Notre Dame Journal of Formal Logic
Volume 30, Number 2, Spring 1989

Models for Modal Syllogisms

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Abstract A semantics is presented for Storrs McCall's separate axiomatizations of Aristotle's accepted and rejected polysyllogisms. The polysyllogisms under discussion are made up of either assertoric or apodeictic propositions. The semantics is given by associating a property with a pair of sets: one set consists of things having the property essentially and the other of things having it accidentally. A completeness proof and a semantic decision procedure are given.

In the opening chapters of [2] Łukasiewicz developed a nonmodal system of logic to illuminate Aristotle's discussion of nonmodal syllogisms. One of the distinctive features of his presentation is his syntactic treatment of the invalid syllogisms. In effect, invalid syllogisms are deduced in his system.[1] Also, a decision procedure for determining the validity and invalidity of Aristotelian nonmodal syllogisms is given in purely syntactic terms. Though Łukasiewicz does not extend his treatment of Aristotelian nonmodal syllogisms to Aristotelian modal syllogisms, Storrs McCall in [3] does, by developing the system L-X-M, which treats syllogisms formed from assertoric and apodeictic propositions. The purpose of this paper is to provide a semantics for L-X-M. (McCall did not provide a semantics for L-X-M.) I shall assume that L-X-M's relationship to Aristotle's modal syllogisms is accurately described in [3]. So my primary interest is mathematical rather than historical. But my hope is that the semantics will provide the reader with an intuitive grasp of Aristotle's thinking about a substantial fragment of Aristotelian syllogisms.

Since modifications, though minor, will be made in McCall's L-X-M, we shall refer to the modified system as LXM, whose presentation will be self-contained. The syntax of LXM is as follows.

*I am grateful to David Bostock and a referee of this journal for comments on an earlier draft of this paper. And I extend my thanks to Timothy Smiley, who read a second version of my paper with remarkable care and made numerous significant corrections and improvements in it.

Received June 29, 1987

Primitive symbols

Variables	a, b, c, etc.
Functions of one argument	N,L,*
Functions of two arguments	C,A,I

Rules of Formation

(i) If x and y are variables, Axy and Ixy are *categorical expressions*.
(ii) If x is a categorical expression, Nx is a *categorical expression*.
(iii) All categorical expressions are *well-formed formulas* (wffs).
(iv) If x is a categorical expression, Lx is a *wff*.
(v) If x and y are wffs, Nx and Cxy are *wffs*.
(vi) If x is a wff, *x is a *starred expression*.
(vii) The only categorical expressions, wffs, and starred expressions are those in virtue of (i)–(vi).[2]

Definitions

Def E E = NI
Def O O = NA
Def M M = NLN
Def K Kxy = NCxNy.

Unstarred axioms

A1 Aaa
A2 LIaa
A3 CKAbcAabAac
A4 CKAbcIbaIac
A5 CKLAbcAabLAac
A6 CKLEcbAabLEac
A7 CKLAbcIabLIac
A8 CKLEbcIabLOac
A9 CKLAcbLOabLOac
A10 CKLObcLAbaLOac
A11 CLIabLIba
A12 CLAabAab
A13 CLIabIab
A14 CLOabOab.

Unstarred rules of inference

R1 x; Cxy; so y.[3]
R2 F(x); so F(x/y). F(x/y) is the result of substituting y for x in each of x's occurrences in F(x). (So, for example, R2 justifies this inference: Aab; so Aaa.)
R3 'x; so y' is a rule of inference if 'x; so y' is valid in the propositional calculus, where x may be the empty symbol. (So, for example, R3 justifies the following inferences: (i) CCAabNAabNAab and (ii) CAabNAab; so NAab.)
R4 F(x); so F(NNx). And F(NNx); so F(x).

Starred axioms

***A1** CLA*aa*MO*aa*
***A2** CLE*ab*LA*aa*
***A5.12** LA*bb*LA*ba*LA*ac*A*ca*MA*cb* → LA*aa* ('*xy* → *z*' abbreviates 'CK*xyz*' etc.)
***A5.21** LA*bb*MA*ab*A*ac*LA*ca*LA*bc* → LA*ac*
***A5.3** LA*aa*LA*cc*MA*ac*LA*ca* → LA*ac*
***A5.41** LA*bb*LA*ff*A*ad*LA*da*MA*ae*LA*cb*LA*bd*LA*ce*A*ec*LA*fc*MA*df* → MA*ac*
***A5.42** LA*bb*LA*cc*MA*ab*A*ad*LA*da*LA*cb*LA*bd*MA*dc* → MA*ac*
***A5.511** LA*ab*A*ca*A*bc* → LI*ac*
***A5.512** LA*dd*LA*ab*A*ba*A*ca*LA*da*MA*bc*MA*bd*LA*cg*A*gc* → LI*ac*
***A5.514** LA*bb*LA*dd*LA*ee*LA*hh*LA*ab*LA*da*LA*af*A*fa*A*ha*MA*bd*MA*bh* MA*be*A*hc*A*gc*LA*cg*LA*ec*LA*cb* → LI*ac*
***A5.6** LA*aa*LA*bb*LA*cc*LA*ab*MA*ba*MA*bc*LA*cb* → I*ac*
***A5.7** LA*aa*LA*bb*LA*cc*LA*dd*LA*ee*LA*ff*LA*ad*MA*da*LA*bd*LA*df*MA *fb*LA*ef*LA*be*MA*ec*LA*ce* → MI*ac*.[4]

Starred rules of inference

***R1** **y*; C*xy*; so **x*.
***R2** *F(*x*/*y*); so *F(*x*). (So, for example, *R2 validates this inference: *CLA*aa*MO*aa*; so *CLA*ab*MO*ab*.)
***R3** *C*xz*; *C*yz*; so *C*x*C*yz*. Here *x* and *y* are simple negative wffs, and *z* is an elementary wff. The *simple negative wffs* are LE*rs*, LO*rs*, E*rs*, O*rs*, ME*rs*, and MO*rs*. The *simple affirmative wffs* are LA*rs*, LI*rs*, A*rs*, I*rs*, MA*rs*, and MI*rs*. The *simple wffs* are the simple negative wffs and the simple affirmative wffs. Simple wffs are *elementary wffs*. And if *x* is a simple wff and *y* is an elementary wff then C*xy* is an *elementary wff*. (So, for example, *R3 justifies this inference: *CE*ac*I*ab*; *CE*bc*I*ab*; so *CE*ac*CE*bc*I*ab*.)

The *axioms* are the starred axioms and the unstarred axioms. The *rules of inference* are the starred rules of inference and the unstarred rules of inference. By a *deduction* we mean a sequence of wffs or starred expressions such that each member of the sequence is an axiom or is entered by one of the rules of inference. A wff *x* is *accepted* if *x* is a member of a deduction. And a wff *x* is *rejected* if **x* is a member of a deduction.

Theorem 1 *Every elementary wff is either accepted or rejected.*

The proof is found in [3], where McCall identifies a finite number of types of elementary wffs and gives a procedure that shows that each formula of each of these types is either accepted or rejected. So, for example, he shows that any elementary wff with affirmative antecedents and a negative consequent is rejected. CLA*ab*CA*bc*MO*ca* is of this form, and we can follow the recipe in his discussion of Case 1 (p. 52) to construct a deduction in which *CLA*ab*CA*bc*MO*ca* is a member. Here is the deduction:

1. CCLA*aa*CLA*aa*MO*aa*CLA*aa*MO*aa* R3
2. *CLA*aa*MO*aa* *A1

3. *CLA*aa*CLA*aa*MO*aa*	*R1, 2, 1
4. CLA*ab*A*ab*	A12
5. CLA*aa*A*aa*	R2,4
6. CCLA*aa*A*aa*CCLA*aa*CA*aa*MO*aa*CL*aa*MO*aa*	R3
7. CCLA*aa*CA*aa*MO*aa*CL*aa*MO*aa*	R1, 5, 6
8. *CLA*aa*CA*aa*MO*aa*	*R1, 2, 7
9. *CLA*ab*CA*bc*MO*ca*.	*R2, 8

By verifying an instance of McCall's Case 3 (p. 53) we shall illustrate *R3. The following deduction shows that CLE*ab*CLE*bc*MO*ac* is rejected.

1. CCLA*ad*MI*ac* (*5.7)[5]	R3
2. *5.7	*5.7
3. *CLA*ad*MI*ac*	*R1, 2, 1
4. *CLA*ac*MI*ab*	*R2, 3
5. CCLA*cc*MI*ac* (*5.7)	R3
6. *CLA*cc*MI*ac*	*R1, 2, 5
7. *CLA*ac*MI*bc*	*R2, 6
8. CCLE*ab*MO*ac*CLNNA*ac*MI*ab*	R3
9. CCLE*ab*MO*ac*CLA*ac*MI*ab*	R4, 8
10. *CLE*ab*MO*ac*	*R1, 4, 9
11. CCLE*bc*MO*ac*CLNNA*ac*MI*bc*	R3
12. CCLE*bc*MO*ac*CLA*ac*MI*bc*	R4, 11
13. *CLE*bc*MO*ac*	*R1, 7, 12
14. *CLE*ab*CLE*bc*MO*ac*.	*R3, 10, 13

By a *model* we mean an ordered quintuple $\langle W, V^e, V^a, V_c^e, V_c^a \rangle$, where V^e, V^a, V_c^e, and V_c^a are functions that map term variables into subsets of the set W. (Think of W as the world, $V^e(x)$ as the things that are essentially x, $V^a(x)$ as the things that are accidentally x, $V_c^e(x)$ as the things that are essentially non-x, and $V_c^a(x)$ as the things that are accidentally non-x.) And we define V so that $V(x) = V^e(x) \cup V^a(x)$. Each of the four functions that make up the model meets the following conditions:

(a) $V^e(x)$ is nonempty
(b) For each x, $V_k^j(x) \cap V_n^m(x) = \varnothing$, if either $j \neq m$ or $k \neq n$; and for each x, $V^e(x) \cup V^a(x) \cup V_c^e(x) \cup V_c^a(x) = W$
(c) If $V(z) \subset V_c^e(y)$ and $V(x) \subset V(y)$ then $V(x) \subset V_c^e(z)$
(d) If $V(y) \subset V^e(z)$ and $V(x) \cap V(y) \neq \varnothing$ then $V^e(x) \cap V^e(z) \neq \varnothing$
(e) If $V(y) \subset V_c^e(z)$ and $V(x) \cap V(y) \neq \varnothing$ then $V^e(x) \cap V_c^e(z) \neq \varnothing$
(f) If $V(z) \subset V^e(y)$ and $V^e(x) \cap V_c^e(y) \neq \varnothing$ then $V^e(x) \cap V_c^e(z) \neq \varnothing$.

The function V maps wffs into the truth values t and f as follows:

(i) $V(\mathrm{A}xy) = t$ iff $V(x) \subset V(y)$
(ii) $V(\mathrm{I}xy) = t$ iff $V(x) \cap V(y) \neq \varnothing$
(iii) $V(\mathrm{N}x) = t$ iff $V(x) = f$
(iv) $V(\mathrm{LA}xy) = t$ iff $V(x) \subset V^e(y)$
(v) $V(\mathrm{LI}xy) = t$ iff $V^e(x) \cap V^e(y) \neq \varnothing$
(vi) $V(\mathrm{LNA}xy) = t$ iff $V^e(x) \cap V_c^e(y) \neq \varnothing$

(vii) $V(\mathrm{LNI}xy) = t$ iff $V(x) \subset V_c^e(y)$
(viii) $V(\mathrm{LNN}x) = t$ iff $V(\mathrm{L}x) = t$
(ix) $V(\mathrm{C}xy) = t$ iff either $V(x) = f$ or $V(y) = t$.

