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WHAT IS A SYLLOGISM?

Łukasiewicz rejected the traditional treatment of syllogisms as arguments and claimed that the authentic Aristotelian syllogism is a conditional whose antecedent is the conjunction of the premisses and whose consequent is the conclusion.[1] For Aristotle, however, a syllogism is essentially something with a deductive structure as well as premisses and conclusions. Consider, for example, his distinction between ostensive and per impossibile syllogisms. This is entirely a matter of how their conclusions are derived and not at all a matter of what conclusions are derivable (*An. Pr.* 45a26, 62b38). Aristotle writes as if he is marking a genuine distinction between two classes of syllogisms, but his way of going about it would be senseless if a syllogism were uniquely determined, as a Łukasiewiczian conditional is, by its premisses and conclusions. Moreover, everything suggests that Aristotle is concerned here with the distinction between direct and indirect patterns of deduction, as exemplified in the contrast between the ostensive argument 'P, Q, so R' and the per impossibile one, 'P, suppose not R, then not Q, so R'.[2]

Łukasiewicz may be equally ready to distinguish between different patterns of deduction, but for him this will as yet have nothing to do with syllogism: syllogisms for him have no more intrinsic connection with deduction than any other conditionals, and in particular Aristotelian 'demonstrations' are mere conditionals and not, ironically, proofs of anything. Thus if Łukasiewicz's treatment is to accommodate Aristotle's distinction, he must both show that the distinction makes sense when interpreted as applying to conditionals and that this sense is such as to establish some connection between ostensive and per impossibile conditionals and ostensive and per impossibile deduction. He attempts neither task, and perhaps it is sufficient to give a bare indication of the difficulties he would have to overcome. If we interpret the distinction as applying to conditionals, then, as was said at the beginning, we must not think that we are dividing conditionals as such into two classes; the grounds for calling a conditional ostensive or per impossibile must be sought outside

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the conditional itself, and the same conditional may be called now ostensive, now per impossibile. Two appropriate distinctions immediately suggest themselves. By the first, a conditional is to be called ostensive or per impossibile according as it is used in an ostensive or a per impossibile proof. Alternatively, a conditional is to be called ostensive or per impossibile according as it is itself established by means of an ostensive proof or a per impossibile one. Both of these succeed in making a connection with ostensive and per impossibile deduction, but neither stands up for a moment against the text of the *Prior Analytics*.[3] If, instead, we start from the text, a third way of drawing the distinction is suggested by 62b29 ff.: a conditional is ostensive when it is used in a derivation of its consequent, per impossibile when it is used in a refutation of one of its antecedents. But this is no better than the other two, for it fails to make any connection with ostensive and per impossibile deduction. A conditional can perfectly well be used in a per impossibile derivation of its consequent, e.g., '$P\&Q \supset R$, P, suppose not R, then not Q, but Q, so R', and the obvious way to use a conditional in a refutation of one of its antecedents is by means of an ostensive argument, viz. '$P\&Q \supset R$, P, not R, so not Q'.

Łukasiewicz's treatment similarly fails to do justice to Aristotle's theory of the reduction of syllogisms. A syllogism may be valid, in that its conclusion follows from its premisses, but it may nonetheless be 'imperfect' because it fails to *show that* the conclusion follows.[4] Aristotle's procedure in such a case is to 'reduce' the imperfect syllogism to a perfect one by filling in its intervals with intermediate steps (*An. Pr.* 24b24, *An. Post.* 79a30). This description makes excellent sense if syllogisms are regarded as arguments – to reduce an imperfect syllogism is to make it perspicuous by expanding it so that it has a finer and hence argumentatively more satisfying structure. We see, too, that the additional material may be inserted so as to produce either a fuller ostensive argument or a per impossibile one. For example, 'P, Q, so R' may be expanded either into 'P, Q, so S, so T, so U, oR' or into 'P, suppose not R, then not Q, but Q, so R'. On the other hand, as Łukasiewicz himself admits (*op. cit.*, p. 44), the proof of a conditional does not fit Aristotle's description of reduction. A further difficulty for Łukasiewicz's treatment is that to prove a conditional we need the deductive machinery of propositional logic – something which is conspicuous by its absence from Aristotle's writings. And even if we read the necessary propositional logic back into the Prior Analytics we must not think that this will bring Aristotle's and Łukasiewicz's ideas of reduction any closer together (Łukasiewicz, *op. cit.*, pp. 54–6). Thus the price of accepting Łukasiewicz's account of syllogisms is his wholesale rejection of Aristotle's account of their reduction.

The traditional treatment is free from these objections, for, as we have seen, an argument has a distinctive structure as well as having premisses and a conclusion. Nevertheless there is a decisive objection to identifying syllogisms with arguments, comparable to Frege's objection to identifying propositions with assertions. Consider as examples, 'P, Q, so R' and 'P, suppose Q, then R, but not R, so not Q'. Frege's point is that if we are to discern the same proposition Q as occurring in the first example (where it is asserted) and in the second (where it is not), then the proposition itself must not be identified with its assertion, but must be something neutral with respect to assertion, supposition, denial, etc. Now it is clear from Aristotle's discussion of the reduction of imperfect syllogisms that if we are to discern a syllogism in the first example, we must be prepared to recognise the same syllogism at the beginning of the second. That is, we must discern the same syllogism in 'P, suppose Q, then R' as in 'P, Q, so R'. But these are different arguments, just as supposition is different from assertion. It follows that the syllogism itself must not be identified with either argument, but must be something neutral with respect to a variety of possible argumentative uses.

