Continuation of global solution curves using global parameters

Philip Korman
Department of Mathematical Sciences
University of Cincinnati
Cincinnati Ohio 45221-0025

Dieter S. Schmidt
Department of Computer Science
University of Cincinnati
Cincinnati, Ohio 45221-0030

Abstract
This paper provides both the theoretical results and numerical calculations of global solution curves, by continuation in global parameters. Each point on the solution curves is computed directly as the global parameter is varied, so that all of the turns that the solution curves make, as well as its different branches, appear automatically on the computer screen. For radial \( p \)-Laplace equations we present a simplified derivation of the regularizing transformation from P. Korman [15], and use this transformation for more accurate numerical computations. While for \( p > 2 \) the solutions are not of class \( C^2 \), we show that they are of the form \( w(r^{\frac{2}{p-1}}) \), where \( w(z) \) is of class \( C^2 \). Bifurcation diagrams are also calculated for non-autonomous problems, and for the fourth order equations modeling elastic beams. We show that the first harmonic of the solution can also serve as a global parameter.

Key words: Global solution curves, global parameters, radial solutions.


1 Introduction

We wish to compute the global bifurcation diagrams for the operator equation

\[ Lu = \lambda f(u) . \]
Here $L$ is a linear operator, $\lambda$ is a parameter. A traditional approach involves continuation in $\lambda$, where one uses Newton’s method, with the initial iterate being the solution at the preceding value of $\lambda$ (or its linear extrapolation to the present value of $\lambda$). These “curve following” methods are very well developed, see e.g., the book of E.L. Allgower and K. Georg [2], but not easy to implement. The problem is that the solution set may consist of multiple pieces, and there may be multiple turns on each curve. If a turn occurs, your computer program must recognize that fact, jump on the other branch, and reverse the direction of continuation in $\lambda$. Here $\lambda$ is a local parameter. If, on the other hand, the problem possesses a global parameter uniquely identifying the solution pair $(\lambda, u)$, then after computing sufficiently many of these pairs directly, one gets the picture of the entire solution set, including all of the turns on each piece.

What quantity can serve as a global parameter? In this paper we present two possibilities: it is either the maximum value of the solution, or the first harmonic of the solution. Next we discuss the first of these possibilities for problems on a ball in $\mathbb{R}^n$.

By classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [5], any positive solution of the semilinear Dirichlet problem (here $u = u(x)$, $x \in \mathbb{R}^n$, $\lambda$ is a positive parameter)

$$\Delta u + \lambda f(u) = 0 \text{ for } |x| < 1, \quad u = 0 \text{ when } |x| = 1$$

is necessarily radially symmetric, i.e., $u = u(r)$, with $r = |x|$, and so the problem turns into an ODE

$$(1.1) \quad u''(r) + \frac{n-1}{r}u'(r) + \lambda f(u(r)) = 0 \text{ for } 0 < r < 1, \quad u'(0) = u(1) = 0.$$  

Moreover, this theorem asserts that the function $u(r)$ is strictly decreasing, and hence $u(0)$ gives the maximum value of solution. A simple scaling argument shows that the value of $u(0) > 0$ is a global parameter, uniquely identifying the solution pair $(\lambda, u(r))$. We continue the solutions in $u(0)$ by developing the natural “shoot-and-scale” method. This method is very easy to implement, even when the solution set of (1.1) consists of multiple curves, with multiple turns on each curve. This method was used previously by R. Schaaf [24] for the one-dimensional case, $n = 1$. We also develop the “shoot-and-scale” method in case of the Neumann boundary conditions. We include very short Mathematica programs, producing the global solution curves for both the Dirichlet and the Neumann boundary conditions. Similar computations were used in P. Korman, Y. Li and D.S. Schmidt [16] to produce nodal ground state solutions of Schrödinger’s equation.
For non-autonomous problems

\[ u''(r) + \frac{n-1}{r} u'(r) + \lambda f(r, u(r)) = 0 \quad \text{for } 0 < r < 1, \quad u'(0) = u(1) = 0 \]

the “shoot-and-scale” method does not apply (one can still shoot, but not scale). In particular, one cannot assert anymore that the value of \(u(0)\) is a global parameter. It is plausible that in many cases the value of \(u(0)\) is still a global parameter (we mention below such a case when \(n = 1\)). Under the assumption that \(u(0)\) is a global parameter we develop a continuation method in \(u(0)\), based on Newton’s method.

We then develop the shoot-and-scale method to compute the curves of positive solutions for the Dirichlet problem in case of the \(p\)-Laplacian

\[ \varphi(u')' + \frac{n-1}{r} \varphi(u') + \lambda f(u) = 0, \quad u'(0) = u(1) = 0, \quad (1.2) \]

where \(\varphi(v) = v|v|^{p-2}, \ p > 1\). As in case \(p = 2\), we show that the value of \(u(0)\) is a global parameter, uniquely identifying the solution pair \((\lambda, u(r))\), so that the global solution curves can be drawn in the \((\lambda, u(0))\) plane. We perform numerical computations, and the corresponding Mathematica program is available from the authors. Writing the equation (1.2) in the form

\[ u'' + \frac{n-1}{(p-1)r} u' + \lambda \frac{f(u)}{(p-1)|u|^{p-2}} = 0, \]

one sees that if \(p > 2\), then \(u''(0)\) does not exist, so that \(u(r) \notin C^2[0,1]\). This singularity also presents a difficulty for computations. Using a regularizing transformation

\[ z = r^{\frac{p}{2(p-1)}}, \]

it was shown in P. Korman [10] that \(u = w \left(r^{\frac{p}{2(p-1)}}\right)\), where the function \(w(z)\) is twice differentiable at \(z = 0\), \(w(z) \in C^2[0,1]\). We present a simplified derivation of this result, and then use the regularizing transformation for more efficient computation of the global solution curves.

