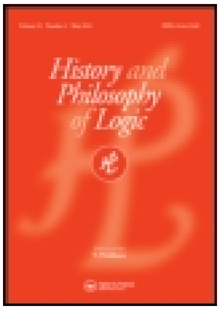

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Aristotle'S natural deduction reconsidered

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Aristotle's Natural Deduction Reconsidered

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John Corcoran's natural deduction system for Aristotle's syllogistic is reconsidered. Though Corcoran is no doubt right in interpreting Aristotle as viewing syllogisms as arguments and in rejecting Lukasiewicz's treatment in terms of conditional sentences, it is argued that Corcoran is wrong in thinking that the only alternative is to construe Barbara and Celarent as deduction rules in a natural deduction system. An alternative is presented that is technically more elegant and equally compatible with the texts. The abstract role assigned by tradition and Lukasiewicz to Barbara and Celarent is retained. The two 'perfect syllogisms' serve as 'basic elements' in the construction of an inductively defined set of valid syllogisms. The proposal departs from Lukasiewicz, and follows Corcoran, however, in construing the construction as one in natural deduction. The result is a sequent system with fewer rules and in which Barbara and Celarent serve as basic deductions. To compare the theory to Corcoran's, his original is reformulated in current terms and generalized. It is shown to be equivalent to the proposed sequent system, and several variations are discussed. For all systems mentioned, a method of Henkin-style completeness proofs is given that is more direct and intuitive than Corcoran's original.

Introduction

My goal in this paper is to reconsider John Corcoran's, now classic, work on the syllogistic.[1] Corcoran's purpose was to argue against two key theses of the interpretation of Lukasiewicz (1957) and others: that syllogisms should be construed as conditional sentences in an object language of the form $(A \wedge B) \rightarrow C$; and that Aristotle's reduction of the valid syllogisms to the 'perfect syllogisms' should be viewed as axiomatic theory in which Barbara and Celarent serve as axioms and the valid syllogisms are derived by rules of inference.[2] Corcoran (1972 and 1974) argues rather that a syllogism is a three-line argument with two premises and a single conclusion, and that Aristotle's reductions should be construed as derivations of a conclusion from premises understood as natural deductions ('perfect syllogisms') in which Barbara and Celarent serve, not as 'axioms', but as rules of natural deduction.

1 See especially Corcoran (1972 and 1974). Smiley (1973) proposes essentially the same axiomatic system as does Corcoran. Both also provide completeness proofs of a non-standard sort. The proof advanced below, in contrast, follows the familiar steps of a Henkin proof, and is offered as part of the paper's general aim of stating Aristotle's natural deduction theory using the concepts that have now become standard in the field. It should be noted that Smiley (1973) remarks on the fact that the rule, Cut, is definable in terms of the more primitive rules. Mention should also be made of Thom (1981) who offers a rich proof theory that contains versions of the systems of Corcoran and Smiley, and makes a step in the direction of this paper by treating syllogisms as syntactic units. It does not, however, include a semantics. The concept of saturation used below is a version of Aristotle's *ecthesis*. Robin Smith 1983 adds a natural deduction version of the rule to Corcoran's system: if z is some term new to X, Y and B, then from $X \vdash Ixy$ and $Y, Azx, Azy \vdash \mathbf{B}$, deduce $X, Y \vdash B$, and from $X \vdash Oxy$ and $Y, Azx, Ezy \vdash \mathbf{B}$, deduce $X, Y \vdash B$, Saturation mandates the presence (not assured by validity) of some Azx, Azy, given that of Ixy, and of some Azx, Ezy, given that of Oxy, much as the notion in first-order logic requires that of some $A[t]$, given that $\exists x A[x]$. See also Paul Thom 1976. For a general survey of natural deduction frameworks, including the sequent version used below, see Göran Sundholm 1983.

2 See Lukasiewicz (1957); Sheperdson (1956); Smiley (1962).

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Corcoran formalizes the relevant natural deduction theory and provides a soundness and completeness proof for the syllogistic under its usual set-theoretic interpretation.

The purpose of my discussion here is threefold.

1. I reformulate Corcoran's theory in a fashion that makes use of more recent developments in natural deduction theory. In doing so, I try to clarify the original discussion. At the same time I shall formulate the proof theory and semantics in somewhat more general terms.
2. I argue that Corcoran's success in fact is conceptually consistent with the 'axiomatic' construal of Barbara and Celarent. In particular, I show that Corcoran's theory is fully equivalent to a proposed natural deduction theory that retains all of Corcoran's natural deduction rules, with the exception of his rule versions of Barbara and Celarent. In the proposed system the two 'perfect syllogisms' function as 'basic deductions' from which all the valid syllogisms are derived, much as they usually have been understood in the history of logic. The possibility of this alternative becomes clear when the background theory is developed in a more current idiom.
3. I offer what I think is a more direct and natural completeness proof of the relevant theories, using the usual Henkin method of extending consistent sets to saturated maximally consistent sets, and then defining for any maximally consistent set a model constructed from the sentences of the set that satisfies the set.

I Background Ideas in Natural Deduction

Syntax and semantics

For the purposes of background theory of natural deduction, it is sufficient to define a *syntax with set Sen of sentences* as any structure, $\langle E_1,\dots,E_n,\mathrm{Sen}\rangle$, containing a distinguished set of expressions, Sen, intended to represent a set of sentences. We shall let *Syn* range over syntaxes. Let $A,\dots,E$ range over Sen, and $X,\dots,Z$ over subsets of Sen.

By *an interpretation* for a syntax, $\mathrm{Syn} = \langle E_1,\dots,E_n,\mathrm{Sen}\rangle$, is meant any function, R, such that, for some sets, $\upsilon_1,\dots,\upsilon_n$, R is a function from $E_1 \cup \dots \cup E_n \cup \mathrm{Sen}$ to $\upsilon_1,\dots,\upsilon_n \cup \{T,F\}$ such that:

1. for $i = 1,\dots n$, if $x \in E_i$, then $R(x) \in \upsilon_i$ and
2. if $A \in \mathrm{Sen}$, then $R(A) \in \{T,F\}$.

A *language* is identified with any pair $\langle \mathrm{Syn},\mathscr{R}\rangle$ such that $\mathscr{R}$ is some set of interpretations for Syn. Let R range over $\mathscr{R}$, and $L = \langle \mathrm{Syn},\mathscr{R}\rangle$ over the languages for Syn. Logical relations are defined relative to a language, L. Let $L = \langle \mathrm{Syn},\mathscr{R}\rangle$:

R '*satisfies*' X in L iff, for all $A \in X$, $R(A) = T$;
X is '*satisfiable*' in L iff, for some R, R satisfies X;
X is '*unassailable*' iff, for any R, there is some $A \in X$, such that $R(A) = T$;
$X \vDash_L A$ iff, for all $R \in \mathscr{R}$, R satisfies X only if $R(A) = T$;
$\vDash_L A$ iff $\varnothing \vDash_L A$.