A wff is *valid* iff in every model $\langle W, V^e, V^a, V_c^e, V_c^a \rangle$ $V(x) = t$.

Theorem 2 *Accepted wffs are valid, and rejected wffs are invalid.*

Proof: We shall show by induction on n that if the nth member of a deduction is unstarred it is valid and if starred it is invalid. The only cases that call for discussion are the starred and unstarred axioms and the rejection rule *R3. First, we shall indicate why unstarred axioms are true in every model. A1: (By set theory since $V(a) \subset V(a)$.) A2: (By condition (a) used to define a model, $V^e(a) \neq \varnothing$.) A3–A5: (Set theory.) A6–A9: (Conditions (c)–(f), respectively, for being a model.) A10–A14: (Set theory.)

Next, for each of the starred axioms we shall specify a model in which it is false. The model for *A1 is:

***A1** $W = \{1\}$, $V^e(a) = \{1\}$, $V^a(a) = V_c^e(a) = V_c^a(a) = \varnothing$

In this model and in those for the other starred axioms W will consist of all and only those objects that belong to the sets $V^e(a)$, $V^a(a)$, $V_c^e(a)$, or $V_c^a(a)$ as specified in the table that corresponds to the axiom. If x is a term that does not occur in the table then $V_k^j(x) = V_k^j(a)$. The job of verifying that each table yields a model is routine, and the reader is invited to check the details.

***A2**

	V^e	V^a	V_c^e	V_c^a
a	1	2	3	
b	3		1,2	

When presenting tables we shall drop brackets and omit the symbol for the empty set to make the assignments of values easier to read. So, according to the above table $V^e(a) = \{1\}$ and $V^a(b) = \varnothing$.

***A5.12**

	V^e	V^a	V_c^e	V_c^a
a	1,2	3		
b	1,2			3
c	1,2,3			

***A5.21**

	V^e	V^a	V_c^e	V_c^a
a	1,2			
b	1			2
c	1	2		

***A5.3**

	V^e	V^a	V_c^e	V_c^a
a	1,2			
c	1			2

***A5.41**

	V^e	V^a	V^e_c	V^a_c
a	1,2,3,4			
b	3,4			1,2
c	4	3	1	2
d	3,4	1,2		
e	3,4			1,2
f	4		2	1,3

***A5.42**

	V^e	V^a	V^e_c	V^a_c
a	1,2,3			
b	2,3			1
c	2		1	3
d	2,3	1		

***A5.511**

	V^e	V^a	V^e_c	V^a_c
a	1	2		
b	1,2			
c	2	1		

***A5.512**

	V^e	V^a	V^e_c	V^a_c
a	1	2		
b	1,2			
c	2			1
d	1			2
g	2			1

***A5.514**

	V^e	V^a	V^e_c	V^a_c
a	1,2	3,4		5,6
b	1,2,3 4,5,6			
c	3,5,6	1,4		2
d	2			1,3,4 5,6
e	6			1,2,3 4,5
f	1,2,3,4			5,6
g	1,3,4 5,6			2
h	1,3,4			2,5,6

***A5.6**

	V^e	V^a	V^e_c	V^a_c
a	1			2
b	1,2			
c	2			1

*A5.7

	V^e	V^a	V_c^e	V_c^a
a	1		2	3,4
b	3			1,2,4
c	2		1	3,4
d	1,3			2,4
e	2,3,4			1
f	1,2,3,4			

Finally, for *R3, assume that Cxz is false in $\langle W, \underline{V}\rangle$, C$yz$ is false in $\langle W', \underline{V}'\rangle$, x and y are simple negative wffs, z is an elementary wff, and $W \cap W' = \varnothing$. (Note that if models $\langle W_1, \underline{V}_1\rangle$ and $\langle W_2, \underline{V}_2\rangle$ are such that W_1 and W_2 are not disjoint one can construct models $\langle W_3, \underline{V}_3\rangle$ and $\langle W_4, \underline{V}_4\rangle$ such that W_3 and W_4 are disjoint, $\langle W_1, \underline{V}_1\rangle$ is isomorphic to $\langle W_3, \underline{V}_3\rangle$, and $\langle W_2, \underline{V}_2\rangle$ is isomorphic to $\langle W_4, \underline{V}_4\rangle$.) We shall show that C$xCyz$ is false in $\langle W'', \underline{V}''\rangle$, where $W'' = W \times W'$, $V''^e(x) = V^e(x) \times V'^e(x)$, $V''^a(x) = V(x) \times V'(x) - V''^e(x)$, $V''^e_c(x) = (V^e_c(x) \times W') \cup (W \times V'^e_c(x))$, and $V''^a_c(x) = W'' - V''(x) - V''^e_c(x)$. (So, for example, let $W = \{1,2\}$, $W' = \{3,4\}$, and let $\underline{V}$ and $\underline{V}'$ be defined by these tables:

$\underline{V}$

	V^e	V^a	V_c^e	V_c^a
a	1	2		
b	2		1	
c	1	2		

$\underline{V}'$

	V'^e	V'^a	V'^e_c	V'^a_c
a	3		4	
b	3		4	
c	4		3	

Then $\underline{V}''$ is defined as follows:

$\underline{V}''$

	V''^e	V''^a	V''^e_c	V''^a_c
a	1;3	2;3	1;4,2;4	
b	2;3		1;3,1;4 2;4	
c	1;4	2;4	1;3,2;3	

We can illustrate the general claim we wish to establish by noting that COabOba is false in $\langle W, \underline{V}\rangle$, CE$bc$O$ba$ is false in $\langle W', \underline{V}'\rangle$, and CO$abCEbc$O$ba$ is false in $\langle W'', \underline{V}''\rangle$, assuming that the various functions treat terms not in the table as they treat the term 'a'.)

First, we shall show that $\langle W'', \underline{V}''\rangle$ meets the six conditions required of all models. *Condition a.* $V^e(x) \times V'^e(x) \neq \varnothing$, since $V^e(x) \neq \varnothing$ and $V'^e(x) \neq \varnothing$. *Condition b.* $(V''(x) \cup V''^e_c(x)) \cap V''^a_c(x) = \varnothing$ (by the definitions). And $V''(x) \cap V''^e_c(x) = \varnothing$. (If $a;b \in V''(x)$ then $a \notin V^e_c(x)$ and $b \notin V'^e_c(x)$. Then $a;b \notin V^e_c(x) \times W'$ and $a;b \notin W \times V'^e_c(x)$. Then $a;b \notin V''^e_c(x)$.) And

$V''^e(x) \cap V''^a(x) = \varnothing$ (by the definitions). So the first part of Condition b is satisfied. The second is immediate from the definition of V''^a_c. *Condition c.* Suppose $V''(z) \subset V''^e_c(y)$ and $V''(x) \subset V''(y)$. Then either $V(z) \subset V^e_c(y)$ or $V'(z) \subset V'^e_c(y)$. And both $V(x) \subset V(y)$ and $V'(x) \subset V'(y)$. So, either $V(x) \subset V^e_c(z)$ or $V'(x) \subset V'^e_c(z)$. So $V''(x) \subset V''^e_c(z)$. *Condition d.* Suppose $V''(y) \subset V''^e(z)$ and $V''(x) \cap V''(y) \neq \varnothing$. Then $V(y) \subset V^e(z)$, $V'(y) \subset V'^e(z)$, $V(x) \cap V(y) \neq \varnothing$, and $V'(x) \cap V'(y) \neq \varnothing$. So $V^e(x) \cap V^e(z) \neq \varnothing$ and $V'^e(x) \cap V'^e(z) \neq \varnothing$. So $V''^e(x) \cap V''^e(z) \neq \varnothing$. *Condition e.* Suppose $V''(y) \subset V''^e_c(z)$ and $V''(x) \cap V''(y) \neq \varnothing$. Then $V(y) \subset V^e_c(z)$ or $V'(y) \subset V'^e_c(z)$. Since $V(x) \cap V(y) \neq \varnothing$ and $V'(x) \cap V'(y) \neq \varnothing$, either $V(x) \cap V^e_c(z) \neq \varnothing$ or $V'(x) \cap V'^e_c(z) \neq \varnothing$. So $V''(x) \cap V''^e_c(z) \neq \varnothing$. *Condition f.* Suppose $V''(z) \subset V''^e(y)$ and $V''^e(x) \cap V''^e_c(y) \neq \varnothing$. Then $V(z) \subset V^e(y)$ and $V'(z) \subset V'^e(y)$. And either $V^e(x) \cap V^e_c(y) \neq \varnothing$ or $V'^e(x) \cap V'^e_c(y) \neq \varnothing$. So $V^e(x) \cap V^e_c(z) \neq \varnothing$ or $V'^e(x) \cap V'^e_c(z) \neq \varnothing$. So $V''^e(x) \cap V''^e_c(z) \neq \varnothing$.

To complete the proof we shall make use of five lemmas.

Lemma 1 *If x is a simple negative wff and is true in $\langle W, \underline{V}\rangle$ then x is true in $\langle W'', \underline{V}''\rangle$.*

We shall prove the lemma by showing that it is true for each of the six forms a simple negative wff may have. Assume that x is true in $\langle W, \underline{V}\rangle$. *Form i*: $x =$ LEyz. So $V(y) \subset V^e_c(z)$. So $V(y) \times V'(y) \subset V^e_c(z) \times W'$. So $V''(y) \subset V''^e_c(z)$. So x is true in $\langle W'', \underline{V}''\rangle$. *Form ii*: $x =$ LOyz. So $V^e(y) \cap V^e_c(z) \neq \varnothing$, and $V'^e(y) \cap W' \neq \varnothing$. So $V''^e(y) \cap V''^e_c(z) \neq \varnothing$. *Form iii*: $x =$ Eyz. Use an argument like that for Form i. *Form iv*: $x =$ Oyz. Use an argument like that for Form ii. *Form v*: $x =$ MEyz. So $V^e(y) \subset \overline{V^e(z)}$. Suppose $a;b \in V^e(y) \times V'^e(z)$. Then $a \notin V^e(z)$. So $a;b \notin V^e(z) \times V'^e(z)$. So $V''^e(y) \subset \overline{V''^e(z)}$. *Form vi*: $x =$ MOyz. So $V(y) \cap V^e(z) \neq \varnothing$. Let $a \in V(y)$, $a \notin V^e(z)$, and $b \in V'(y)$. Then $a;b \in V(y) \times V'(y)$ and $a;b \notin V^e(z) \times V'^e(z)$. So x is true in $\langle W'', \underline{V}''\rangle$.

Lemma 2 *If x is a simple negative wff and is true in $\langle W', \underline{V}'\rangle$, then x is true in $\langle W'', \underline{V}''\rangle$.*

Proof: Modify the proof of Lemma 1.

Lemma 3 *If x is a simple affirmative wff and is true in both $\langle W, \underline{V}\rangle$ and $\langle W', \underline{V}'\rangle$, then x is true in $\langle W'', \underline{V}''\rangle$.*

