathbf{R_{FOL}}$ meeting the following conditions:

We stipulate that the following sets exist, called sets of ***atomic*** (or ***basic***) ***expressions***:

- the (infinite) set of ***variables***: $\mathbf{Vbls_{FOL}} = \{v_1,\ldots,v_n,\ldots\}$;
- some set $\mathbf{Cons_{FOL}}$ of ***constants*** (***proper names***) drawn from (*i.e.* subset of): $\{c_1,\ldots,c_n,\ldots\}$;
- some set $\mathbf{Preds_{FOL}}$ of ***predicates*** drawn from: $\{P^0_1,\ldots,P^0_n,\ldots;P^1_1,\ldots,P^1_n,\ldots;\ldots;P^m_1,\ldots,P^m_n,\ldots;\ldots\}$; it is stipulated that P^0_1 is $\perp$, called the ***contradiction symbol*** (it's meaning is explained below);
- some set $\mathbf{Funcs_{FOL}}$ of ***functors*** drawn from: $\{f^1_1,\ldots,f^1_n,\ldots;\ldots;f^m_1,\ldots,f^m_n,\ldots;\ldots\}$.

The set $\mathbf{R_{FOL}}$ of ***grammatical rules*** includes $R_{\sim}$,$R_{\wedge}$,$R_{\vee}$,$R_{\rightarrow}$, and $R_{\leftrightarrow}$ from sentential syntax and three new rules:

- R_{AF} takes a symbol x and a string of n items $y_1\ldots y_n$ and makes up the string $xy_1\ldots y_n$:
  - *i.e.* $R_{AF}(x,y_1,\ldots,y_n)= xy_1\ldots y_n$;
- R_{Func} takes a symbol x and a string of n items $y_1\ldots y_n$ and makes up the string $x(y_1\ldots y_n)$;
  - *i.e.* $R_{Func}(x,y_1,\ldots,y_n)= x(y_1\ldots y_n)$;
- $R_{\forall}$ takes the sign x and string y and makes up the string $\forall xy$; *i.e.* $R_{\forall}(x,y)=\forall xy$.

The set $\mathbf{Trms_{FOL}}$ of ***terms*** is defined inductively:

1. **Basis Clause.** All constants and variables are terms (i.e. $\mathbf{Vbls_{FOL}}$ and $\mathbf{Cons_{FOL}}$ are subsets $\mathbf{Trms_{FOL}}$).
2. **Inductive Clause.** If $t_1,\ldots,t_n$ are terms and f^n is a functor, then the result $f^n(t_1\ldots t_n)$ made up by apply to them the rule R_{Func} is a term.
3. Nothing else is a term.

The set $\mathbf{AF_{FOL}}$ of ***atomic formulas*** of **FOL** generated by $\mathbf{Trms_{FOL}}$, $\mathbf{Preds_{FoL}}$, and $\mathbf{Funcs_{FOL}}$ is the set of all $P^n_m t_1\ldots t_n$ made up by applying the rule R_{AF} to a predicate P^n_m and the string of terms $t_1,\ldots,t_n$.

The set $\mathbf{F_{FOL}}$ of ***formulas*** (also called ***the language***) generated by $\mathbf{Trms_{FOL}}$, $\mathbf{Preds_{FoL}}$ and $\mathbf{Funcs_{FOL}}$ is inductively defined:

1. **Basis Clause.** If P is in $\mathbf{AF_{FOL}}$ then P is in $\mathbf{F_{FOL}}$ (i.e. $\mathbf{AF_{FOL}}$ is a subset of $\mathbf{F_{FOL}}$).
2. **Inductive Clause.** If P and Q are in $\mathbf{F_{FOL}}$, and v is a variable, then the strings $\sim P$, $(P\wedge Q)$, $(P\vee Q)$, $(P\rightarrow Q)$, $(P\leftrightarrow Q)$, and $\forall vP$ that result from applying to them the rules $R_{\sim}$,$R_{\wedge}$,$R_{\vee}$,$R_{\rightarrow}$, $R_{\leftrightarrow}$, and $R_{\forall}$ are all in $\mathbf{F_{FOL}}$.
3. Nothing else is in $\mathbf{F_{FOL}}$.