Proof theory

Inductive sets

Since this paper contrasts various attempts to define the notion of 'provable syllogism' in terms of axiom and natural deduction systems, it is useful to begin with concepts from the theory of induction. For present purposes it is sufficient to identify an *inductive* system with a structure $\langle B, C, \{R_1, \ldots, R_n\} \rangle$ such that:

1. B (the set of *basic elements* of the system) and C (the set *constructed* by the system) are at most denumerable sets;
2. each R_i (a *construction rule* of the system) is a finite relation on $B \cup C$;
3. C is the least set X such that $B \subseteq X$, for any R_i, if R_i is an $m+1$-place relation, $\langle e_1, \ldots, e_{m+1} \rangle \in R$ and $\langle e_1, \ldots, e_m \rangle \in C$, then $e_{m+1} \in C$.

Relative to an at most denumerable set B and a set of finitary relations $\{R_1, \ldots, R_n\}$ defined for tuples in B, a *derivation* (*tree*) relative to B and $\{R_1, \ldots, R_n\}$ is defined as any finite labelled tree Π such that:

1. Every leaf node of Π is labelled by an element in B.
2. For any node, n, of Π with immediate predecessor nodes $m_1, \ldots, m_k$,:
   (a) each m_i (for $i \leqslant k$) is labelled by some element e_i, and
   (b) n is labelled by e and some rule R_i, such that $\langle e_1, \ldots, e_k, e \rangle \in R_i$.

If the leaf nodes of a deduction tree Π are labelled, respectively, $e_1, \ldots, e_k$, its root node is labelled by e, and if $\{R_1, \ldots, R_n\} = \{R \mid R$ is a finitary relation on Sen that labels some node of $\Pi\}$, we say Π is a *derivation* (*tree*) of e from $\langle e_1, \ldots, e_k \rangle$ relative to $\{R_1, \ldots, R_n\}$. If, in addition, all the leaf nodes of Π are in B, then Π is called a *proof* (*tree*) of e from $\langle e_1, \ldots, e_k \rangle$ relative to B and $\{R_1, \ldots, R_n\}$. It is an elementary fact of induction theory that $e \in C$ for an inductive system $\langle B, C, \{R_1, \ldots, R_n\} >$ iff there is some proof tree of e relative to B and some subset of $\{R_1, \ldots, R_n\}$.

Axiom systems

For our purposes here, an *axiom system* may be identified with an inductive system, $\langle Ax, \vdash, \{R_1, \ldots, R_n\} \rangle$, such that for some syntax, $Syn = \langle E_1, \ldots, E_n, Sen \rangle$, Ax and $\vdash$ are subsets of Sen.

Natural deduction systems

By a *deduction* in Syn in meant any pair $\langle X, A \rangle$ such that $A \in$ Sen and X is a finite subset of Sen. Here X is called the '*premise set*' of the deduction and A the '*conclusion*'. It will suffice to define an '*inference rule*' for Syn as any finitary relation on deductions in Syn. In addition, a special set, BD, of deductions is distinguished, called the '*set of basic deductions*'. By a '*natural deduction system*' for Syn is meant any inductive system $\langle BD, \vdash, RL \rangle$ such that BD is a set of deductions distinguished for Syn, and RL is a set of derivation rules for Syn. The inductively defined relation $\vdash$ is called the 'set of *provable deductions*' for Syn relative to BD and RL. We write $X \vdash A$ for $\langle X, A \rangle \in \vdash$, and adopt the customary abbreviations:

$X, A \vdash B$	means	$X \cup \{A\} \vdash B$;
$A_1, \ldots, A_n \vdash B$	means	$\{A_1, \ldots, A_n\} \vdash B$;
$\vdash A$	means	$\varnothing \vdash A$.

It follows from induction theory that $\langle X, A \rangle$ is a provable deduction in $\langle BD, \vdash, RL \rangle$ iff there is some proof tree of Syn relative to BD and some subset of RL such that its root node is labelled by $\langle X, A \rangle$.

In cases in which the notion of a uniform substitution, S, is defined for Syn, it is customary to define a derivation rule, R, for Syn by a tree diagram. If $S(A)$ is the result of uniformally substituting elements of Sen for sentence letters of A, the notion is extended to sets and deductions as follows: $S(X) = \{S(A) \mid A \in X\}$ and $S(\langle X, A \rangle) = \langle S(X), S(A) \rangle$. Then, the tree

$$R\colon \frac{\langle X_1, A_1 \rangle, \quad \ldots \quad \langle X_n, A_n \rangle}{\langle Y, B \rangle}$$

defines the relation $R = \{\langle d_1, \ldots, d_{n+1} \rangle \mid$ for some uniform substitution S, $d_1 = S(\langle X_1, A_1 \rangle)$ & ... & $d_{n+1} = S(\langle Y, B \rangle)\}$.

A relation, R, is said to be '*definable*' relative to rules $R_1, \ldots, R_m$ and is called a '*derived rule*' in $\langle BD, \vdash, RL \rangle$, where $\{R_1, \ldots, R_n\} \subseteq RL$, iff there is a derivative tree Π of d_{n+1} from $d_1, \ldots, d_n$ relative to BD and $\{R_1, \ldots, R_m\}$, and $R = \{S(d_1), \ldots, S(d_{n+1}) \mid S$ is a uniform substitution for Sen$\}$. A natural deduction system, $\langle BD, \vdash, RL \rangle$, is said to be '*reducible*' to a natural deduction system, $\langle BD', \vdash', RL' \rangle$, iff $BD \subseteq BD'$ and every $R \in RL$ is a derivable rule in $\langle BD', \vdash', RL' \rangle$. Two systems are '*strictly equivalent*' iff they are mutually reducible. Let two systems, $\langle BD, \vdash, RL \rangle$ and $\langle BD', \vdash', RL' \rangle$, be called '*constructively equivalent*' iff $\vdash = \vdash'$.

II The Syllogistic

The syntax for the syllogistic

An at most denumerable set of common nouns is posited:

$$\text{Terms} = \{\boldsymbol{t}_1, \ldots, \boldsymbol{t}_n, \ldots\}.$$

Two notions of sentence, one narrow and one wide, are defined according to whether we abstract from Aristotle's restriction against allowing the same term to occur simultaneously as the subject and predicate of a sentence. (It follows from the more general forms of the completeness results below that the restriction is unnecessary and plays no significant theoretical role in the semantics or proof theory.)

$$\text{Sen}^+ = \{z \mid \text{for some } x, y \in \text{Terms},\ z = \boldsymbol{A}xy \text{ or } z = \boldsymbol{E}xy \text{ or } z = \boldsymbol{I}xy \text{ or } z = \boldsymbol{O}xy\}.$$

$$\text{Sen}^- = \{z \mid \text{for some } x, y \in \text{Terms},\ x \neq y \text{ and, } z = \boldsymbol{A}xy \text{ or } z = \boldsymbol{E}xy \text{ or } z = \boldsymbol{I}xy \text{ or } z = \boldsymbol{O}xy\}.$$

From this point, let Sen range over $\{\text{Sen}^+, \text{Sen}^-\}$. A '*syllogistic syntax*' is defined as any pair, $\langle$Terms, Sen$\rangle$; and let Syn range over syllogistic syntaxes. Let variables x, y, z range over Terms. A syntactic negation is introduced by eliminative definition:

$$\sim \boldsymbol{A}xy =_{\text{def}} \boldsymbol{O}xy \qquad \sim \boldsymbol{E}xy =_{\text{def}} \boldsymbol{I}xy$$
$$\sim \boldsymbol{I}xy =_{\text{def}} \boldsymbol{E}xy \qquad \sim \boldsymbol{O}xy =_{\text{def}} \boldsymbol{A}xy$$

By an '*interpretation for a syntax, Syn*' is meant any R such that, for some $\mathscr{U}$, R is a function from Terms $\cup$ Sen to $\mathscr{U} \cup \{T, F\}$ such that:

1. If $x \in$ Terms, then $R(x) \in \mathscr{U}$.
2. If $A \in$ Sen, then $R(A) \in \{T, F\}$.

A '*syllogistic language*' is identified with any pair, $\langle \text{Syn}, \mathscr{R} \rangle$, such that $\mathscr{R}$ is some set of interpretations for Syn. Let R range over $\mathscr{R}$, and $L = \langle \text{Syn}, \mathscr{R} \rangle$ over the languages for Syn. Logical relations are defined relative to a language, L. Let $L = \langle \text{Syn}, \mathscr{R} \rangle$.

R '*satisfies*' X in L iff for all $A \in X$, $\mathrm{R}(A) = \mathrm{T}$;
X is '*satisfiable*' in L iff, for some R, R satisfies X;
X is '*unassailable*' iff, for any R, there is some $A \in X$, $\mathrm{R}(A) = \mathrm{T}$;
$X \vDash_{\mathrm{L}} A$ iff, for all $\mathrm{R} \in \mathscr{R}$, R satisfies X only if $\mathrm{R}(A) = \mathrm{T}$;
$\vDash_{\mathrm{L}} A$ iff $\varnothing \vDash_{\mathrm{L}} A$.

This study will be concerned only with a particular type of language in which terms are interpreted over non-empty sets—as in the accounts of Aristotle and Corcoran—or, more abstractly, in which terms are assigned non-minimal elements of a special variety of semi-lattice. By an '*order theoretic model structure for Syn*' is meant any structure, $\langle \mathscr{U}, \leqslant, \wedge, 0 \rangle$, such that:

1. $\langle \mathscr{U}, \leqslant \rangle$ is a partially ordered structure with least element 0; and
2. $\langle \mathscr{U}, \wedge \rangle$ is the meet semi-lattice determined by $\langle \mathscr{U}, \leqslant \rangle$.

Some direct properties of such model structures are now listed. Let $\langle \mathscr{U}, \leqslant, \wedge, 0 \rangle$ be such a structure and let $x, y \in \mathscr{U}$. The properties numbered 4–7 are the semantic correlates of the trivial ('immediate') relations of the traditional square of opposition:

1. $x \leqslant y$ iff $x \wedge y = x$;
2. $\mathscr{U}$ is closed under $\wedge$, and $\wedge$ is idempotent, commutative and associative;
3. $x \wedge 0 = 0$;
4. $x \leqslant y$ only if $x \wedge y \neq 0$, if $x \neq 0$;
5. $x \wedge y = 0$ only if not $(x \leqslant y)$, if $x \neq 0$;
6. not $(x \leqslant y$ iff not $(x \leqslant y))$;
7. not $(x \wedge y \neq 0$ iff $x \wedge y = 0)$.

An '*order theoretic interpretation*' of Syn relative to an order theoretic model structure, $\langle \mathscr{U}, \leqslant, \wedge, 0 \rangle$, is defined as any interpretation, R, of Syn mapping Terms $\cup$ Sen to $\mathscr{U} \cup \{\mathrm{T}, \mathrm{F}\}$ such that:

1. If $x \in$ Terms, $\mathrm{R}(x) \in \mathscr{U}$ and $\mathrm{R}(x) \neq 0$.
2. If $A \in$ Sen, then:
   (a) if A is some $\boldsymbol{A}xy$, then $\mathrm{R}(A) = \mathrm{T}$ iff $\mathrm{R}(x) \leqslant \mathrm{R}(y)$;
   (b) if A is some $\boldsymbol{E}xy$, then $\mathrm{R}(A) = \mathrm{T}$ iff $\mathrm{R}(x) \wedge \mathrm{R}(y) = 0$;
   (c) if A is some $\boldsymbol{I}xy$, then $\mathrm{R}(A) = \mathrm{T}$ iff $\mathrm{R}(x) \wedge \mathrm{R}(y) \neq 0$;
   (d) if A is some $\boldsymbol{O}xy$, then $\mathrm{R}(A) = \mathrm{T}$ iff not $(\mathrm{R}(x) \leqslant \mathrm{R}(y))$.

Let $\mathscr{R}^+$ be the set of all order-theoretic interpretations for Syn^+, and $\mathscr{R}^-$ the set of order-theoretic interpretations for Syn^-. Let an order-theoretic model structure $\langle \mathscr{U}, \leqslant, \wedge, 0 \rangle$ be called '*set theoretic*' iff $\mathscr{U}$ is a family of sets, $\leqslant$ is set inclusion in $\mathscr{U}$, $\wedge$ is set intersection and 0 is the empty set. Let $\mathscr{R}\!\downarrow$ be the restriction of a set of order-theoretic interpretations to those which are set theoretic. Then four languages—$\mathrm{L}^+ = \langle \mathrm{Syn}^+; \mathscr{R}^+ \rangle$ and $\mathrm{L}^- = \mathrm{Syn}^-; \mathscr{R}^- \rangle$—and their restrictions—$\mathrm{L}\!\downarrow^+ = \langle \mathrm{Syn}^+; \mathscr{R}\!\downarrow^+ \rangle$ and $\mathrm{L}\!\downarrow^- \langle \mathrm{Syn}^-; \mathscr{R}\!\downarrow^- \rangle$—may be distinguished. From this point, let $\mathscr{R}$ range over these four sets of interpretations and L over these four languages.

The semantic version of the theory of 'immediate' inference follows trivially for all four languages:

Theorem: 1. $\boldsymbol{A}xy \vDash \boldsymbol{I}xy$ and $\boldsymbol{E}xy \vDash \boldsymbol{O}xy$;
2. $\{\boldsymbol{A}xy, \boldsymbol{E}xy\}$ is not satisfiable, and $\{\boldsymbol{I}xy, \boldsymbol{O}xy\}$ is unassailable;
3. $\boldsymbol{A}xy \vDash \sim \boldsymbol{O}xy$, $\boldsymbol{O}xy \vDash \sim \boldsymbol{A}xy$, $\boldsymbol{E}xy \vDash \sim \boldsymbol{I}xy$, $\boldsymbol{I}xy \vDash \sim \boldsymbol{E}xy$.

Natural deduction concepts

A series of rules necessary for the statement of Corcoran's original 1972 theory and for abstractions from it will be defined first.

$$\text{C1: } \frac{X \vdash \boldsymbol{E}xy}{X \vdash \boldsymbol{E}xy}$$

$$\text{C2: } \frac{X \vdash \boldsymbol{A}xy}{X \vdash \boldsymbol{I}xy}$$

$$\text{RD: } \frac{X \vdash A \quad Y \vdash \sim A}{X \cup Y - \{B\} \vdash \sim B}$$

$$\text{ID: } \frac{X \vdash B}{X \vdash \boldsymbol{A}xx}$$

The first three rules are used for deductions in Sen^-. ID is added for deductions in Sen^+. The rule RD is a general version of *reductio* and may be partitioned into three disjoint subrelations:

$$\text{RD1: } \frac{XB \vdash A \quad Y \vdash \sim A}{X, Y \vdash \sim B}$$

$$\text{RD2: } \frac{X \vdash A \quad Y \vdash \sim A}{X, Y \vdash B}$$

$$\text{RAI: } \frac{X, \sim B \vdash A \quad Y, \sim B \vdash \sim A}{X, Y \vdash B}$$

The first rule is said to '*discharge*' B. The second is a version of *ex falso quodlibet*. The third is a version of Aristotle's rule *reductio ad impossible* and is said to '*discharge*' $\sim B$.