Proof: We shall show that the lemma holds for each of the six types of simple affirmative wffs. Assume that x is true in $\langle W, \underline{V}\rangle$ and $\langle W', \underline{V}'\rangle$. *Form i*: $x =$ LAyz. Then $V(y) \subset V^e(z)$ and $V'(y) \subset V'^e(z)$. So $V''(y) \subset V''^e(z)$. So x is true in $\langle W'', \underline{V}''\rangle$. *Form ii*: $x =$ LIyz. So $V^e(y) \cap V^e(z) \neq \varnothing$ and $V'^e(y) \cap V'^e(z) \neq \varnothing$. Let $a \in V^e(y) \cap V^e(z)$ and $b \in V'^e(y) \cap V'^e(z)$. Then $a;b \in V''^e(y) \cap V''^e(z)$. So x is true in $\langle W'', \underline{V}''\rangle$. *Form iii* and *Form iv*: $x =$ Aab and $x =$ Iab. Use arguments like those for Forms i and ii. *Form v*: $x =$ MAyz. So $V^e(y) \subset \overline{V^e_c(z)}$ and $V'^e(y) \subset \overline{V'^e_c(z)}$. Suppose $a;b \in V^e(y) \times V'^e(y)$. Then $a;b \notin V^e_c(z) \times W'$ and $a;b \notin W \times V'^e_c(z)$. So $a;b \notin V''^e_c(z)$. So $V''^e(y) \subset \overline{V''^e_c(z)}$. *Form vi*: $x =$ MIyz. So $V(y) \cap \overline{V^e_c(z)} \neq \varnothing$ and $V'(y) \cap \overline{V'^e_c(z)} \neq \varnothing$. Suppose $a \in V(y) \cap \overline{V^e_c(z)}$ and $b \in V'(y) \cap \overline{V'^e_c(z)}$. Then $a;b \in V(y) \times$

$\underline{V}'(y)$, $a;b \notin V_c^e(z) \times W'$, and $a;b \notin W \times V'^e_c(z)$. So $V''(y) \cap \overline{V''^e_c(z)} \neq \varnothing$. So, x is true in $\langle W'', \underline{V}'' \rangle$.

Lemma 4 *If x is a simple wff and is true in both $\langle W, \underline{V} \rangle$ and $\langle W', \underline{V}' \rangle$, then x is true in $\langle W'', \underline{V}'' \rangle$.*

Proof: It is an immediate consequence of Lemmas 1 and 3.

Lemma 5 *If x is a simple wff and is false in both $\langle W, \underline{V} \rangle$ and $\langle W', \underline{V}' \rangle$, then x is false in $\langle W'', \underline{V}'' \rangle$.*

Proof: Note that LAxy is true in a model iff MOxy is false in it. And the following pairs of wffs are such that the former is true in a model iff the latter is false in it: LIxy and MExy, Axy and Oxy, Ixy and Exy, MAxy and LOxy, and MIxy and LExy (the so-called octagon of opposition). So, suppose x is a simple wff and is false in $\langle W, \underline{V} \rangle$ and $\langle W', \underline{V}' \rangle$. Then there is a simple wff y (the contradictory of x) which is true in these models. By Lemma 4, y is true in $\langle W'', \underline{V}'' \rangle$. So, x is false in $\langle W'', \underline{V}'' \rangle$.

We shall invoke the above lemmas in a proof by induction on the number n of occurrences of simple wffs in the elementary wff z. Assume that x and y are simple negative wffs and z is an elementary wff. And assume that Cxz is false in $\langle W, \underline{V} \rangle$ and Cyz is false in $\langle W', \underline{V}' \rangle$.

Basis step: $n = 1$. So z is a simple wff. By Lemmas 1 and 2, x and y are true in $\langle W'', \underline{V}'' \rangle$, and, by Lemma 5, z is false in $\langle W'', \underline{V}'' \rangle$. So, C$xCyz$ is false in $\langle W'', \underline{V}'' \rangle$.

Induction step: $n = k + 1$. So $z = Cz_1Cz_2C\dots z_{k+1}$. $CxCz_2C\dots z_{k+1}$ is false in $\langle W, \underline{V} \rangle$ and $CyCz_2C\dots z_{k+1}$ is false in $\langle W', \underline{V}' \rangle$. So, by the induction hypothesis, $CxCyCz_2C\dots z_{k+1}$ is false in $\langle W'', \underline{V}'' \rangle$.Since z_1 is true in $\langle W'', \underline{V}'' \rangle$, C$xCyz$ is false in $\langle W'', \underline{V}'' \rangle$.

An immediate consequence of the first two theorems is:

Theorem 3 *Elementary valid wffs are accepted, and elementary invalid wffs are rejected.*

Theorem 4 *Valid wffs are accepted, and invalid wffs are rejected.*

Proof: We shall exploit Theorem 3 by linking wffs that may not be elementary to wffs that are elementary. We shall call these links OE-chains. An *OE-chain* is a sequence of sets $S_1, S_2, \dots$ of wffs such that $\langle S_j, S_{j+1} \rangle$ $(1 \leq j)$ is an instance of one of the following pairs:

(i) $\langle \{P\text{NN}X, \dots\}, \{PX, \dots\} \rangle$ (NN)
(ii) $\langle \{Q\text{CNC}XYZ, \dots\}, \{Q\text{C}X\text{CN}YZ, \dots\} \rangle$ (CNC)
(iii) $\langle \{Q\text{CC}XYZ, \dots\}, \{Q\text{CN}XZ, Q\text{C}YZ, \dots\} \rangle$ (CC)
(iv) $\langle \{Q\text{NC}XY, \dots\}, \{QX, Q\text{N}Y, \dots\} \rangle$ (NC)

where P is any string of symbols and Q has the form $Cx_1Cx_2\dots Cx_n$, where each x_i $(1 \leq i \leq n)$ is a simple wff, or Q is the empty symbol. And we shall say that S is *OE-connected* to T if there is an OE-chain $S_1, \dots, S_n$ such that $S_1 = \{S\}$ and $S_n = \{T\}$.

Lemma 1 *If* $\{w\}$ *is OE-connected to* $\{w_1,\ldots,w_j\}$ *then if* x *is a wff then* $\{Cxw\}$ *is OE-connected to* $\{Cxw_1,\ldots,Cxw_j\}$.

Proof: We shall use induction on n, where $S_1,\ldots,S_n$ is the OE-chain in virtue of which $\{w\}$ is OE-connected to $\{w_1,\ldots,w_j\}$, assuming that the former is OE-connected to the latter.

Basis step: $n = 1$. Note that every set is OE-connected to itself.

Induction step: $n = k + 1$. There are two cases. *Case 1:* $\langle S_k, S_{k+1}\rangle$ is equal to $\langle\{v, v_2,\ldots,v_j\},\{v_1,\ldots,v_j\}\rangle$, where each v_m $(1 \leq m \leq j)$ is equal to some w_n $(1 \leq n \leq j)$ and each w_n is equal to some v_m, and $\langle S_k, S_{k+1}\rangle$ is an instance of (NN) or (CNC). Then $\langle T_k, T_{k+1}\rangle$ is an instance of (NN) or (CNC). Since, by the induction hypothesis, $\{Cxw\}$ is OE-connected to $\{Cxv, Cxv_2,\ldots,Cxv_j\}$, it follows that $\{Cxw\}$ is OE-connected to $\{Cxv_1,\ldots,Cxv_j\}$. So $\{Cxw\}$ is OE-connected to $\{Cxw_1,\ldots,Cxw_j\}$. *Case 2:* $\langle S_k, S_{k+1}\rangle$ is an instance of (CC) or (NC). This case is treated like Case 1.

Lemma 2 *If* w *is a wff then there are elementary wffs* $w_1,\ldots,w_j$ *such that* $\{w\}$ *is OE-connected to* $\{w_1,\ldots,w_j\}$.

Proof: We shall use induction on the number n of C's in w.

Basis step: $n = 0$. Then $w = N_1 c$ or $w = N_1 L N_2 c$, where N_1 and N_2 are (possibly empty) strings of N's, and c is a categorical expression. By (NN) and the definitions of M, E, and O, $\{w\}$ is OE-connected to $\{s\}$, where s is a simple wff (and hence an elementary wff).

Induction step: $n = k + 1$. We consider two cases.

Case 1: There is exactly one C prior to the leftmost occurrence of a categorical expression c in w. Then $w = N_1 C N_2 cx$ or $w = N_1 C N_2 L N_3 cx$, where N_1, N_2, and N_3 are (possibly empty) sequences of N's. So, by (NN) and the definitions of M, E, and O, $\{w\}$ is OE-connected either to $\{Csx\}$ or to $\{NCsx\}$, where s is a simple wff. So, there are two subcases to consider. *Subcase i:* $\{w\}$ is OE-connected to $\{Csx\}$. By the induction hypothesis $\{x\}$ is OE-connected to $\{x_1,\ldots,x_n\}$, where each x_i $(1 \leq i \leq n)$ is an elementary wff. By Lemma 1 $\{Csx\}$ is OE-connected to $\{Csx_1,\ldots,Csx_n\}$. So $\{w\}$ is OE-connected to a set of elementary wffs. *Subcase ii:* $\{w\}$ is OE-connected to $\{NCsx\}$. Since $\{NCsx\}$ is OE-connected to $\{s, Nx\}$ and since, by the induction hypothesis, $\{Nx\}$ is OE-connected to $\{x_1,\ldots,x_n\}$, where each x_i $(1 \leq i \leq n)$ is an elementary wff, it follows that $\{w\}$ is OE-connected to a set of elementary wffs.

Case 2: There are at least two C's to the left of the leftmost categorical expression in w. Then $w = N_1 C N_2 Cxyz$. There are three subcases to consider.

Subcase i: $\{w\}$ is OE-connected to $\{CCxyz\}$. By (CC), $\{CCxyz\}$ is OE-connected to $\{CNxz, Cyz\}$. So, by the induction hypothesis, $\{w\}$ is OE-connected to a set containing only elementary wffs. *Subcase ii:* $\{w\}$ is OE-connected to $\{CNCxyz\}$. By (CNC), $\{CNCxyz\}$ is OE-connected to $\{CxCNyz\}$. By the induction hypothesis $\{CNyz\}$ is OE-connected to $\{x_1,\ldots,x_n\}$, where each x_i $(1 \leq i \leq n)$ is an elementary wff. By Lemma 1, $\{CxCNyz\}$ is OE-connected to $\{Cxx_1,\ldots,Cxx_n\}$, whose members are elementary wffs. *Subcase iii:* $\{w\}$ is OE-connected to

{NCC*xyz*} or to {NCNC*xyz*}. So, {*w*} is OE-connected either to {C*xy*,N*z*} or to {NC*xy*,N*z*}. So, by the induction hypothesis, {*w*} is OE-connected to a set that has only elementary wffs as members.

To illustrate OE-chains note that {CCNA*ab*A*ba*I*ab*} is OE-connected to {CNNA*ab*I*ab*,CA*ba*I*ab*} (by (CC)), which is OE-connected to {CA*ab*I*ab*, CA*ba*I*ab*},[6] whose members are elementary wffs.

As another example note that (by (NC)) {NCI*ab*NI*cd*} is OE-connected to {I*ab*,NNI*cd*}, which (by (NN)) is OE-connected to {I*ab*,I*cd*}. By two applications of Lemma 1, {CA*ab*CA*cd*NCI*ab*NI*cd*} is OE-connected to {CA*ab*CA-*cd*I*ab*,CA*ab*CA*cd*I*cd*}.

Lemma 3 *If* $\{w\}$ *is OE-connected to* $\{w_1, \ldots, w_j\}$ *then: (i) if w is valid, each* w_i *is valid; (ii) if w is invalid, some* w_i *is invalid; (iii) if each* w_i *is accepted, w is accepted; and (iv) if some* w_i *is rejected, w is rejected.*

Proof: The proof is by induction on n where $S_1, \ldots, S_n$ is the OE-chain in virtue of which $\{w\}$ is OE-connected to $\{w_1, \ldots, w_j\}$.

Basis step: $n = 1$. Tautological.

Induction step: $n = k + 1$. There are two cases to consider. *Case 1:* $\langle S_k, S_{k+1}\rangle$ is an instance of (NN) or (CNC). Then $\langle S_k, S_{k+1}\rangle$ is an instance of $\langle\{x, \ldots\}, \{y, \ldots\}\rangle$, where C*xy* and C*yx* may be entered as members of a deduction, given R4 and R3. So, if S_k has only valid members so does S_{k+1}; if S_k has an invalid member so does S_{k+1}; if S_{k+1} has only accepted members so does S_k (by R1); and if S_{k+1} has a rejected member so does S_k (by *R3). *Case 2:* $\langle S_k, S_{k+1}\rangle$ is an instance of (CC) or (NC). The reasoning is similar to that for Case 1.

Now we are ready to prove Theorem 4. Suppose w is valid. Then by Lemma 2 there are elementary wffs $w_1, \ldots, w_j$ such that $\{w\}$ is OE-connected to $\{w_1, \ldots, w_j\}$. By Lemma 3 each w_i is valid. By Theorem 3 each w_i is accepted. And by Lemma 3 w is accepted. By similar reasoning it follows that if w is invalid then w is rejected.

With the proof of Theorem 4 and the earlier proof of Theorem 2 we have reached our main goal: providing a semantics for LXM. We shall conclude our discussion by considering a semantic decision procedure for LXM. To this end we shall show that:

Theorem 5 *If a wff* $S(a_1, \ldots, a_n)$ *is false in a model M then there is a model* M', *where* $M' = \langle W', V'^e, V'^a, V'^e_c, V'^a_c\rangle$, *such that* W' *is a subset of* $\{1, 2, \ldots, 4^n\}$ *and* $S(a_1, \ldots, a_n)$ *is false in* M'.

Theorem 5 yields a decision procedure for validity, since in virtue of it only a finite number of models need to be examined to test a wff for validity.

To show that Theorem 5 is true we shall specify a procedure for constructing a model M' of the sort specified:

(a) Let $M = \langle W, V^e, V^a, V^e_c, V^a_c\rangle$. Enumerate the 4^n sets $\bigcap_{i=1}^{n} V^{j_i}_{k_i}(a_i)$, where $j_i = e$ or $j_i = a$, and $k_i = c$ or $k_i = \varnothing$. Call these 4^n sets the *basic sets of M relative to* $a_1, a_2, \ldots, a_n$. (Here are three of the sixteen basic sets relative to a_1 and a_2: $V^e(a_1) \cap V^e(a_2)$, $V^e(a_1) \cap V^a(a_2)$, and $V_c^e(a_1) \cap V_c^a(a_2)$.)

(b) Let E be an enumeration of the basic sets relative to $a_1, \ldots, a_n$. Then let $x \in W'$ iff the x-th set in the enumeration E is nonempty.

(c) Let $x \in V'^{j_i}_{k_i}(a_i)$ ($j_i = e$ or a, $k_i = c$ or $\emptyset$) iff

(i) '$V^{j_i}_{k_i}(a_i)$' is used to form the expression that denotes the x-th set in E, and

(ii) the x-th set in E is nonempty.

Let us illustrate the procedure. 'C*IabAab*' is falsified in this model: $W =$ the set of natural numbers, $V^e(a) =$ the set of odd numbers, $V^a(a) = \emptyset$, $V_c^e(a) =$ the set of even numbers, $V_c^a(a) = \emptyset$, $V^e(b) =$ the set of prime numbers, $V^a(b) = \emptyset$, $V_c^e(b) =$ the set of natural numbers that are not prime, and $V_c^a(b) = \emptyset$. The basic sets relative to a and b that are nonempty are listed as: (1) $V^e(a) \cap V^e(b)$, (2) $V^e(a) \cap V_c^e(b)$, (3) $V_c^e(a) \cap V^e(b)$, and (4) $V_c^e(a) \cap V_c^e(b)$. By part (b) of the procedure, $W' = \{1,2,3,4\}$. By part (c) of the procedure, $V'^e(a) = \{1,2\}$, $V'^a(a) = \emptyset$, $V'^e_c(a) = \{3,4\}$, $V'^a_c(a) = \emptyset$, $V'^e(b) = \{1,3\}$, $V'^a(b) = \emptyset$, $V'^e_c(b) = \{2,4\}$, and $V'^a_c(b) = \emptyset$. So $V'(\mathrm{I}ab) = t$ and $V'(\mathrm{A}ab) = f$. So $V'(\mathrm{CI}ab\mathrm{A}ab) = f$.

To prove Theorem 5, we shall first verify that M' is a model if M is a model. Let us assume that M is a model and let us examine the conditions (a)–(f) used to define a model. *Condition* (a). $V^e(a) \neq \emptyset$. So one of the basic sets of M relative to $a, \ldots$ *denoted by using* '$V^e(a)$' is nonempty. So $V'^e(a) \neq \emptyset$. *Condition* (b). Suppose $V'^j_k(a) \cap V'^m_n(a) \neq \emptyset$ for $j \neq m$ and $k \neq n$. Then '$V^j_k(a)$' and '$V^m_n(a)$' are used to denote the same basic set of M relative to $a, \ldots$. But this is impossible. So the $V'^j_k(a)$'s are mutually exclusive. To see that they are mutually exhaustive suppose there is an x such that $x \in W'$ and $x \notin V'^e(a) \cup V'^a(a) \cup V'^e_c(a) \cup V'^a_c(a)$. Since $x \in W'$, the x-th basic set of M relative to $a, \ldots$ is nonempty. This set must be denoted by using one of the expressions '$V^e(a)$', '$V^a(a)$', '$V_c^e(a)$', or '$V_c^a(a)$'. But then x must belong to one of $V'^e(a)$, $V'^a(a)$, $V'^e_c(a)$ or $V'^a_c(a)$. But this is impossible. *Condition* (c). Suppose $V'(z) \subset V'^e_c(y)$ and $V'(x) \subset V'(y)$. Then $V(z) \subset V_c^e(y)$ and $V(x) \subset V(y)$. By the assumption that M is a model, $V(x) \subset V_c^e(z)$. But then $V'(x) \subset V'^e_c(z)$. (For suppose $V'(x) \not\subset V'^e_c(z)$. Then one of these cases obtains: (i) $V'^e(x) \cap V'^e(z) \neq \emptyset$; (ii) $V'^e(x) \cap V'^a(z) \neq \emptyset$; (iii) $V'^e(x) \cap V'^a_c(z) \neq \emptyset$; (iv) $V'^a(x) \cap V'^e(z) \neq \emptyset$; (v) $V'^a(x) \cap V'^a(z) \neq \emptyset$; or (vi) $V'^a(x) \cap V'^a_c(z) \neq \emptyset$. In the first case it follows that $V^e(x) \cap V^e(z) \neq \emptyset$, which contradicts the claim that $V(x) \subset V_c^e(z)$. This claim is also contradicted in the other five cases.) *Conditions* (d), (e), and (f). Use the same type of argument as that used for Condition (c).