We also develop the shoot-and-scale method to compute the curves of positive solutions for a class of fourth order equations (\(\lambda\) is a positive parameter)

\[ u^iv(x) = \lambda f(u(x)), \quad \text{for } x \in (-1,1) \]

\[ u(-1) = u'(-1) = u(1) = u'(1) = 0, \quad (1.3) \]
modeling the displacements of an elastic beam, clamped at the end points. Under the assumption that \( f(u) > 0 \) and \( f'(u) > 0 \) for \( u > 0 \), it was shown in P. Korman [9] that any positive solution is an even function, decreasing on \((0, 1)\), so that \( u(0) \) is the maximal value of the solution. Quite surprisingly, it was also shown in P. Korman [9] that the value of \( u(0) \) is a global parameter, uniquely identifying the solution pair \((\lambda, u(x))\) of (1.3). This opens the way for continuation in \( u(0) \), but the shoot-and-scale method requires a major adjustment. The reason is that the knowledge of \( u(0) = \alpha \) and \( u'(0) = 0 \) does not identify the solutions of the initial value problem corresponding to the equation in (1.3). We develop a method to compute the bifurcation diagrams for the problem (1.3). Such curves have never been computed previously, to the best of our knowledge.

What other quantities can be used as a global parameter? Consider the Dirichlet problem

\[
\Delta u + f(u) = g(x), \quad x \in D, \quad u(x) = 0 \text{ on } \partial D,
\]

on a smooth domain \( D \subset \mathbb{R}^n \). Decompose the given function \( g(x) = \mu \varphi_1(x) + e(x) \), where \( \varphi_1(x) \) is the principal eigenfunction of the Laplacian with zero boundary conditions, \( \mu \in \mathbb{R} \), and \( \int_D e(x) \varphi_1(x) \, dx = 0 \). Similarly, decompose the solution \( u(x) = \xi \varphi_1(x) + U(x) \), with \( \xi \in \mathbb{R} \) and \( \int_D U(x) \varphi_1(x) \, dx = 0 \). It was shown in P. Korman [14] that \( \xi \) is a global parameter, uniquely identifying the solution pair \((u(x), \mu)\) of (1.4), provided that

\[
f'(u) < \lambda_2, \quad \text{for all } u \in \mathbb{R},
\]

where \( \lambda_2 \) is the second eigenvalue of the Laplacian on \( D \), with zero boundary conditions. We implement numerical computations of the solution curve \((u(x), \mu)(\xi)\) for the one-dimensional case, and apply them to illustrate some recent multiplicity results of A. Castro et al [3]. Our computations also show that the condition (1.5) cannot be removed, in general.

We see absolutely no need to ever use the traditional method of finite differences (or finite elements) for the problem (1.1). To explain why, it suffices to consider the one-dimensional version of the problem (1.1):

\[
u''(x) + \lambda f(u(x)) = 0 \quad \text{for } -1 < x < 1, \quad u(-1) = u(1) = 0.
\]

If we divide the interval \((-1, 1)\) into \( N \) pieces, with the step \( h = 2/N \), and the subdivision points \( x_i = ih \), and denote by \( u_i \) the numerical approximation of \( u(x_i) \), then the finite difference approximation of (1.6) is

\[
\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \lambda f(u_i) = 0, \quad 1 \leq i \leq N - 1, \quad u_0 = u_N = 0.
\]
This is a system of nonlinear algebraic equations, more complicated in every way than the original problem (1.6). In particular, this system often has more solutions than the corresponding differential equation (1.6). The existence of the extra solutions of difference equations (not corresponding to the solutions of the differential equation (1.6)) has been recognized for a while, and a term spurious solutions has been used. When studying the algebraic nonlinear system of equations (1.7), one can no longer rely on the familiar tools from differential equations. Even for the simplest nonlinearity \( f(u) = u^k \), the analysis of the finite difference problem (1.7) is very involved, see E.L. Allgower [1]. We avoid spurious solutions with the methods used in this paper.

2 Preliminary results

The study of radial solutions begins with the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [5], which states that in case of a ball in \( \mathbb{R}^n \), say a unit ball, any positive solution of the semilinear Dirichlet problem

\[
\Delta u + \lambda f(u) = 0 \quad \text{for} \quad |x| < 1, \quad u = 0 \quad \text{when} \quad |x| = 1
\]

(2.1) is necessarily radially symmetric, i.e., \( u = u(r) \), with \( r = |x| \), and so the problem turns into an ODE

\[
u''(r) + \frac{n-1}{r} u'(r) + \lambda f(u(r)) = 0 \quad \text{for} \quad 0 < r < 1, \quad u'(0) = u(1) = 0.
\]

(2.2) