Non-singular solutions of \( p \)-Laplace problems, allowing multiple changes of sign in the nonlinearity

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Abstract
For the \( p \)-Laplace Dirichlet problem (where \( \varphi(t) = t|t|^{p-2}, \ p > 1 \))

\[
\varphi(u'(x))' + f(u(x)) = 0 \quad \text{for } -1 < x < 1, \ u(-1) = u(1) = 0
\]

assume that \( f'(u) > (p-1)\frac{f(u)}{u} > 0 \) for \( u > \gamma > 0 \), while \( \int_{u}^{\infty} f(t) \, dt < 0 \) for all \( u \in (0, \gamma) \). Then any positive solution, with \( \max_{(-1,1)} u(x) = u(0) > \gamma \), is non-singular, no matter how many times \( f(u) \) changes sign on \((0, \gamma)\). Uniqueness of solution follows.

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We consider positive solutions of

\[
\varphi(u'(x))' + f(u(x)) = 0 \quad \text{for } -1 < x < 1, \ u(-1) = u(1) = 0,
\]

where \( \varphi(t) = t|t|^{p-2}, \ p > 1 \), so that \( \varphi'(t) = (p-1)|t|^{p-2} \). The linearized problem is

\[
(\varphi'(u'(x))w'(x))' + f'(u(x))w(x) = 0 \quad \text{for } -1 < x < 1, \\
\quad w(-1) = w(1) = 0.
\]

Recall that any positive solution of (1) is an even function \( u(-x) = u(x) \), satisfying \( xu'(x) < 0 \) for \( x \neq 0 \) so that \( \max_{(-1,1)} u(x) = u(0) \), and that any non-trivial solution of (2) is of one sign, so that we may assume that \( w(x) > 0 \) for \( x \in (-1,1) \), see e.g., P. Korman [5], [6].
If \( f'(u) > (p - 1) \frac{f(u)}{u} > 0 \) for \( u > 0 \), it is well known that any positive solution of (1) is non-singular, i.e., the problem (2) admits only the trivial solution \( w(x) \equiv 0 \). Now suppose that \( f'(u) > (p - 1) \frac{f(u)}{u} > 0 \) holds only for \( u > \gamma \), for some \( \gamma > 0 \). It turns out that positive solutions of (1), with maximum value greater than \( \gamma \) are still non-singular, provided that \( \int_\gamma^u f(t) \, dt < 0 \) for all \( u \in (0, \gamma) \). The main result is stated next. It is customary to denote \( F(u) = \int_0^u f(t) \, dt \).

**Theorem 1** Assume that \( f(u) \in C^1(\bar{R}_+) \), and for some \( \gamma > 0 \) it satisfies

(3) \quad f(\gamma) = 0, \quad \text{and} \quad f(u) > 0 \quad \text{on} \quad (\gamma, \infty),

(4) \quad f'(u) > (p - 1) \frac{f(u)}{u}, \quad \text{for} \quad u > \gamma,

(5) \quad F(\gamma) - F(u) = \int_u^\gamma f(t) \, dt < 0, \quad \text{for} \quad u \in (0, \gamma).

Then any positive solution of (1), satisfying

(6) \quad u(0) > \gamma, \quad \text{and} \quad u'(1) < 0,

is non-singular, which means that the linearized problem (2) admits only the trivial solution.

In case \( p = 2 \) this result was proved in P. Korman [7], while for general \( p > 1 \) a weaker result, requiring that \( f(u) < 0 \) on \( (0, \gamma) \), was given in J. Cheng [3] (and before that by R. Schaaf [10] for \( p = 2 \) case), see also P. Korman [5], [6] for a different proof, and a more detailed description of the solution curve. Other multiplicity results on \( p \)-Laplace equations include [1], [2], [4] and [9].

**Proof:** Assume, on the contrary, that the problem (2) admits a non-trivial solution \( w(x) > 0 \). Let \( x_0 \in (0, 1) \) denote the point where \( u(x_0) = \gamma \). Define

\[
q(x) = (p - 1)(1 - x)\varphi(u'(x)) + \varphi'(u'(x))u(x).
\]

We claim that

(7) \quad q(x_0) < 0.

Rewrite (using that \( (p - 1)\varphi(t) = t\varphi'(t) \))

\[
q(x) = \varphi'(u'(x)) [(1 - x)u'(x) + u(x)]
\].
Since \( \varphi'(t) > 0 \) for all \( t \neq 0 \), it suffices to show that the function \( z(x) \equiv (1 - x)u'(x) + u(x) < 0 \) satisfies \( z(x_0) < 0 \). Indeed,

\[
z(x_0) = \int_{x_0}^{1} [u'(x_0) - u'(x)] \, dx < 0,
\]

which implies the desired inequality (7), provided we can prove that

\[
(8) \quad u'(x_0) - u'(x) < 0, \quad \text{for } x \in (x_0, 1).
\]

The “energy” function \( E(x) = \frac{p-1}{p} |u'(x)|^p + F(u(x)) \) is seen by differentiation to be a constant, so that \( E(x) = E(x_0) \), or

\[
\frac{p-1}{p} |u'(x)|^p + F(u(x)) = \frac{p-1}{p} |u'(x_0)|^p + F(\gamma), \quad \text{for all } x.
\]

By the assumption (5), it follows that

\[
\frac{p-1}{p} [|u'(x)|^p - |u'(x_0)|^p] = F(\gamma) - F(u(x)) < 0, \quad \text{for } x \in (x_0, 1),
\]

justifying (8), and then giving (7).