To complete the proof we shall use:

Lemma 1 *If c is equal to any of the expressions* A*xy*, I*xy*, LA*xy*, LI*xy*, LNA*xy*, *or* LNI*xy*, *where x and y need not be distinct, then c is true in M iff c is true in M′.*

Proof: A*xy* is true in M iff $V(x) \subset V(y)$. And $V(x) \subset V(y)$ iff $V'(x) \subset V'(y)$. (First, suppose $V(x) \subset V(y)$ and $V'(x) \not\subset V'(y)$. Then one of these cases obtains: (i) $V'^e(x) \cap V'^e_c(y) \neq \emptyset$; (ii) $V'^e(x) \cap V'^a_c(y) \neq \emptyset$; (iii) $V'^a(x) \cap$

$V'^e_c(y) \neq \varnothing$; or (iv) $V'^a(x) \cap V'^a_c(y) \neq \varnothing$. In the first case $V^e(x) \cap V^e_c(y) \neq \varnothing$. But then $V(x) \not\subset V(y)$. And the other three cases yield the same result. Secondly, suppose $V'(x) \subset V'(y)$ and $V(x) \not\subset V(y)$. Then one of these cases obtains: (i) $V^e(x) \cap V^e_c(y) \neq \varnothing$; (ii) $V^e(x) \cap V^a_c(y) \neq \varnothing$; (iii) $V^a(x) \cap V^e_c(y) \neq \varnothing$; or (iv) $V^a(x) \cap V^a_c(y) \neq \varnothing$. In the first case $V'^e(x) \cap V'^e_c(y) \neq \varnothing$. But then $V'(x) \not\subset V'(y)$. And the other three cases yield the same result.) So A*xy* is true in *M* iff A*xy* is true in *M′*. And the arguments for the other expressions follow the same pattern.

Now we use strong induction on the number *m* of C's in $S(a_1, \ldots, a_n)$.

Basis step: $m = 0$. Then $S = \mathrm{N}_1 c$ or $\mathrm{S} = \mathrm{N}_1\mathrm{LN}_2 c$, where N_1 and N_2 are (possibly empty) strings of N's, and *c* is either A*xy* or I*xy*. So, by Lemma 1, *S* is true in *M* iff *S* is true in *M′*.

Induction step: $m = k + 1$. Then it is an immediate consequence of the induction hypothesis and the truth-conditions for C that *S* is true in *M* iff S is true in *M′*. And thus, *a fortiori*, Theorem 5 holds.

A natural extension of the above development of Aristotle's logic would provide a semantics for a logical system that captured Aristotle's insights about contingent propositions. But so far I have been unable to give a satisfactory interpretation of his contingency operator.[7]

And another direction to move the discussion is into the area of natural deduction systems. Smiley in [4] gives very persuasive arguments to show that syllogisms should not be treated as conditional statements in the style of Łukasiewicz, but should be treated as deductive structures. And Corcoran in [1] also presents Aristotle's logic by using a natural deduction system. The differences in the systems of Smiley and Corcoran suggest that many natural deduction complements to LXM are possible.[8]

NOTES

1. Whether his treatment of syllogisms as conditional statements is proper is debatable. See the conclusion of this paper and see [1] and [4].
2. Note that LLA*ab* is not a wff. Since LA*ab* is not a categorical expression, though it is a wff, (iv) cannot be applied to it to generate a wff. And note that *A*ab* is not a wff, and **A*ab* is not a starred expression.
3. Here *x* and *y* range over wffs, and in some other places they range over term variables, as in R2 below. But they never range over starred expressions.
4. Note that the top line on page 65 of [3], where the proof of *A5.7 is underway, should have 'MA*ec*' in place of 'LA*ec*'. On the preceding page McCall has shown that his substitutions do not permit the occurrence of 'A*ec*'. So there could be no occurrence of 'LA*ec*'.
5. The full line is found by replacing '(*A5.7)' by *A5.7. And we shall also abbreviate other lines by using names of axioms.
6. Łukasiewicz also reduces CCNA*ab*A*ba*I*ab* to these elementary wffs (see p. 119 of [2]). But OE-chains provide a simpler means of accomplishing the reduction. Note

that Łukasiewicz uses nine reduction rules, whereas only four are used to construct OE-chains.

7. See [3] for a purely syntactical system that is designed to illuminate Aristotle's contingency operator. And note McCall's strong arguments to show that the contingency operator cannot be defined in terms of the syntactical connectives of LXM.

8. Smiley (p. 140 of [4]) says that the following arguments are not Aristotelian syllogisms: (1) 'A*ab*, O*ab*; so I*cd*' and (2) 'A*ab*, E*ab*; so O*ca*'. But the following annotated deductions in Corcoran's system show that Corcoran counts them as valid Aristotelian syllogisms: (1) +A*ab*, +O*ab*, ?I*cd*, *h*E*cd*, *a*A*ab*, B*a*O*ab* and (2) +A*ab*, +E*ab*, ?O*ca*, *h*A*ca*, *s*A*cb*, *c*I*cb*, *s*E*cb*.

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