Notes on Tarski's Definition of Satisfaction

Here are two statements of the semantics of FOL. The first is what is now the one commonly found in logic texts. The second is closer to one of Tarski's in the 1950's in which it is perfectly clear that the notion of truth is extensional, i.e. that truth of a formula, even that of quantified formulas, is defined as a function of the truth of its parts.

Let $\mathbf{F_{FOL}}$ be a first-order language.

A ***model*** or ***structure for basic expressions of*** $\mathbf{F_{FOL}}$ ***relative to a non-empty domain*** D and an interpretation operation $\Im$ is any A=<D,$\Im$>, that meets the following conditions:

- $D\neq\varnothing$.
- Every constant *c* is assigned by $\Im$ to an object in D. That is, $\Im(c)\in D$.
- Every *n*-place predicate is assigned by $\Im$ to an *n*-place relation on objects in D. There are three special cases:
  1. If n=1, then $\Im(P^1)\subseteq D$, *i.e.* a one place predicate P^1 stands for a subset of D.
  2. If n>2, then $\Im(P^n)\subseteq D^n$, *i.e.* $\Im(P^n)$ is an *n*-place relation on elements of D, *i.e.* some set of *n*-tuples drawn from D^n. If the syntax specifies that first 2-place predicate P^2_1 is the identity predicate =, then it is required that $\Im(P^2_1)$ be the identity relation on D.
  3. If n=0, then $\Im(P^0_i)\in\{T,F\}$, *i.e.* 0-place predicates function semantically like sentence letters of sentential logic in that they do require any terms to their right and they stand for a truth-value. Furthermore, the syntax specifies that the first 0-place predicate P^0_1 is the contradiction sign $\perp$. It is required that $\Im(\perp)$=F, *i.e.* $\perp$ always takes the value F.
- Every *n*-place functor f^n is assigned by $\Im$ to an *n*-place function (also called an operation) on objects in D. That is, $\Im(f^n)\in D^{D^n}$.

Let A=<D,$\Im$> be a model.

A ***variable assignment*** over D for $\mathbf{F_{FOL}}$ is any function ***s*** of mapping the set of variables into D.

An ***interpretation*** $\Im$ ***relative to a model*** $\mathbf{A}$=<D,$\Im$> ***and an assignment s of the variable over*** D for $\mathbf{F_{FOL}}$, briefly $\Im^{\mathrm{A}}_{s}$, is defined inductively:

- **Basis Clause.** If *t* is a constant, then $\Im^{\mathrm{A}}_{s}(t)=\Im(t)$.
  If *v* is a variable, then $\Im^{\mathrm{A}}_{s}(v)=\boldsymbol{s}(v)$.
- **Inductive Clause.** If *t* is some complex term, then $f^n(t_1...t_n)$, $\Im^{\mathrm{A}}_{s}(f^n(t_1...t_n))=\Im(f^n)(\Im^{\mathrm{A}}_{s}(t_1),...,\Im^{\mathrm{A}}_{s}(t_n))$.

P ***is satisfied in model*** A=<D,$\Im$> ***relative to an a variable assignment s over*** D for $\mathbf{F_{FOL}}$ (abbreviated equivalent as $\mathrm{A}_{\boldsymbol{s}}\models P$ or $\Im^{\mathrm{A}}_{s}(P)$=T) is defined inductively:

- **Basis Clause.** An atomic formula $P^n t_1,...,t_n$ is T in $\Im^{\mathrm{A}}_{s}$, iff the objects picked out by its terms under $\Im^{\mathrm{A}}_{s}$ (in order) stand in the relation picked out in $\Im$ by its predicate. In symbols

  $\mathrm{A}_{\boldsymbol{s}}\models P^n t_1,...,t_n$ iff $<\Im^{\mathrm{A}}_{s}t_1),..., \Im^{\mathrm{A}}_{s}(t_n)>\in \Im(P^n)$

  or equivalently,

  $\Im^{\mathrm{A}}_{s}(P^n t_1,...,t_n)$=T) iff $< \Im^{\mathrm{A}}_{s}(t_1),..., \Im^{\mathrm{A}}_{s}(t_n)>\in \Im(P^n)$
- **Inductive Clauses.** The satisfaction of a molecular formula relative to variable assignment is broken down into case one for each formation rule of the syntax:

  $\mathrm{A}_{\boldsymbol{s}}\models\sim P$ iff not $\mathrm{A}_{\boldsymbol{s}}\models P$; or $\quad\Im^{\mathrm{A}}_{s}(\sim P)$=T iff $\Im^{\mathrm{A}}_{s}$ $P)\neq$T

  $\mathrm{A}_{\boldsymbol{s}}\models P\wedge Q$ iff ($\mathrm{A}_{\boldsymbol{s}}\models P$ and $\mathrm{A}_{\boldsymbol{s}}\models Q$); or $\quad\Im^{\mathrm{A}}_{s}(P\wedge Q)$=T iff, $\Im^{\mathrm{A}}_{s}(P)$=T and $\Im^{\mathrm{A}}_{s}(Q)$=T

  $\mathrm{A}_{\boldsymbol{s}}\models P\vee Q$ iff ($\mathrm{A}_{\boldsymbol{s}}\models P$ or $\mathrm{A}_{\boldsymbol{s}}\models Q$);or $\quad\Im^{\mathrm{A}}_{s}(P\vee Q)$=T iff, $\Im^{\mathrm{A}}_{s}(P)$=T or $\Im^{\mathrm{A}}_{s}(Q)$=T

  $\mathrm{A}_{\boldsymbol{s}}\models P\rightarrow Q$ iff (not $\mathrm{A}_{\boldsymbol{s}}\models P$ or $\mathrm{A}_{\boldsymbol{s}}\models Q$); or $\quad\Im^{\mathrm{A}}_{s}(P\rightarrow Q)$=T iff, $\Im^{\mathrm{A}}_{s}(P)\neq$T or $\Im^{\mathrm{A}}_{s}(Q)$=T

  $\mathrm{A}_{\boldsymbol{s}}\models P\leftrightarrow Q$ iff ($\mathrm{A}_{\boldsymbol{s}}\models P$ iff $\mathrm{A}_{\boldsymbol{s}}\models Q$);or $\quad\Im^{\mathrm{A}}_{s}(P\leftrightarrow Q)$=T iff, $\Im^{\mathrm{A}}_{s}(P)$=T iff $\Im^{\mathrm{A}}_{s}(Q)$=T

  $\mathrm{A}_{\boldsymbol{s}}\models\forall xPx$ iff for any *x*-variant $\boldsymbol{s}'$ of $\boldsymbol{s}$, $\mathrm{A}_{\boldsymbol{s}'}\models Px$, or $\quad\Im^{\mathrm{A}}_{s}(\forall xPx)$=T iff for any *x*-variant $\boldsymbol{s}'$ of $\boldsymbol{s}$, $\Im^{\mathrm{A}}_{s}(Px)$=T

P ***is true*** **(simpliciter)** *in* A=<D,$\Im$> (abbreviated $\mathrm{A}\models P$) iff, for all ***s*** over D, $\mathrm{A}_{\boldsymbol{s}}\models P$.

X ***logically entails*** *P* (briefly $X\models P$) iff, for all A, if (for all Q in X, $\mathrm{A}\models Q$), then $\mathrm{A}\models P$).

P is ***valid*** (briefly $\models P$) iff, for all A, $\mathrm{A}\models P$.

A model or ***structure for basic expressions of*** $\mathbf{F_{FOL}}$ ***relative to a non-empty domain*** D and an interpretation operation $\Im$ is any A=<D,$\Im$>, that meets the following conditions:

- $D\neq\varnothing$.
- Every constant *c* is assigned by $\Im$ to an object in D. That is, $\Im(c)\in D$.
- Every *n*-place predicate is assigned by $\Im$ to an *n*-place relation on objects in D. There are three special cases:
  1. If n=1, then $\Im(P^1)\subseteq D$, *i.e.* a one place predicate P^1 stands for a subset of D.
  2. If n>2, then $\Im(P^n)\subseteq D^n$, *i.e.* $\Im(P^n)$ is an *n*-place relation on elements of D, *i.e.* some set of *n*-tuples drawn from D^n. If the syntax specifies that first 2-place predicate $P^2{}_1$ is the identity predicate =, then it is required that $\Im(P^2{}_1)$ be the identity relation on D.
  3. If n=0, then $\Im(P^0{}_i)\in\{T,F\}$, *i.e.* 0-place predicates function semantically like sentence letters of sentential logic in that they do require any terms to their right and they stand for a truth-value. Furthermore, the syntax specifies that the first 0-place predicate $P^0{}_1$ is the contradiction sign $\perp$. It is required that $\Im(\perp)$=F, *i.e.* $\perp$ always takes the value F.
- Every *n*-place functor f^n is assigned by $\Im$ to an *n*-place function (also called an operation) on objects in D. That is, $\Im(f^n)\in D^{D^n}$.

Let A=<D,$\Im$> be a model.

A ***satisfaction sequence*** over D for $\mathbf{F_{FOL}}$ is any denumerable sequence **s** of elements in D, i.e. any element of D^ω.

An ***interpretation*** $\Im$ of ***terms relative to a model*** $\mathbf{A}$=<D,$\Im$> ***and a satisfaction sequence s over*** D for $\mathbf{F_{FOL}}$, briefly $\Im^{\mathrm{A}}_{s}$, is defined inductively:

- **Basis Clause.** If *t* is a constant, then $\Im^{\mathrm{A}}_{s}(t)=\Im(t)$.
  If *v* is a variable, then $\Im^{\mathrm{A}}_{s}(v)=\mathbf{s}_i$, the *i*-th element of the sequence **s**.
- **Inductive Clause.** If *t* is some complex term, then $f^n(t_1...t_n)$, $\Im^{\mathrm{A}}_{s}(f^n(t_1...t_n))=\Im(f^n)(\Im^{\mathrm{A}}_{s}(t_1),...,\Im^{\mathrm{A}}_{s}(t_n))$.

The set $\Im^{\mathrm{A}}(P)$ ***of sequences that satisfies*** *P* ***in model*** A=<D,$\Im$> $\mathbf{F_{FOL}}$ $\Im^{\mathrm{A}}(P)$=T) is defined inductively:

- **Basis Clause.** For an atomic formula $P^n t_1,...,t_n$,
  $\Im^{\mathrm{A}}(P^n t_1,...,t_n) = \{ \mathbf{s}\in D^\omega \mid <\Im^{\mathrm{A}}_{s}(t_1),..., \Im^{\mathrm{A}}_{s}(t_n)>\in \Im(P^n)\}$.

- **Inductive Clauses.** The definition of $\Im^{\mathrm{A}}(P)$ a molecular formula *P* is broken down into case one for each formation rule of the syntax:
  $\Im^{\mathrm{A}}(\sim P)= D^\omega-\Im^{\mathrm{A}}(P)$,
  $\Im^{\mathrm{A}}(P\wedge Q)=\Im^{\mathrm{A}}(P)\cap\Im^{\mathrm{A}}(Q)$,
  $\Im^{\mathrm{A}}(P\vee Q) =\Im^{\mathrm{A}}(P)\cup\Im^{\mathrm{A}}(Q)$,
  $\Im^{\mathrm{A}}(P\rightarrow Q)= [D^\omega-\Im^{\mathrm{A}}(P)]\cup\Im^{\mathrm{A}}(Q)$,
  $\Im^{\mathrm{A}}(\forall xP)=D^\omega$ if $\Im^{\mathrm{A}}(P)=D^\omega$, $\Im^{\mathrm{A}}(\forall xP)=\varnothing$ otherwise.

P ***is true*** **(simpliciter)** ***in*** A=<D,$\Im$> (abbreviated $\mathrm{A}\models P$) iff $\Im^{\mathrm{A}}(P)=D^\omega$.
X ***logically entails*** *P* (briefly $X\models P$) iff, for all A, if (for all Q in X, $\mathrm{A}\models Q$), then $\mathrm{A}\models P$).
P is ***valid*** (briefly $\models P$) iff, for all A, $\mathrm{A}\models P$.