Three important additional rules are:

$$\text{Thinning: } \frac{X \vdash A}{X, Y \vdash A}$$

$$\text{Cut: } \frac{X \vdash A \quad Y, A \vdash B}{X, Y \vdash B}$$

$$\text{Transposition: } \frac{XA \vdash B}{X, \sim B \vdash \sim A}$$

Theorem: 1. Thinning is definable in terms of RD1.
2. Cut is definable in terms of RD1.
3. Transposition is definable in terms of RD1.
4. RD1 is definable in terms of Cut and Transposition.
5. RD2 is definable in terms of Cut, Thinning and Transposition.
6. RAI is definable in terms of RD1 and RD2.
7. RD1 is definable in terms of RAI and Thinning.
8. RD2 is definable in terms of RAI and Thinning.

Proof: Proofs of parts 2–8 are straightforward; part 1 is shown by induction on the nested subsets of $Y = \{B_1, \ldots, B_n\}$.

Basis step. Let $Y = \{B_1\}$:

$$\frac{X \vdash A \quad X, B, \sim A \vdash \sim A \text{ (basic)}}{X, B \vdash A \text{ (RDI)}}$$

Inductive step: Assume that $X, B_1, \ldots, B_i \vdash A$ is definable. Then, $X, B_1, \ldots, B_{i+1} \vdash A$ is derived as follows:

$$\frac{X, B_1, \ldots, B_i \vdash A \quad X, B_1, \ldots, B_i, \sim A \vdash \sim A \text{ (basic)}}{X, B_1, \ldots, B_{i+1} \vdash A \text{ (RDI)}}$$

Though it will be useful later to expand the set of basic deductions, I begin the discussion of natural deduction theories for the syllogistic by employing the definition of basic deduction that is standard in natural deduction theory. It is this notion that is appropriate for Corcoran's treatment. Let BD_s be $\{\langle X, A \rangle \mid \langle X, A \rangle$ is a deduction and $A \notin X\}$.

Given the previous theorem, four equivalent natural deduction theories for Sen^- may be defined as sufficient for the theory of immediate inference:

$\text{IM1} = \langle \text{BD}_s, \vdash_1, \{\text{C1}, \text{C2}, \text{RD}\} \rangle$;
$\text{IM2} = \langle \text{BD}_s, \vdash_2, \{\text{C1}, \text{C2}, \text{RD2}, \text{Cut}, \text{Transposition}\} \rangle$;
$\text{IM3} = \langle \text{BD}_s, \vdash_3, \{\text{C1}, \text{C2}, \text{Thinning}, \text{Cut}, \text{Transposition}\} \rangle$;
$\text{IM4} = \langle \text{BD}_s, \vdash_4, \{\text{C1}, \text{C2}, \text{Thinning}, \text{RAI}\} \rangle$ (Corcoran's system).

Since the four systems are strictly equivalent, the four sets of provable deductions are co-extensive.

Theorem: Let $\vdash$ be the set of provable deductions relative to BD_s and some rule, RL, that includes Cut. Then:

$$\frac{X_1 \vdash A_1 \quad \ldots \quad X_n \vdash A_n}{X_1, \ldots, X_n \vdash B} \quad \text{iff} \quad A_1, \ldots A_n \vdash B$$

Proof: $\Rightarrow$:

$$\frac{A_1 \vdash A_1 \text{ (basic)} \quad \ldots \quad A_n \vdash A_n \text{ (basic)}}{A_1, \ldots, A_n \vdash B \text{ (by assumed rule)}}$$

$\Leftarrow$: Let $A_1, \ldots, A_n \vdash B$ be provable by tree Π. Then the following is a derivation tree that defines the required rule:

$$\cfrac{\cfrac{\cfrac{\cfrac{\overset{\Pi}{A_1, \ldots, A_n \vdash B \text{(given)}} \quad X_1 \vdash A_1 \text{(given)}}{X_1, A_2, \ldots, A_n \vdash B \text{(Cut)}} \quad X_2 \vdash A_2 \text{(given)}}{X_1, X_2, A_3, \ldots, A_n \vdash B \text{(Cut)} \qquad X_3 \vdash A_3 \text{(given)}}}{\vdots \qquad\qquad \vdots}}{\cfrac{X_1, \ldots, X_{n-1} A_n \vdash B \text{(Cut)} \quad X_n \vdash A_n \text{(given)}}{X_1, \ldots, X_n \vdash B \text{ (Cut)}}}$$

■

Corcoran's natural deduction theory

Corcoran's original theory is defined in terms of several additional rules which in standard theory would be defined now as follows:

$$\text{C1}': \frac{X \vdash \boldsymbol{I}xy}{X \vdash \boldsymbol{I}yx}$$

$$\text{C2}': \frac{X \vdash \boldsymbol{A}xy}{X \vdash \boldsymbol{I}yx}$$

$$\text{C2}'': \frac{X \vdash \boldsymbol{E}xy}{X \vdash \boldsymbol{O}yx}$$

$$\text{PS1}: \frac{X \vdash \boldsymbol{A}zy \quad Y \vdash \boldsymbol{A}xz}{X, Y \vdash \boldsymbol{A}xy}$$

$$\text{PS2}: \frac{X \vdash \boldsymbol{E}zy \quad Y \vdash \boldsymbol{A}xz}{X, Y \vdash \boldsymbol{E}xy}$$

$$\text{PS3}: \frac{X \vdash \boldsymbol{A}zy \quad Y \vdash \boldsymbol{I}xz}{X, Y \vdash \boldsymbol{I}xy}$$

$$\text{PS4}: \frac{X \vdash \boldsymbol{E}zy \quad Y \vdash \boldsymbol{I}xz}{X, Y \vdash \boldsymbol{O}xy}$$

Rules C1 and C1′ are natural deduction versions of the traditional rules of simple conversion, and C2′ and C2″ are versions of conversion *per accidens*. Rules PS1–4 are versions of the traditional first figure syllogisms Barbara, Celarent, Darii and Ferio, in which the syllogisms are treated 'epitheoretically' as natural deduction rules.

Theorem: Relative to the set BD_S,
1. C2′ is definable in terms of C1, C2 and RAI and Thinning;
2. C1′ is definable in terms of C1, Cut and Transposition;
3. C2″ is definable in term of C2′, Cut and Transposition;
4. both PS3 and PS4 are definable in terms of PS2.

In his original presentation Corcoran 1972 develops the theory of immediate inference by taking as primitive natural deduction versions of the simple conversion of *E*-statements: conversion *per accidens* of *A*-statements; and *argumentum per impossible*. He observes that simple conversion of *I*-statements and conversion *per accidens* of *E*-statements are then definable. He extends the theory to embrace the full syllogistic by adding versions of Barbara and Celarent treated as natural deduction rules, and observes that Darii and Ferio are then definable.