The object that appears to combine the requisite argumentative structure with the requisite neutrality is a *proof-sequence* or *deduction*. A proof-sequence may be either formal or informal, according as its component statements (and the relation of implication which holds between them) belong to a formalised language or to ordinary informal mathematics. Thus to equate syllogisms with proof-sequences or deductions is not to prejudge the question of formalisation; it is only to revive Aristotle's own definition of a syllogism as 'discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so' (24b18), while noting that τεθέντων (stated) is neutral with regard to assertion. Among possible deductions in formalised languages we may however distinguish between those of which it is merely required that each succeeding wff should be implied by previous ones, and those in which each step must be justified by one of a given number of primitive rules of inference. It is, of course, the latter kind which is familiar from the construction of axiomatic calculi, but the distinction provides the formal counterpart of Aristotle's distinction between syllogisms in general and those 'perfect' syllogisms in which each step is self-evident. We may note too that the definition of a formal deduction is easily made to cover indirect patterns of proof like reductio ad impossibile; see Definition 1 below.

Given that Aristotle is concerned with deductions, i.e., with how conclusions may be derived, we should expect him to be equally concerned with deducibility, i.e., with what conclusions are derivable. We should also bear in mind that deducibility can be discussed either by means of verbs such as '... implies ...' or '... follows from ...', or by means of conditionals such as 'if ... then necessarily ...' or plain 'if ... then ...'; the difference between the verbal form and the conditional form being merely the difference between mention and use. In this way I think we can explain Aristotle's frequent use of conditionals in his discussions of syllogistic without needing to identify, as Łukasiewicz does, the conditionals with the syllogisms themselves. Moreover, since deducibility is equivalent to the existence of a deduction, for to say that Q follows from P is equivalent to saying that there exists a deduction of Q from P, we shall at the same time be able to explain the frequent occurrence of such phrases as 'there will be a syllogism' or its opposite, 'no syllogism will be possible'.

It remains to enquire what it might mean for the premisses of a syllogism to imply the conclusion. By building onto the propositional calculus[5] Łukasiewicz in effect equates syllogistic implication with strict implication and thereby commits himself to embracing the novel moods corresponding to such theorems as $Aab \;\&\; Oab \supset Icd$ or $Aab \;\&\; Acd \supset Aee$. On the other hand Aristotle's own omission of these syllogisms of strict implication, as they may be called, can hardly be written off as an oversight. For they violate his dictum that 'a syllogism relating this to that proceeds from premisses which relate this to that' (41a6). This dictum is part of a principle which is absolutely fundamental to his syllogistic, namely the principle that the premisses of a syllogism must form a chain of predications linking the terms of the conclusion. Thus his doctrine of the figures, which provides the framework for his detailed investigation of syllogistic, is founded on this principle (40b30 ff.) Not less important is that the chain principle is essential to the success of his attempt at a completeness proof for the syllogistic. By this I mean his attempt to show that every valid syllogistic inference, regardless of the number of premisses, can be carried out by means of a succession of two-premiss syllogisms.[6] The proof turns on the argument that if neither of two pairs of premisses imply a conclusion, nor do all four premisses taken together. At first sight this looks like a very poor argument indeed, for it appears to overlook the obvious point that the logical strength of a number of premisses taken together is not limited to the sum of their separate strengths. If however we are assuming from the start that our premisses, if they are to be usable, have got to fit together in accordance with the chain principle, we are thereby placing a severe restriction on the way different pairs of premisses can genuinely augment one another; and we obtain an argument which if not absolutely conclusive is no longer despicable.

One is thus led to ask what account of implication, if any, will harmonise with Aristotle's chain principle for syllogisms. The question invites a logical rather than a historical answer, and there are two constraints governing any possible answer. Firstly, we must either exclude irrelevant premisses or else restrict reductio ad impossibile. For otherwise by using $P, Q \vdash P$ in a per impossibile argument we could derive anything from a pair of contradictory premisses. For example, we could validate the distinctly non-Aristotelian mood $Aab, Oab \vdash Icd$ by arguing 'Aab, suppose Ecd, then Aab, but Oab, so Icd'. Secondly, even to permit a change in the multiplicity of occurrence of a relevant premiss will be incompatible with the free use of reductio ad impossibile. For otherwise we could start with $Aca, Aca \vdash Iaa$, which is an instance of Darapti, and by ignoring the repetition of the premiss obtain $Aca \vdash Iaa$. But $Aab, Eab \vdash Eaa$ by Cesare, and putting these together in a reductio ad impossibile yields the non-Aristotelian mood $Aab, Eab \vdash Oca$.

I shall offer my answer to my question in the shape of a formal system in which I shall put into practice the idea of treating syllogisms as deductions, and which is intended to match as closely as possible Aristotle's own axiomatisation of the syllogistic by means of conversion, reductio ad impossibile, and the two universal moods of the first figure. The definition of deducibility will ensure that in counting premisses attention is paid to whether and how often they are used in a deduction. This is in order to satisfy the constraints mentioned in the preceding paragraph, and also to harmonise with Aristotle's own remarks about the numbers of premisses (cf. *An. Pr.* 42b1, *An. Post.* 86b13); though it should be stressed that the case for treating syllogisms as deductions is independent of the case for this particular treatment of deducibility. The principal result proved is that the system is complete with respect to the valid moods, where these are defined in the spirit of the Aristotelian figures but without his restriction to the special case of two premisses. I shall also show that the system possesses a simple decision procedure. Finally I shall test the system's harmony with Aristotle's ideas by considering it in relation to his well-known discussion of syllogisms with false premisses.

The vocabulary of the system consists of the symbols A, E, I, O, together with an infinite stock of terms. The wffs are Aab, Eab, Iab and Oab, for all terms a and b. Aab and Oab will be said to be each other's *contradictory*; likewise Eab and Iab. To indicate the contradictory of an otherwise unspecified wff P I shall write $\bar{P}$, with the remark that this notation is part of the metatheory and not a connective belonging to the system itself. A corollary of the definition that will be taken for granted in the sequel, is that if $P=\bar{Q}$ then $Q=\bar{P}$. Lower-case variables will be used to stand for terms, reserving P, Q, R for wffs and X, Y, Z for sets of wffs. Commas will be used to indicate the union or augmentation of sets, e.g., X, Y or X, P, and angled brackets will be used for sequences, e.g., $\langle P, P, Q\rangle$. Since it is going to be essential to our treatment of deducibility that we should be able to distinguish between cases where the same premiss occurs a different number of times, we shall want to construe the notion of a set of wffs so as to take account of their multiplicity of occurrence. This is most easily done by taking 'set of wffs' always to mean 'set of *occurrences* of wffs', and counting the number of members accordingly. For example, P, P, Q will be a different set from P, Q, and the former will have three members while the latter has only two.