Next, we claim that

\[
(9) \quad (p - 1)w(x_0)\varphi(u'(x_0)) - u(x_0)w'(x_0)\varphi'(u'(x_0)) > 0,
\]

which implies, in particular, that

\[
(10) \quad w'(x_0) < 0.
\]

Indeed, by a direct computation, using (1) and (2),

\[
[(p - 1)w(x)\varphi(u'(x)) - u(x)w'(x)\varphi'(u'(x))]' = \left[ f'(u) - (p - 1)\frac{f(u)}{u} \right] uw.
\]

The quantity on the right is positive on \((0, x_0)\), in view of our condition (4). Integration over \((0, x_0)\), gives (9).

We have for all \( x \in [-1, 1] \)

\[
(11) \quad \varphi'(u') (u'w' - u''w) = \text{constant} = \varphi'(u'(1))u'(1)w'(1) > 0,
\]

as follows by differentiation, and using the assumption \( u'(1) < 0 \). Hence

\[
(12) \quad u'(x)w'(x) - u''(x)w(x) > 0, \quad \text{for } x \in (x_0, 1).
\]
Since \( f(u(x_0)) = 0 \), it follows from the equation (1) that \( u''(x_0) = 0 \). Then (11) implies

\[
\varphi'(u'(1))u'(1)w'(1) = \varphi'(u'(x_0))u'(x_0)w'(x_0) = (p-1)\varphi(u'(x_0))w'(x_0).
\]

We need the following function, motivated by M. Tang [11] (which was introduced in P. Korman [5], and used in Y. An et al [2])

\[ T(x) = x[\varphi'(u'(x))w'(x) + f(u(x))w(x)] - (p-1)\varphi(u'(x))w(x). \]

One verifies that

\[
T'(x) = pf(u(x))w(x).
\]

Integrating (14) over \((x_0, 1)\), and using (5) and (12), obtain

\[
T(1) - T(x_0) = p \int_{x_0}^1 f(u(x))w(x) \, dx
\]

\[
= p \int_{x_0}^1 [F(u(x)) - F(\gamma)] \frac{w(x)}{u'(x)} \, dx
\]

\[
= -p \int_{x_0}^1 [F(u(x)) - F(\gamma)] \frac{u'(x)u'(x) - w(x)u''(x)}{u^2(x)} \, dx < 0,
\]

which implies that

\[
L \equiv (p-1)\varphi(u'(1))w'(1) - (p-1)x_0\varphi(u'(x_0))w'(x_0) = (p-1)\varphi(u'(x_0))w(x_0) < 0.
\]

On the other hand, using (13), then (9), followed by (10) and (7), we estimate the same quantity as follows

\[
L > (p-1)\varphi(u'(x_0))w'(x_0) - (p-1)x_0\varphi(u'(x_0))w'(x_0) + u(x_0)w'(x_0)\varphi'(u'(x_0)) = w'(x_0)q(x_0) > 0,
\]

a contradiction.

\[\diamondsuit\]

We remark that in case \( f(0) < 0 \) it is possible to have a singular positive solution with \( u'(1) = 0 \), so that the assumption \( u'(1) < 0 \) is necessary.

We now consider the problem (where \( \varphi(t) = t|t|^{p-2}, \ p > 1 \))

\[
\varphi(u'(x))' + \lambda f(u(x)) = 0 \quad \text{for } -1 < x < 1, \ u(-1) = u(1) = 0,
\]

depending on a positive parameter \( \lambda \). The following result follows the same way as the Theorem 3.1 in [5].
Figure 1: The curve of positive solutions for the problem (15), in case $p = 3$ and $f(u) = u(u - 1)(u - 2)(u - 4)$.

**Theorem 2** Assume that $f(u) \in C^1(\bar{R}_+)$, and the conditions (3), (4) and (5) hold. Then there exists $0 < \lambda_0 \leq \infty$ so that the problem (15) has a unique positive solution for $0 < \lambda < \lambda_0$. All positive solutions, satisfying $u(0) > \gamma$, lie on a continuous solution curve that is decreasing in the $(\lambda, u(0))$ plane (see Figure 1). In case $f(0) < 0$, one has $\lambda_0 < \infty$, and at $\lambda = \lambda_0$ a positive solution with $u'(\pm 1) = 0$ exists, and no positive solutions exist for $\lambda > \lambda_0$.

**Example** In Figure 1 we present the solution curve of the problem (15) in case $p = 3$ and $f(u) = u(u - 1)(u - 2)(u - 4)$. Here $\gamma = 4$, and one verifies that the Theorem 2 applies. The *Mathematica* program to perform numerical computations for this problem is explained in detail in [8] (it uses the shoot-and-scale method). The solution curve in Figure 1 exhausts the set of all positive solutions (since $\int_0^2 f(u) \, du < 0$, there are no solutions with $u(0) = \max_{(-1,1)} u(x) \in (1, 2)$).

**References**