In this presentation I depart from Corcoran in several ways. First, the modern statement of the theory demands that we make explicit that proof constructions begin with elements in the set BD_S of basic deduction. The appeal to BD_S is left implicit in Corcoran's presentation. Second, for the conversion rules I prefer to take C1 and C2 as primitive, inasmuch as doing so demarcates by distinct rules the operations of conversion (C1) and subalternation (C2).

Third, for the theory of immediate inference, I make explicit, in a way Corcoran does not, that RAI together with conversion rules are not sufficient. If RAI is taken as primitive then Thinning must be added. The result is the theory IM4 which I propose as the formal reconstruction of Corcoran's theory of immediate inference. Equally, one of the alternative rules' sets of the strictly equivalent theories IM1–3 could be used.

Corcoran's main idea concerns how to extend the natural deduction theory of immediate inference so as to embrace the full syllogistic. His proposal consists of augmenting the rule set by adding Barbara and Celarent, treated as natural deduction rules. He observes that Darii and Ferio, also treated as natural deduction rules, then become definable. Accordingly, I take

$$\mathrm{SYL_C} = \langle \mathrm{BD_{S,}} \vdash_{\mathrm{C}}, \mathrm{C1}, \mathrm{C2\,Thinning}, \mathrm{RAI}, \mathrm{PS1}, \mathrm{PS2} \rangle$$

as the formal version of Corcoran's full syllogistic theory.

Corcoran's rejection of the traditional interpretation of Aristotle

Corcoran's main conceptual claim, which is also a claim about how Aristotle should be interpreted, is that the tradition has been wrong to construe the reduction of valid syllogisms to Barbara and Celarent as some sort of implicit axiomatic theory in which Barbara and Celarent serve as 'axioms'. For my purposes here it will be useful to be more precise and divide the view that Corcoran criticizes into two theses:

1. that syllogisms are object language sentences (of the form $(A \wedge B) \rightarrow C)$;
2. that the set of valid syllogisms is to be construed as an axiom system, defined as the inductive closure of the basic sentences Barbara and Celarent under some inference rules.

It is clear that if thesis 1 is false, so is thesis 2. Syllogisms cannot be *axiomatized* if they are not sentences. There is no question but that thesis 1 is controversial, and indeed implausible. Lukasiewicz (1957) explicitly argues that syllogisms are sentences rather than three-line arguments, and he does so in order to treat Aristotle's reductions as proofs in an axiom system. It is clear also that Lukasiewicz has not convinced many scholars that thesis 1 is true.[3] Aristotle's text consistently treats syllogisms as consisting of three sentences, and the logical tradition from ancient through mediaeval times has done likewise. Moreover, historians of logic subsequent to Lukasiewicz generally have not found his axiomatization persuasive enough to reject the traditional reading. On the whole, then, most commentators agree with Corcoran in rejecting thesis 1 in favour of its alternative.

1′. that syllogisms consist of three-line arguments in the object language.

In rejecting thesis 1, however, Corcoran seems to think that he must reject also any inductive construction of the syllogisms from Barbara and Celarent as basic elements. In particular, he seems to reject this alternative to thesis 2:

2′. that the set of valid syllogisms is to be construed as an inductive set, defined as the closure of the basic elements Barbara and Celarent are under some construction rules.

He seems to reason as follows. Since syllogisms are arguments and not sentences, and since the valid syllogisms may be 'proven' as arguments in a natural deduction theory

3 See, for example, Patzig (1968) and Rose (1968). In section 4.2, 'The Syllogistic', (Martin, 1987), the author offers a natural deduction version of the syllogistic in which the valid syllogisms are defined as a constructive set of arguments (rather than sentences); *viz*: the closure of basic deductions, including Barbara and Celarent, under natural deduction rules suggested by the Square of Opposition. The point of the discussion there, however, is more to illustrate natural deduction techniques than to interpret Aristotle.

in which Barbara and Celarent are not taken as basic deductions, Aristotle should not be construed as having intended Barbara and Celarent to serve as basic elements in a reduction.

It is, however, difficult to accept Corcoran's conclusion. He is forced to depart from the standard reading in which Aristotle's notion of 'perfect syllogism' is understood to refer to the two basic syllogisms, Barbara and Celarent, distinguished for being particularly self-evident. He is forced also to construe the various reductions given by Aristotle, so that they are no longer step-wise reductions of syllogisms to Barbara and Celarent construed as basic syllogisms. Rather, he demands the reading, new to the tradition, that in the manner of natural deduction a reduction presupposes the premises of the syllogism to be 'reduced', and then derives the conclusion in the manner of a natural deduction using Barbara and Celarent as rules of deduction. Admittedly, Aristotle's text is terse and his explanations few, and Corcoran's reading is consistent with the text.

What seems to have influenced Aristotelian scholars, however, is not the text, but a technical point. It is the perceived inconsistency of taking syllogisms as arguments and accepting Barbara and Celarent as basic elements in an inductive construction. Corcoran is supported in his view, moreover, by his technical success in developing a natural deduction construction of the valid syllogisms in which Barbara and Celarent are understood, not as basic deductions, but as natural deduction rules of construction.

The point I want to make here is that any perceived inconsistency between syllogisms as arguments and the perfect syllogisms as basic elements in a construction is specious. It is perfectly possible to develop a natural deduction theory of the syllogism that conforms to both thesis 1′ and thesis 2′. Indeed the resulting theory is fully equivalent, in the constructive sense, to Corcoran's 1974 natural deduction theory.

Let $\mathrm{BD_A}$ be $\mathrm{BD_c} \cup \{\{A, B\} \vdash C \mid$ either (for some $x, y, x \in$ Terms, $A = Azy$, $B = Axz$ and $C = Axy$) or (for some $x, y, x \in$ Terms, $A = Ezy$, $B = Axz$ and $C = Exy$)$\}$. Let $\mathrm{SYL_A} = \langle \mathrm{BD_A}, \vdash_A, \mathrm{C1}, \mathrm{C2}, \text{Thinning}, \mathrm{RAI} \rangle$.

Theorem: $\mathrm{SYL_A}$ is constructively equivalent to $\mathrm{SYL_C}$.

Proof: $\mathrm{SYL_C}$ is reducible to $\mathrm{SYL_A}$, because PS1 and PS2 are easily definable in $\mathrm{SYL_A}$. Thus $\vdash_C \subseteq \vdash_A$. Conversely it is straightforward to show that $\mathrm{BS_A} \subseteq \vdash_C$, from which it follows by induction that $\vdash_A \subseteq \vdash_C$. ■.