The system has no axioms but has the following rules of inference;

Rule 1. From Aab, Abc infer Aac
Rule 2. From Aab, Ebc infer Eac
Rule 3. From Eba infer Eab
Rule 4. From Aba infer Iab

The definition of formal deduction is best given inductively:

DEFINITION 1. (i) $\langle Q\rangle$ is a deduction of Q from itself. (ii) If, for each i,

$\langle \ldots P_i \rangle$ is a deduction of P_i from X_i, and if Q follows from $P_1, \ldots, P_n$ by a rule of inference, then $\langle \ldots P_1, \ldots, \ldots P_n, Q \rangle$ is a deduction of Q from $X_1, \ldots, X_n$. (iii) If $\langle \ldots P \rangle$ is a deduction of P from $X_1, \bar{Q}$, and $\langle \ldots \bar{P} \rangle$ is a deduction of $\bar{P}$ from X_2, then $\langle \ldots P, \ldots \bar{P}, Q \rangle$ is a deduction of Q from X_1, X_2.

To signify that there exists a deduction of Q from X we write $X \vdash Q$.

The first two clauses in Definition 1 are intended to resemble the familiar idea of a formal deduction of Q from X as a finite sequence of wffs ending in Q, each wff either belonging to X or following from preceding wffs by a rule of inference. The third clause is intended to accommodate reductio ad impossibile arguments. For example, since $\langle Axn, Anm, Axm \rangle$ is a deduction of Axm from Axn, Anm by rule 1, and since $\langle Oxm \rangle$ is a deduction of Oxm from itself, and given that A and O wffs are contradictories, it follows that $\langle Axn, Anm, Axm, Oxm, Oxn \rangle$ is a deduction of Oxn from Anm, Oxm. The per impossibile structure of this deduction is brought out in the corresponding argument, 'suppose Axn, then since Anm, Axm; but Oxm, so Oxn'. The terms have been lettered to make it easier to compare this way of validating Baroco with that of Aristotle (27a37); it is instructive to compare it with Łukasiewicz's treatment of the same passage (Łukasiewicz, *op. cit.*, p. 54ff.).

The omission of axioms from the system is intentional, but it would be a straightforward matter to allow for axioms in Definition 1, and to allow for theorems by supplementing the definition of 'deduction of Q from X' with a definition of 'proof of Q'. This would incidentally supply the analogue, to the limited extent that it is possible to do so in purely formal terms, to the distinction between syllogisms in general and demonstrations (25b30, 71b18). If in particular Aaa is added as a sole axiom scheme, the results proved in the sequel will all carry over with minor modifications, e.g., the omission of 'has more than one member' from Definition 2 and Theorem 2, and the omission of 'non-empty' from-Definition 3 and Theorem 5.

THEOREM 1. (i) $P \vdash P$. (ii) if each $X_i \vdash P_i$ and $P_1, \ldots, P_n \vdash Q$ then $X_1, \ldots, X_n \vdash Q$. (iii) If $X, \bar{Q} \vdash \bar{P}$ then $X, P \vdash Q$. (iv) $Aba \vdash Iab$. (v) $Iba \vdash Iab$. (vi) $Aab \vdash Iab$. (vii) $Aac_1, Ac_1c_2, \ldots, Ac_nb \vdash Aab$; where $n \geqslant 0$. (viii) Aca, $Icb \vdash Iab$. (ix) Aca, $Ibc \vdash Iab$. (x) Adb, $Iad \vdash Iab$. (xi) Adb, $Ida \vdash Iab$. (xii) $Aca, Adb, Icd \vdash Iab$. (xiii) $Aca, Adb, Idc \vdash Iab$. (xiv) $Aca, Acb \vdash Iab$.

The proof of this theorem, as of the subsequent ones, is given in the appendix.

The intended interpretation of the system is the familiar one in which the terms are understood as ranging over non-null classes, while A, E, I, O stand respectively for class-inclusion, exclusion, overlap and non-inclusion. Satisfiability and logical consequence are defined with respect to this interpretation in the standard way. Thus a set of wffs will be said to be *satisfiable* if there is some way of assigning non-null classes as values to the terms so as to make all the members of the set true simultaneously. And a wff Q will be said to be a *logical consequence* of a set X if there is no way of assigning values to the terms so as to make all the members of X true and Q false.

DEFINITION 2. A set of wffs is an *antilogism* if it can be derived by substitution of terms for terms from a set which (i) is unsatisfiable, (ii) has no unsatisfiable proper subset, and (iii) has more than one member.

By an *A-chain* Ac_1-c_n I mean primarily a sequence of wffs of the form $\langle Ac_1c_2, Ac_2c_3, \ldots, Ac_{n-1}c_n\rangle$; but it is convenient to be able to count any term as being linked to itself by an empty A-chain, and the same notation can be used to cover both cases. Thus to say that a set of wffs is of the form $Aa-b$ is to mean that either its members can be arranged into a sequence of the kind described, with a as c_1 and b as c_n, or else that it is empty and $a=b$.

THEOREM 2. A set of wffs is an antilogism if and only if it has more than one member and is of one of the following forms

(1) less $Aa-b$, Oab (2) $Ac-a, Ac-b, Eab$ (3) $Ac-a, Ad-b$, Icd (or Idc), Eab.

THEOREM 3. If X, $\bar{Q}$ is an antilogism then $X \vdash Q$.

THEOREM 4. If $X \vdash Q$ then X, $\bar{Q}$ is an antilogism.

These theorems will be needed later as steps in the proof of Theorem 6, but they serve incidentally to establish the existence of an extremely simple decision procedure for the system. For by Theorems 3 and 4, whether $X \vdash Q$ depends on whether X, $\bar{Q}$ is an antilogism, and Theorem 2 provides an effective method for deciding this. In other words:

THEOREM 5. The system is decidable, and possesses the following decision procedure: $X \vdash Q$ if and only if X is non-empty and X, $\bar{Q}$ is of the form $Aa-b$, Oab or $Ac-a$, $Ac-b$, Eab or $Ac-a$, $Ad-b$, Icd (or Idc), Eab.
Aristotle's definition of the syllogistic figures is confined to the case where there are just two premisses, but otherwise it gives the maximum of generality to the idea that the premisses should form a chain of predications linking the terms of the conclusion (40b30 ff.). That is to say, the variables he uses for the major, middle and minor terms are all distinct from one another, so that none occurs more than twice in the statement of the premisses and conclusion; though when it comes to substituting actual terms in the resulting forms we are of course at liberty to replace different variables by the same term (64a1). The definitions given below are intended to reproduce these features of the Aristotelian figures, but freed however from their restriction to the special case of just two premisses.

Let *Vab* stand indifferently for any of the wffs *Aab*, *Aba*, *Eab*, *Eba*, *Iab*, *Iba*, *Oab*, *Oba*. Then by a *chain* of wffs I intend primarily a sequence of the form $\langle Vc_1c_2, Vc_2c_3, \dots, Vc_{n-1}c_n \rangle$; but it is also convenient to be able to count any term as linked to itself by an empty chain. Thus to say that a set of wffs is a chain linking a and b is to mean that either its members can be arranged in a sequence of the kind described, with a as c_1 and b as c_n, or else that it is empty and $a=b$. The idea of a chain of wffs thus includes that of an A-chain as a special case, but it also includes chains that are not A-chains; e.g., the premisses in Cesare form a chain *Aab*, *Ecb* linking the terms a and c of the conclusion *Eac*.

DEFINITION 3. X and Q belong to an *Aristotelian mood* if they can be derived by simultaneous substitution of terms from X_1 and Q_1 such that (i) X_1 is a non-empty chain of wffs linking the terms of Q_1, and (ii) no term occurs more than twice in X_1, Q_1. If in addition Q_1 is a logical consequence of X_1 the mood is *valid*.

THEOREM 6. $X \vdash Q$ if and only if X and Q belong to a valid Aristotelian mood.

The idea of a valid Aristotelian mood combines a syntactic ingredient (the idea of a chain of wffs) with a semantic one (logical consequence). Theorem 6 establishes its equivalence to a purely syntactic idea of deducibility. Theorems 3 and 4, on the other hand, relate it to the purely semantic idea of antilogism. Antilogism was used in preference to implication because its symmetry makes it easier to work with, but it ought to be possible to define an implication relation out of the same material, just as the classical consequence relation is defined out of the same material as the technically simpler idea of unsatisfiability. If this were done Theorems 3 and 4 could then be used to assert a direct equivalence between the idea of a valid Aristotelian mood and this purely semantic idea of implication. The relevant definition may be of some independent interest to a reader who is concerned with the controversies over entailment and strict implication.

DEFINITION 4. *X implies Q* if X and Q can be derived by simultaneous substitution from X_1 and Q_1 such that (i) Q_1 is a logical consequence of X_1 but not of any proper subset of it, (ii) X_1 is satisfiable, (iii) Q_1 is not logically true. On comparing this definition with Definition 2 it will be seen that X implies Q if and only if X, $\bar{Q}$ is an antilogism. This makes it easy to restate Theorems 3, 4 and 6 as follows.

THEOREM 7. These are equivalent to each other: (i) $X \vdash Q$, (ii) X implies Q, (iii) X and Q belong to a valid Aristotelian mood.

Discussing syllogisms with false premisses, Aristotle says that even where both premisses are false the conclusion may be true, 'but it is not necessitated' and it is true 'only in respect to the fact, not to the reason' (57a40, 53b8). What this means can only be explained by looking at the argument Aristotle uses in support of it. Namely, he argues that since the conclusion of a syllogism follows from the premisses, and since 'it is impossible that the same thing should be necessitated by the being and by the not-being of the same thing' (57b3), it cannot be the case that the conclusion should also follow from the falsity of the premisses. Since the falsity of a premiss is equivalent to the truth of its contradictory, this is to say in effect that *if a conclusion follows syllogistically from false premisses it cannot also follow from their contradictories.* This is why, when we are given that the premisses of a syllogism are false, we are according to Aristotle never able to make use of this to show that the conclusion is true (i.e., by deriving it from the true contradictories of the original premisses), even though the conclusion may be true in fact.

This is essentially the same reading of Aristotle as has been more fully defended by Geach.[7] Geach goes on to establish that Aristotle's 'metatheorem of his syllogistic' does indeed hold good for the traditional moods with two premisses. The following theorem removes the restriction on the number of premisses. It also generalises the result so as to apply to subsets of the premisses, the case where all are false being covered by taking X to be empty.

THEOREM 8. If every member of X is true and each of $P_1, \ldots, P_n$ is false ($n \geqslant 1$), then not both $X, P_1, \ldots, P_n \vdash Q$ and $X, \bar{P}_1, \ldots, \bar{P}_n \vdash Q$.

APPENDIX

Proof of Theorem 1. Part (i) follows immediately from Definition 1 (i). The proof of (ii) is by induction on the length of the assumed deduction of Q from $P_1, \ldots, P_n$. There are three cases to consider, corresponding to the three clauses of Definition 1. *Case 1*: the deduction is $\langle Q \rangle$, in which case $n=1$ and $P_1 = Q$, and the result follows immediately. *Case 2*: the deduction is $\langle \ldots R_1, \ldots, \ldots R_m, Q \rangle$, made up of deductions of each R_j from Y_j, where Q follows from $R_1, \ldots, R_m$ by a rule of inference and $Y_1, \ldots, Y_m = P_1, \ldots, P_n$. By the induction hypothesis there exist deductions of each R_j from Z_j, where Z_j represents the same selection from $X_1, \ldots, X_n$ as Y_j is from $P_1, \ldots, P_n$. Substituting these deductions for the originals produces the required deduction of Q from $Z_1, \ldots, Z_m$, i.e., from $X_1, \ldots, X_n$. *Case 3*: the deduction is $\langle \ldots R, \ldots \bar{R}, Q \rangle$, made up of a deduction of R from $Y_1, \bar{Q}$ and a deduction of $\bar{R}$ from Y_2, where $Y_1, Y_2 = P_1, \ldots, P_n$. By the induction hypothesis there exist deductions of R from $Z_1, \bar{Q}$ and of $\bar{R}$ from Z_2, where Z_1 and Z_2 represent the same partition of $X_1, \ldots, X_n$ as Y_1 and Y_2 are of $P_1, \ldots, P_n$. Substituting these deductions for the originals produces the required deduction of Q from Z_1, Z_2, i.e., from $X_1, \ldots, X_n$.

To prove (iii) we recall that $\langle P \rangle$ is a deduction of P from itself. Then if $\langle \ldots \bar{P} \rangle$ is the assumed deduction of $\bar{P}$ from $X, \bar{Q}$, we see that $\langle \ldots \bar{P}, P, Q \rangle$ is the required deduction of Q from X, P in accordance with Definition 1 (iii).

The proof of (iv) is immediate from rule 4. (v) follows from rule 3 and (iii). (vi) follows from (iv) and (v), by (ii). (vii) is proved by induction on n, using (i) for the basis and using (ii) and rule 1 in the induction step. (viii) follows from rule 2 and (iii). (ix) follows from (viii) and (v), by (ii). (x) follows likewise from (ix) and (v). (xi) follows likewise from (x) and (v).

(xii) follows likewise from (viii) and (x). (xiii) follows likewise from (ix) and (x). (xiv) follows likewise from (viii) and (vi).

Proof of Theorem 2. We first show that a set of wffs cannot be an antilogism unless it satisfies the conditions of the theorem. If X is not of any of the three forms listed in the theorem, neither is any X_1 from which it might be derived by substitution. There are then two possibilities: either (i) X_1 contains a proper subset which is of the required form, or (ii) neither X_1 nor any subset of it is of the required form. In case (i), since any set of the required form is evidently unsatisfiable, X_1 fails to satisfy Definition 2 (ii). In case (ii) consider the assignment of values in which the value of each term a is the class (call it the a-class) consisting of every term c for which X_1 contains a chain $Ac-a$, together with every pair $\{c, d\}$ for which X_1 contains Icd and $Ac-a$ (or $Ad-a$). These classes are non-null since every a belongs to its own class by virtue of the empty chain $Aa-a$. If a wff Aab belongs to X_1 then whenever X_1 contains a chain $Ac-a$ it contains a chain $Ac-b$, namely $Ac-a$, Aab. Every element of the a-class is therefore a member of the b-class, and so Aab is true under this assignment A wff Eab can only be false under the assignment if the a-class and the b-class have a common element. This can only happen if, for some c, X_1 contains $Ac-a$ and $Ac-b$, or if, for some c and d, X_1 contains Icd and $Ac-a$ (or $Ad-a$) and $Ac-b$ (or $Ad-b$). if X_1 also contained Eab it would in each of these cases contain a subset of one of the forms listed in the theorem, contrary to hypothesis. Thus if Eab belongs to X_1 it is true. If Iab belongs to X_1 the pair $\{a, b\}$ will be in the a-class and the b-class, and Iab will therefore be true. If Oab belongs to X_1 then X_1 cannot contain any chain $Aa-b$, for otherwise it would contain a subset of one of the forms listed in the theorem, namely $Aa-b$, Oab. Hence a is not in the b-class; but it is in the a-class and Oab is therefore true. So every wff in X_1 is true under this assignment of values, contrary to Definition 2 (i). Thus in neither case does X_1 satisfy the conditions required for X to be an antilogism. Finally we note that if X does not have more than one member then, since we are taking account of multiplicity of occurrence, neither does any set from which it might be derived by substitution; and so again, by Definition 2 (iii), X cannot be an antilogism.

To prove the converse we note that whenever a set is of one of the forms listed in the theorem its members can be arranged so as to link all the terms concerned in a single non-empty closed chain of wffs. Thus in $Aa-b$, Oab the ends of the A-chain are joined by Oab (if $Aa-b$ is empty there is present only the one term a, linked back to itself by Oaa). Similarly in $Ac-a$, $Ac-b$, Eab the A-chains join up directly at one end through c and are linked at the other end by Eab; while in $Ac-a$, $Ad-b$, Icd (or Idc), Eab they are linked at one end by Icd or Idc and at the other end by Eab. If, therefore, X is of any of the required forms, by starting at any point in the chain and replacing where necessary each successive term by a term different from any that have preceded it, we obtain a set X_1 which is also of the required form but in which no term occurs more than twice. On the other hand no proper subset of this X_1 can be of the required form, for otherwise its members too would make up a non-empty closed chain of wffs and then some terms would occur more than twice in X_1, namely the term or terms through which those members of X_1 that are not members of the subset link on to those that are. And since every subset of a proper subset is itself a proper subset of X_1, it follows that no proper subset of X_1 contains a subset of the required form. Therefore, by the argument of the first part of the proof, every such set is satisfiable. On the other hand X_1 itself is unsatisfiable, and X can of course be derived from it by substitution. Finally, if X has more than one member so does X_1, and so X_1 satisfies all the conditions of Definition 2 required for X to be an antilogism.

Proof of Theorem 3. For each of the three types of antilogism set out in Theorem 2 we shall show that the contradictory of the last wff in the list is deducible from the remaining ones. If $\bar{Q}$ is the last wff when X, $\bar{Q}$ is listed in accordance with Theorem 2, this will establish $X \vdash Q$ directly. If the last wff is not $\bar{Q}$ let it be P, where $X = Y, P$. Then the proof below will establish that $Y, \bar{Q} \vdash \bar{P}$, whence $X \vdash Q$ by Theorem 1-iii.

Case 1: The antilogism is $Aa-b$, Oab. $Aa-b$ is not empty, since an antilogism must have more than one member, and so by Theorem 1-vii it follows that $Aa-b \vdash Aab$, as required.

Case 2: The antilogism is $Ac-a$, $Ac-b$, Eab. As before, the A-chains cannot both be empty. If $Ac-a$ is empty, what we have to show is (i) $Aa-b \vdash Iab$, where $Aa-b$ is non-empty. Similarly if $Ac-b$ is empty we have to show (ii) $Ab-a \vdash Iab$, where $Ab-a$ is non-empty. Where neither A-chain is empty we have to show (iii) $Ac-a, Ac-b \vdash Iab$. We have seen already from case 1 that when $Aa-b$ is non-empty $Aa-b \vdash Aab$, and also $Aab \vdash Iab$ by

Theorem 1-vi; whence (i) follows by theorem 1-ii. (ii) and (iii) follow likewise from Theorems 1-iv and 1-xiv respectively.

Case 3: The antilogism is $Ac-a$, $Ad-b$, Icd (or Idc), Eab. According as both or one or other or neither A-chain is empty we have to show (i) $Iab \vdash Iab$ and $Iba \vdash Iab$, or (ii) $Ad-b, Iad \vdash Iab$ and $Ad-b, Ida \vdash Iab$, or (iii) $Ac-a, Icb \vdash Iab$ and $Ac-a, Ibc \vdash Iab$, or (iv) $Ac-a, Ad-b, Icd \vdash Iab$ and $Ac-a, Ad-b, Idc \vdash Iab$. Of these (i) follows directly from Theorems 1-i and 1-v. (ii) follows from Theorems 1-x and 1-xi taken in turn with Theorem 1-ii and the result established in case 1. (iii) follows similarly from Theorems 1-viii and 1-ix, and (iv) from Theorems 1-xii and 1-xiii.

Proof of Theorem 4 is by induction on the length of the assumed deduction of Q from X.

Case 1: the deduction is $\langle Q \rangle$. The premiss is Q itself and the pair Q, $\bar{Q}$ is an antilogism by Theorem 2 – of the first type if Q is an A or O wff, of the third type if it is an E or I wff.

Case 2: the deduction is $\langle \ldots P_1, \ldots, \ldots P_n, Q \rangle$, where Q follows from $P_1, \ldots, P_n$ by a rule of inference and each P_i is deduced from X_i, where $X = X_1, \ldots, X_n$. We consider the rules of inference in turn.

If the rule in question is rule 1 then $n=2$ and Q must be of the form Aac, while the two premisses of the rule are of the form Aab and Abc. Without loss of generality we may take P_1 to be Aab and P_2 to be Abc. Then by the induction hypothesis X_1, Oab and X_2, Obc are antilogisms. By Theorem 2, X_1 must therefore be of the form $Aa-b$ and X_2 must be of the form $Ab-c$. But then X_1, X_2, $\bar{Q}$ is $Aa-b$, $Ab-c$, Oac, which is of the form $Aa-c$, Oac. Also X_1, X_2, $\bar{Q}$ has more than one member since both X_1 and X_2, are non-empty, by Theorem 2. Hence X_1, X_2, $\bar{Q}$ is an antilogism by Theorem 2.

If the rule is rule 2 then n=2 and P_1, P_2, Q may be taken to be of the forms Aab, Ebc, Eac respectively. By the induction hypothesis X_1, Oab and X_2, Ibc are antilogisms. By Theorem 2, X_1 must therefore be of the form $Aa-b$ and X_2 must be of the form $Ab-e$, $Ac-f$, Eef (or Efe). But then X_1, X_2, $\bar{Q}$ is $Aa-b$, $Ab-e$, $Ac-f$, Iac, Eef (or Efe), which is of the form $Aa-e$, $Ac-f$, Iac, Eef (or Efe), and so is an antilogism by Theorem 2.

If the rule is rule 3 then n=1 and P_1 and Q are of the form Eba and Eab respectively. By the induction hypothesis X, Iba is an antilogism. But then X, $\bar{Q}$, which is X, Iab, is also an antilogism by Theorem 2.

If the rule is rule 4 then n=1 and P_1 and Q are of the form Aba and Iab

respectively. By the induction hypothesis X, *Oba* is an antilogism and by Theorem 2 X must therefore be non-empty and of the form $Ab-a$. But then X, $\bar{Q}$, i.e., $Ab-a$, *Eab*, is an antilogism of the second type of Theorem 2.

Case 3: the deduction is $\langle \dots P, \dots \bar{P}, Q \rangle$, made up of a deduction of P from X_1, $\bar{Q}$ and a deduction of $\bar{P}$ from X_2, where $X = X_1, X_2$. There are four subcases to consider.

Subcase 1: P is *Aab*. By the induction hypothesis X_1, $\bar{Q}$, $\bar{P}$ is an antilogism, so by Theorem 2 X_1, $\bar{Q}$ must be non-empty and of the form $Aa-b$. But then the only difference between X_2, P and X_1, X_2, $\bar{Q}$ is that the latter contains a non-empty chain $Aa-b$ where the former has the single wff *Aab*. Since by the induction hypothesis X_2, P is an antilogism it follows from Theorem 2 that X_1, X_2, $\bar{Q}$ is also an antilogism.