III Soundness and Completeness

Corcoran's main metalogical result may be stated in terms of $\mathrm{SYL_C} = \langle \mathrm{BD_S}, \vdash_C, \mathrm{C1}, \mathrm{C2}, \text{Thinning}, \mathrm{RAI}, \mathrm{PS1}, \mathrm{PS2} \rangle$. He showed that $\mathrm{SYL_C}$ is sound and complete for the syllogistic interpreted over sets, in the narrower syntax that does not allow the same terms to occur as both subject and predicate:

Corcoran's Theorem: Relative to $\mathrm{L}{\downarrow}^- = \langle \mathrm{Syn}^-, \mathscr{R}{\downarrow}^- \rangle$ and natural deduction theory $\langle \mathrm{BD_S}, \vdash_C, \mathrm{C1}, \mathrm{C2}, \text{Thinning}, \mathrm{RAI}, \mathrm{PS1}, \mathrm{PS2} \rangle$ for Syn^-, the two relations $\vDash_{\mathrm{L}{\downarrow}^-}$ and $\vdash_C$ are co-extensive.

In what follows I offer what I think is a more direct and intuitive proof of the completeness theorem and of some generalizations from it. By using Henkin methods, the treatment shares the advantage of the earlier restatement of the general natural deduction framework, in that it helps situate syllogistic metalogic within the mainstream of current natural deduction theory. The relatively simple idea of the Henkin-style proof is to construct a model for a saturated maximally consistent set, M, by defining the extension of a term, x, so that it includes the term x itself and all terms y that 'fall under it', in the sense that the sentence Ayx is in M.

The main result will be shown for $L^- = \langle \mathrm{Syn}^-, \mathscr{R}^- \rangle$ and, due to the convenience of its larger set of primitive rules, for the system $\mathrm{SYL}_C = \langle \mathrm{BD}_C, \vdash_C, \mathrm{C1}, \mathrm{C2}, \text{Thinning}, \mathrm{RAI}, \mathrm{PS1}, \mathrm{PS2} \rangle$. The result holds equally well for the family of systems equivalent to SYL_C, including SYL_A. Subscripts will be employed only to avoid ambiguity in contexts where other languages are discussed. We employ without proof some standard syllogistic results:

Theorem: 1. (Soundness) if $X \vdash A$, then $X \vDash A$.
2. (Reducibility of three-premise syllogisms) If $A, B, C, \vdash D$, then there is some E such that $A, B \vdash E$ and $E, C \vdash D$.

Soundness is proven by the standard induction. The second result is proven by a review of the small finite number of cases involved. (If Sen = Sen^+ the number of cases is larger but still finite and manageable.)

Definitions. Let X be a finite subset of Sen.

$X \vdash A$	means	for some finite subset Y of X, $Y \vdash A$.
X is inconsistent	means	for some A, $X \vdash A$ and $X \vdash \sim A$.
X is consistent	means	X is not inconsistent.

Theorem: 1. If X is inconsistent, then $X \vdash A$.
2. If X is finite, then X is inconsistent iff, for some A, $X \vdash A$ and $X \vdash \sim A$.
3. If X is finite, then $X \vdash A$ iff $X \cup \{\sim A\}$ is inconsistent.
4. $X \vdash A$ iff, for some B, $X, \sim A \vdash B$ and $X, \sim A \vdash \sim B$.
5. If $A_1, \ldots, A_n, B \vdash C$, then, for some D, $A_1, \ldots, A_n \vdash D$ and $D, B \vdash C$.

The first four results are standard and follow directly from the definitions. Result 5 follows by induction using the reducibility to two of three-premised syllogisms.

Definition: 1. If Sen = Sen^-, y is '*new to X*', means y does not occur in any $A \in X$.
2. If Sen = Sen^+, y is '*new to X*' means y does not occur in any A in X other than Ayy and Iyy.

Theorem: For $X = \{A_1, \ldots, A_n\}$, if $X, B \vdash C$ and $X \cup \{B\}$ is consistent, then no term occurring in C is new to $X \cup \{B\}$.

Proof by induction:

Basis step: *Let* $X = \{A_1\}$. Assume $A_1, B \vdash C$ and that $\{A, B\}$ is consistent. Since A_1, $B \vdash C$, then by an earlier theorem following is a proof tree:

$$\frac{A_1 \vdash A_1 \quad B \vdash B}{A_1, B \vdash C}$$

But this obtains only if $A_1 = C$ or $B = C$ or the descent is by PS1 or PS2. In all cases the theorem follows.

Inductive step: *Let* $X = \{A_1, \ldots, A_n\}$. Assume: (1) that $X = \{A_1, \ldots, A_n\}, B \vdash C$ and (2) that $X \cup \{B\}$ is consistent. By (1) and an earlier theorem, there is a D such that the following is a derivation tree:

$$\frac{A_1, \ldots, A_n \vdash D \quad D, B \vdash C}{A_1, \ldots, A_n, B \vdash C\,(\mathrm{Cut})}$$

Moreover, by (2), both $\{A_1, \ldots, A_n\}$ and $\{D, B\}$ are consistent, and the induction hypothesis applies to the two leaves of the tree. Hence all the terms in C are in either D or B, and hence in either X or B. ■

Corollary: If $X \cup \{A\}$ is consistent and there is some term occurring in B that is new to $X \cup \{A\}$, then $X \cup \{A, B\}$, is consistent.

Definition: X is '*maximally consistent*' iff X is consistent and for any A, either $A \in X$ or $\sim A \in X$.

Theorem: Let X be maximally consistent. Then:

1. $A \in X$ iff $X \vdash A$.
2. $A \in X$ iff $\sim A \notin X$.
3. If $\boldsymbol{A}xy \in X$, then $\boldsymbol{I}xy \in X$
4. If $\boldsymbol{E}xy \in X$, then $\boldsymbol{O}xy \in X$
5. Not both $\boldsymbol{A}xy$ and $\boldsymbol{E}xy$ are in X.
6. Either $\boldsymbol{I}xy$ or $\boldsymbol{O}xy$ is in X.

Definition: X is a '*saturated maximally consistent*' set iff X is maximally consistent and

1. $\boldsymbol{I}xy \in X$ iff, for some z, $\boldsymbol{A}zx$, $\boldsymbol{A}zy \in X$;
2. $\boldsymbol{O}xy \in X$ iff, for some z, $\boldsymbol{A}zx$, $\boldsymbol{E}zy \in X$.

Lemma 1: Every consistent set is extendible to a saturated maximally consistent set.

Proof: Let Sen be ordered in a series $\{A_i\}$. A series $\{X_i\}$ of subsets of Sen is defined inductively:

$$X_1 = X.$$

If A_n is some $\boldsymbol{A}xy$ or $\boldsymbol{E}xy$:

$X_{n+1} = X_n \cup \{A_n\}$, if $X_n \cup \{A_n\}$ is consistent;
$= X_n$ otherwise.

If A_n is some $\boldsymbol{I}xy$:

$X_{n+1} = X_n \cup \{A_n, \boldsymbol{A}zx, \boldsymbol{A}zy\}$, if z is some term new to $X_n \cup \{A_n\}$, and $X_n \cup \{A_n\}$ is consistent;
$= X_n$ otherwise.

If A_n is some $\boldsymbol{O}xy$:

$X_{n+1} = X_n \cup \{A_n, \boldsymbol{A}zx, \boldsymbol{A}zy\}$, if z is some term new to $X_n \cup \{A_n\}$, and $X_n \cup \{A_n\}$ is consistent;
$= X_n$ otherwise.

Claim 1: All X_i are consistent.

Proof by induction.

Basis step: By definition $X_1 = X$ and X is consistent.
Inductive step:

Case 1. A_n is some $\boldsymbol{A}xy$ of $\boldsymbol{E}xy$.
Then either $X_{n+1} = X_n \cup \{A_n\}$ and $X_n \cup \{A_n\}$ is consistent, or $X_{n+1} = X_n$ and X_n is consistent.

Case 2. A_n is some $\boldsymbol{I}xy$.
Then either $X_n \cup \{A_n\}$ is consistent or it is not. Suppose first that is is not. Then $X_{n+1} = X_n$, which is consistent. Suppose $X_n \cup \{A_n\}$ is consistent. Then, for some

term, z, new to $X_n \cup \{A_n\}$, $X_{n+1} = X_n \cup \{A_n, \boldsymbol{Azx}, \boldsymbol{Azy}\}$. Since z of $\boldsymbol{Azx}$ and $\boldsymbol{Azy}$ is new to X_n and to $\boldsymbol{Ixy}$, it follows from an earlier theorem that $X_n \cup \{\boldsymbol{Ixy}, \boldsymbol{Azx}\}$ and $X_n \cup \{\boldsymbol{Ixy}, \boldsymbol{Azy}\}$ are both consistent. Suppose for a *reductio* that $X_n \cup \{A_n, \boldsymbol{Azx}, \boldsymbol{Azy}\}$ is inconsistent. Then, $X_n, \boldsymbol{Ixy}, \boldsymbol{Axy} \vdash \sim \boldsymbol{Azy}$. Then, by a previous theorem, there is some $\boldsymbol{B}$ such that $X_n, \boldsymbol{Ixy} \vdash \boldsymbol{B}$ and $\boldsymbol{B}, \boldsymbol{Azx} \vdash \boldsymbol{Ozy}$. Moreover, since $X_n \cup \{\boldsymbol{Ixy}, \boldsymbol{Azx}\}$ is consistent and $X_n, \boldsymbol{Ixy} \vdash \boldsymbol{B}$, $\{\boldsymbol{B}, \boldsymbol{Azx}\}$ is consistent. But the only $\boldsymbol{B}$ such that both $\{\boldsymbol{B}, \boldsymbol{Azx}\}$ is consistent and $\boldsymbol{B}, \boldsymbol{Azx} \vdash \boldsymbol{Ozy}$ is $\boldsymbol{Ozy}$. But then X_n, $\boldsymbol{Ixy} \vdash \boldsymbol{Ozy}$. Since $X_n \cup \{\boldsymbol{Ixy}\}$ is consistent by a previous theorem, z must occur in some sentence in X or in $\boldsymbol{Ixy}$. Since it does not occur in $\boldsymbol{Ixy}$, it must occur in X_n, contrary to the earlier stipulation that z is new to X_n and $\boldsymbol{Ixy}$. Hence, $X_n \cup \{A_n, \boldsymbol{Azx}, \boldsymbol{Azy}\}$ is inconsistent.

Case 3. A_n is some $\boldsymbol{Oxy}$.

The argument is similar to Case 2. Either $X_n \cup \{A_n\}$ is consistent or it is not. Suppose it is not. Then $X_{n+1} = X_n$, which is consistent. Suppose $X_n \cup \{A_n\}$ is consistent. Then, for some term, z, new to $X_n \cup \{A_n\}$, $X_{n+1} = X_n \cup \{A_n, \boldsymbol{Azx}, \boldsymbol{Ezy}\}$. Since z of $\boldsymbol{Azx}$ and $\boldsymbol{Ezy}$ is new to X_n and to $\boldsymbol{Oxy}$, it follows from an earlier theorem that $X_n \cup \{\boldsymbol{Oxy}, \boldsymbol{Azx}\}$ and $X_n \cup \{\boldsymbol{Oxy}, \boldsymbol{Ezy}\}$ are both consistent. Suppose for a *reductio* that $X_n \cup \{A_n, \boldsymbol{Azx}, \boldsymbol{Ezy}\}$ is inconsistent. Then, X_n, $\boldsymbol{Oxy}$, $\boldsymbol{Axy} \vdash \sim \boldsymbol{Ezy}$. Then by a previous theorem, for some $\boldsymbol{B}$, X_n, $\boldsymbol{Oxy} \vdash \boldsymbol{B}$ and $\boldsymbol{B}, \boldsymbol{Azx} \vdash \boldsymbol{Izy}$. Moreover, since $X_n \cup \{\boldsymbol{Oxy}, \boldsymbol{Azx}\}$ is consistent and $X_n\, \boldsymbol{Oxy} \vdash \boldsymbol{B}$, $\{\boldsymbol{B}, \boldsymbol{Azx}\}$ is consistent. But the only $\boldsymbol{B}$ such that both $\{\boldsymbol{B}, \boldsymbol{Azx}\}$ is consistent and $\boldsymbol{B}, \boldsymbol{Azx} \vdash \boldsymbol{Izy}$ is $\boldsymbol{Izy}$. But then z is not new to X_n and $\boldsymbol{Ixy}$, contrary to the earlier stipulation. Hence: $X_n \cup \{A_n, \boldsymbol{Azx}, \boldsymbol{Azy}\}$ is inconsistent.

It follows from the three cases that X_{n+1} is consistent. Hence, by induction, all X_i are consistent.

Claim 2: $X \subseteq \bigcup\{X_i\}$. This trivially follows because $X = X_1$.

Claim 3: $\bigcup\{X_i\}$ is consistent. Proof is by standard *reductio*.

Claim 4: $\bigcup\{X_i\}$ is maximal.
Proof is by *reductio*. Assume for some B, neither $B \in \bigcup\{X_i\}$ nor $\sim B \in \bigcup\{X_i\}$. By standard reasoning, it is easily shown that for some finite $Y \subseteq X_m$ and some C, both $Y, A_n \vdash C$ and $Y, A_n \vdash \sim C$. The following then is a proof tree:

$$\frac{Y, A_n \vdash C \quad Y, A_n \vdash \sim C}{Y \vdash \sim A_n\,(\mathrm{RAI})}$$

Hence, $X_m \vdash \sim A_n$ and $X_m \vdash \sim B$. Reasoning likewise, $X_m \vdash \sim A_m$ and $X_m \vdash B$. Hence X_m is inconsistent, contradicting what was shown earlier.

Claim 5: $\bigcup\{X_i\}$ is saturated.
We illustrate the case if $\boldsymbol{Ixy} \in \bigcup\{X_i\}$ then, for some z, $\boldsymbol{Azx}, \boldsymbol{Azy} \in \bigcup\{X_i\}$. Assume $\boldsymbol{Ixy} \in \bigcup\{X_i\}$. Then, for some n, $\boldsymbol{Ixy} = A_n$, and, since $\bigcup\{X_i\}$ is consistent, it follows that $X_n \cup \{\boldsymbol{Ixy}\}$ is consistent and that, for some variable, z, new to X_n and $\boldsymbol{Ixy}$, $X_{n+1} = X_n \cup \{A_n, \boldsymbol{Ixy}, \boldsymbol{Azx}, \boldsymbol{Azy}\} \subseteq \bigcup\{X_i\}$. Hence, for some z, $\boldsymbol{Azx}$, $\boldsymbol{Azy} \in \bigcup\{X_i\}$.