Subcase 2: P is *Eab*. By the induction hypothesis X_1, $\bar{Q}$, $\bar{P}$ and X_2, P are antilogisms. By Theorem 2 therefore, X_1, $\bar{Q}$ must be of the form $Aa-e$, $Ab-f$, *Eef* (or *Efe*); while X_2 must be either of the form $Ac-a$, $Ac-b$ or else of the form $Ag-a$, $Ah-b$, *Igh* (or *Ihg*). In the former case X_1, X_2, $\bar{Q}$ is $Ac-a$, $Aa-e$, $Ac-b$, $Ab-f$, *Eef* (or *Efe*), which in turn is of the form $Ac-e$, $Ac-f$, *Eef* (or *Efe*) and so is an antilogism of the second type of Theorem 2. In the latter case it is $Ag-a$, $Aa-e$, $Ah-b$, $Ab-f$, *Igh* (or *Ihg*, *Eef* (or *Efe*), which in turn is of the form $Ag-e$, $Ah-f$, *Igh* (or *ihg*), *Eef* (or *Efe*) and so is an antilogism of the third type.

Subcases 3 and 4: P is *Iab* or *Oab*. $\bar{P}$ is then *Eab* or *Aab*, and we have merly to repeat the proofs of subcases 1 and 2, writing $\bar{P}$ for P and X_1, $\bar{Q}$ for X_2 and vice-versa.

Proof of Theorem 6. If $X \vdash Q$ then, by Theorems 2 and 4, X, $\bar{Q}$ is of the form required by Theorem 2. Following the procedure described in the proof of Theorem 2 we obtain a set X_1, $\bar{Q}_1$ in which no term occurs twice and whose members form a closed chain of wffs. If therefore we break the circle by removing $\bar{Q}_1$ the remaining members will form a chain linking the terms of $\bar{Q}_1$; but these are the same as the terms of Q_1. Since an antilogism must have more than one member, X_1 is non-empty. Finally, since X_1, $\bar{Q}_1$ is unsatisfiable, Q_1 is a logical consequence of X_1. Thus X_1 and Q_1 meet all the requirements of Definition 3 for X and Q to belong to a valid Aristotelian mood. Conversely, suppose X and Q belong to a valid Aristotelian mood and let X_1 and Q_1 be as in Definition 3. Since the members of X_1 form a chain linking the terms of Q_1 and since these are the same as the terms of $\bar{Q}_1$, the members of X_1, $\bar{Q}_1$ form a closed chain of wffs, in which moreover no term occurs more than twice. We can then argue exactly as in the proof of Theorem 2 to conclude that every proper subset of X_1, $\bar{Q}_1$ is satisfiable. But X_1, $\bar{Q}_1$ itself is unsatisfiable, since by hypothesis Q_1 is a logical consequence of X_1; and it has more than one member by hypothesis X_1 is non-empty. Thus it satisfies all the requirements of Definition 2 for X, Q to be an antilogism. Hence $X \vdash Q$ by Theorem 3.

Proof of Theorem 8 is by reductio ad absurdum. Suppose that every member of X is true while each P_i is false. Also suppose that both $X, P_1, \ldots, P_n \vdash Q$ and $X, \bar{P}_1, \ldots, \bar{P}_n \vdash Q$. The corresponding antilogisms must both be of one of the forms required by Theorem 2. Inspection of the quality and quantity of the wffs involved shows that this will at most be possible if $n=2$ and then only if P_1 and P_2 are of opposite quality and quantity. There are thus two cases to consider.

Case 1: P_1 is Aab and P_2 is Ocd. By Theorem 2 the antilogism $X, P_1, P_2, \bar{Q}$ must be of the form $Ac-d$, Ocd and $\bar{Q}$ must therefore be an A wff. Let it be Aef; then X must be either (i) $Ac-a$, $Ab-e$, $Af-d$, or (ii) $Ac-e$, $Af-a$, $Ab-d$. By the same theorem the antilogism $X, \bar{P}_1, \bar{P}_2, \bar{Q}$ must be of the form $Aa-b$, Oab, and so X must be either (iii) $Aa-c$, $Ad-e$, $Af-b$, or (iv) $Aa-e$, $Af-c$, $Ad-b$. In dealing with the four cross-combinations which thus arise, let us say that an A-chain is *true* if it is empty or composed entirely of true wffs. Either way, if $Aa-b$ is true so is Aab. We note that any A-chains contained in X are true, since by hypothesis every members of X is true; also that Acd is true since by hypothesis its contradictory, P_2, is false. We shall show that Aab must also be true, contradicting the hypothesis that it, viz. P_1, is false.

Subcase (*i*−*iii*): Let $Ae-b$ be the reverse of $Ab-e$, that is, if $Ab-e=Abb_1, Ab_1b_2, \ldots, Ab_me$ let $Ae-b=Aeb_m, \ldots, Ab_1b$. If $Ab-e$ is empty $Ae-b$ is also empty and so is true. If $Ab-e$ is not empty its first member Abb_1 cannot belong to $Aa-c$, otherwise this true A-chain would have a true subchain $Aa-b$, and Aab would be true, contrary to hypothesis. Likewise Abb_1 cannot belong to $Ad-e$, otherwise this would contain a true subchain $Ad-b$, and the truth of $Aa-c$, Acd and $Ad-b$ would imply the truth of Aab. Hence Abb_1 must belong to the remaining A-chain listed in (iii), $Af-b$. This therefore has a true subchain Ab_1-b and so Ab^1b is true. But since if Ab_1b is true and Aab is false Aab_1 must be false, we can repeat for Ab_1-e the argument so far given for $Ab-e$, and so on until we have successively established the truth of Ab_1b, $Ab_2b_1, \ldots, Aeb_m$. Thus whether $Ab-e$ is empty or not, $Ae-b$ is true. But the truth of $Aa-c$, Acd, $Ad-e$, $Ae-b$ implies the truth of Aab.

Subcase (i–iv) is similar. We show that if $Ab-e$ is not empty its members cannot belong to $Aa-e$ but must all belong to $Af-c$, $Ad-b$, and we thereby show as before that $Ae-b$ is true. But the truth of $Aa-e$, $Ae-b$ implies the truth of Aab.

Subcases (ii–iii) and (ii–iv) are the mirror images of (i–iv) and (i–iii). In subcase (ii–iii), for example, we show that if $Af-a$ is not empty then (but this time working backwards from its last member) its members cannot belong to $Af-b$ but must all belong to $Aa-c$, $Ad-e$. We thereby show that the reverse chain $Aa-f$ is true; but the truth of $Aa-f$, $Af-b$ implies the truth of Aab.

Case 2: P_1 is Iab and P_2 is Ecd. By Theorem 2, $X, P_1, P_2, \bar{Q}$ must be of the form $Aa-c$, $Ab-d$, Iab, Ecd or else $Ab-c$, $Aa-d$, Iab, Ecd. Let $\bar{Q}$ be Aef; then X must be either (i) $Aa-c$, $Ab-e$, $Af-d$ or (ii) $Aa-e$, $Af-c$, $Ab-d$ or (iii) $Ab-c$, $Aa-e$, $Af-d$ or (iv) $Ab-e$, $Af-c$, $Aa-d$. By the same theorem $X, \bar{P}_1, \bar{P}_2, \bar{Q}$ must be of the form $Ac-a$, $Ad-b$, Icd, Eab or else $Ad-a$, $Ac-b$, Icd, Eab. X must therefore be either (v) $Ac-a$, $Ad-e$, $Af-b$ or (vi) $Ac-e$, $Af-a$, $Ad-b$ or (vii) $Ad-a$, $Ac-e$, $Af-b$ or (viii) $Ad-e$, $Af-a$, $Ac-b$. There are thus sixteen cross-combinations to consider, for each of which we shall show that Iab must be true, contradicting the hypothesis that it, viz. P_1, is false.

Subcase (i–v): If $Ab-e$ is empty the reverse chain $Ae-b$ is also empty and so true. If $Ab-e$ is not empty its first member Abb_1 cannot belong to $Ac-a$, otherwise this would have a true subchain $Ab-a$, which would imply the truth of Aba and hence of Iab. Likewise it cannot belong to $Ad-e$, otherwise this would have a true subchain $Ad-b$, and the truth of $Ac-a$, $Ad-b$, Icd would imply the truth of Iab. Hence Abb_1 must belong to the remaining chain listed in (v), $Af-b$. This therefore has a true subchain Ab_1-b and so Ab_1b is true. But since if Ab_1b is true and Iab is false Iab_1 must be false, we can repeat for Ab_1-e the argument so far given for $Ab-e$, and so on until we have established the truth of each member of $Ae-b$. But the truth of $Ac-a$, $Ad-e$, $Ae-b$, Icd implies the truth of Iab. Interchanging a and b in this proof turns it into a proof of subcase (iii–viii), interchanging c and d turns it into a proof of subcase (iv–vii), and interchanging both pairs turns it into a proof of (ii–vi).

Subcase (i–vi): Here the truth of $Af-a$, $Af-d$, $Ad-b$ directly implies the truth of Iab. Subcases (i–vii), (ii–v), (ii – viii), (iii–vi), (iii–vii), (iv–v), and (iv–viii) are similar.

Subcase (i–viii): Here the truth of $Aa-c$, $Ac-b$ directly implies the truth of Iab. Subcases (ii–vii), (iii–v) and (iv–vi) are similar.

In Theorem 8 an assertion about syllogistic validity is made to follow from an assumption about truth-values. In common with all similar results, the theorem can be strengthened by weakening its hypothesis, so that it is no longer assumed that the wffs involved actually have the truth-values in question, but merely the possibility of their having those truth-values. In the present case the weakened hypothesis will thus be that X, $\bar{P}_1, \ldots, \bar{P}_n$ is satisfiable. This implies that there is some assignment of values in which each member of X would be true and each P_i false. All we then need to do is to read 'true' and 'false' throughout the present proof as meaning true or false in this assignment of values, and the proof will go through without further change.[8]

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NOTES

[1] J. Łukasiewicz, 1957, *Aristotle's syllogistic*, 2nd ed., Oxford.

[2] There is more than one kind of per impossibile argument. The one illustrated is the 'hypothetical' version, in which the falsity of the 'impossibile' (here, the truth of Q) is presupposed as a 'hypothesis' rather than posited as a premiss.

[3] For example, the first distinction is flatly contradicted by Aristotle's statement that "the falsity is established in reductions ad impossibile by an ostensive syllogism" (41a32). The second distinction is incompatible with the whole of *An. Pr.*, Book 2, Chapters 11–14; e.g., it would be nonsense to say of a per impossibile conditional that it 'posits what it wishes to refute' (62b29), or to say that a universal affirmative cannot be proved by a per impossibile first figure conditional (61b9), if 'per impossibile' referred to the manner of establishing the conditional in question.

[4] *An. Pr.* 24b23. Compare the distinction between 'valid' and 'satisfactory' arguments in J. F. Bennett, 'Entailment,' *Philosophical Review* **78** (1969), 219, 231ff.

[5] G. Patzig, 1968, *Aristotle's theory of the syllogism*, Dordrecht, follows Łukasiewicz in treating syllogisms as conditionals in the propositional calculus, but on second thoughts Patzig indicates a preference for the approach of K. Ebbinghaus, 1964, 'Ein formales Modell der Syllogistik des Aristoteles,' *Hypomnemata* **9**. This is still to treat syllogisms as conditionals, but as conditionals set in Lorenzen's 'operative logic', in which they are established in natural deduction fashion. The resulting system does not have all the non-Aristotelian moods of Łukasiewicz's, but it does have those described in the next paragraph.

[6] *An Pr.* 41b36. I read the terminology of 'whole' and 'part' at 42a10 as a variation on

the terminology of 'major' and 'minor' used earlier in the explanation of the syllogistic figures.

[7] P. T. Geach, 1963, 'Aristotle on conjunctive propositions', *Ratio* 5, 39ff.

[8] **(Added in proof.)** Since this article was written I have learnt of concurrent work by Prof. J. P. Corcoran (to appear in *Journal of Symbolic Logic, Archiv für Geschichte der Philosophie* and *Mind.*) Corcoran's approach to the syllogistic is very similar to that advocated here, but his treatment is independent and distinctive and provides further strong support for the new approach.