Lemma 2: If X is a saturated maximally consistent set, then there is some $\mathrm{R} \in \mathscr{R}^-$ such that, for any A, $A \in X$ iff $\mathrm{R}(A) = \mathrm{T}$.

Proof: Let X be as specified. R is defined so as to be $\mathscr{R}^-$. Let $\langle \mathscr{P}(\text{Terms}), \subseteq, \cap, \varnothing \rangle$ be the order-theoretic structure determined by the power set $\mathscr{P}$(Terms) of Terms, the set inclusion relation $\subseteq$ on $\mathscr{P}$(Terms), the meet operation $\cap$ determined by $\subseteq$, and the least element $\varnothing$. Let R be a function from Terms $\cup$ Sen$^-$ into $\mathscr{P}$(Terms) $\cup$ {T, F} such that:

if $x \in$ Terms, $\quad \mathrm{R}(x) = \{y \in \text{Terms} \mid Ayx \in X\}$

for $A \in \text{Sen}^-$,

if A is some Axy, $\mathrm{R}(A) = \mathrm{T}$ iff $\mathrm{R}(x) \subseteq \mathrm{R}(y)$
if A is some $\boldsymbol{E}xy$, $\mathrm{R}(A) = \mathrm{T}$ iff $\mathrm{R}(x) \cap \mathrm{R}(y) = \varnothing$
if A is some $\boldsymbol{I}xy$, $\mathrm{R}(A) = \mathrm{T}$ iff $\mathrm{R}(x) \cap \mathrm{R}(y) \neq \varnothing$,
if A is some $\boldsymbol{O}xy$, $\mathrm{R}(A) = \mathrm{T}$ iff $\mathrm{R}(x) \nsubseteq \mathrm{R}(y)$

To show $\mathrm{R} \in \mathscr{R}^-$, it remains to show that for any $x \in$ Terms, $\mathrm{R}(x) \neq \varnothing$. Since X is maximal, there are two cases: either $\boldsymbol{I}xy \in X$ or $\boldsymbol{E}xy \in X$.

Case 1. Suppose $\boldsymbol{I}xy \in X$.

Since X is satisfiable, there is some z such that $Azx \in X$. Hence $z \in \mathrm{R}(x)$ and $\mathrm{R}(x) \neq \varnothing$.

Case 2. Suppose $\boldsymbol{E}xy \in X$.
Since X is closed under $\vdash$, $\boldsymbol{O}xy \in X$. Since X is satisfiable, for some z, $Azx \vDash X$. Hence $z \in \mathrm{R}(x) \neq \varnothing$.

It is shown now that for any $A, A \in X$ iff $\mathrm{R}(A) = \mathrm{T}$. There are four cases:

Case 1: A is some Axy. $\Rightarrow$: Assume $Axy \in X$.
For an arbitrary z assume $z \in \mathrm{R}(x)$. Then $Azx \in X$. Since $Azx, Axy \vdash Azy$ and X is closed under $\vdash, Azy \in X$. Hence $z \in \mathrm{R}(y)$. Since z is arbitrary, $\mathrm{R}(x) \subseteq \cap(y)$. $\Leftarrow$: Suppose $\mathrm{R}(x) \subseteq \mathrm{R}(y)$ but (for *reductio*) that $Axy \notin X$. Since X is maximal, $\boldsymbol{O}xy \in X$. Since A is *saturated*, there is some z such that $Azx, \boldsymbol{E}zy \in X$. Hence $z \in \mathrm{R}(x)$. Hence $z \in \mathrm{R}(y)$ and $Azy \in X$. But it follows from the facts that $\boldsymbol{E}zy \in X$, $\boldsymbol{E}zy \vdash \boldsymbol{O}zy$, and X is closed under $\vdash$, that $\boldsymbol{O}zy \in X$. But then X is inconsistent, contrary to the original assumption.

Case 2. A is some $\boldsymbol{E}xy$. $\Rightarrow$: Let $\boldsymbol{E}xy \in X$.
Suppose $z \in \mathrm{R}(x) \cap \mathrm{R}(y)$. Hence $Azx, Azy \in X$. But $Azx, Azy \vdash \boldsymbol{I}xy$. Hence $\boldsymbol{I}xy \in X$ and X is inconsistent. Absurd. $\Leftarrow$: Let $\mathrm{R}(x) \cap \mathrm{R}(y) = \varnothing$. Suppose $\boldsymbol{E}xy \notin X$. Then $\boldsymbol{I}xy \in X$ and, for some z, Azx, $Azy \in X$. Thus $z \in \mathrm{R}(x) \cup \mathrm{R}(y)$. Absurd.

Case 3. A is some $\boldsymbol{I}xy$. $\Rightarrow$: Let $\boldsymbol{I}xy \in X$.
Hence, for some $z, Azx, Azy \in X$. Thus, $z \in \mathrm{R}(x) \cap \mathrm{R}(y) \neq \varnothing$. $\Leftarrow$: Suppose $\mathrm{R}(x) \cap \mathrm{R}(y) \neq \varnothing$. Then, for some $z, z \in \mathrm{R}(x), R(y)$. Hence $Azx, Azy \in X$. Further, $Azx, Azy \vdash \boldsymbol{I}xy$. Hence $\boldsymbol{I}xy \in X$.

Case 4. A is some Oxy. $\Rightarrow$: Let $\boldsymbol{O}xy \in X$.
Then, for some $z, Azx, \boldsymbol{E}zx \in X$. Thus, $z \in \mathrm{R}(x)$. Suppose (for *reductio*) $z \in \mathrm{R}(y)$. Then $Azy \in X$. Hence $\boldsymbol{I}zy \in X$. But then X is inconsistent, which is absurd. $\Leftarrow$: Let $\mathrm{R}(x) - \mathrm{R}(y) \neq \varnothing$. Suppose $\boldsymbol{O}xy \notin X$. Let $z \in \mathrm{R}(x), z \notin \mathrm{R}(y)$. Then, $Azx \in X, Azy \notin X$. Hence $\boldsymbol{O}zy \in X$. Hence $\boldsymbol{O}xy \in X$. Absurd. ■

Theorem: If $X \vDash A$, then $X \vdash A$.

Proof: Let $X \vDash A$. Then for some $\mathrm{R} \in \mathscr{R}^-$, R satisfies X and $\mathrm{R}(\sim A) = \mathrm{T}$. Clearly $X \cup \{\sim A\}$ is inconsistent. For, if not, there is by Lemma 1 some maximally consistent

Y such that $X \subseteq \{\sim A\} \subseteq Y$. Then, by Lemma 2, there is some $R \in \mathscr{R}^-$ such that R satisfies Y, including $X \cup \{\sim A\}$. But then R satisfies X and $\sim A$ contrary to the assumption. Since $X \cup \{\sim A\}$ is inconsistent, there are some finite subsets Z and Z' and some B such that $Z, \sim A \vdash B$ and $Z', \sim A \vdash \sim B$. Hence by RAI, $Z, Z' \vdash A$. Since $Z \cup Z'$ is a finite subset of $X, X \vdash A$. ■

The proof is easily adapted for $L^+ = \langle Syn^+, \mathscr{R}^+ \rangle$ and $SYL_C = \langle BD_C, \vdash_C, C1, C2,$ Thinning, RAI, PS1, PS2, ID $\rangle$, and its equivalents.

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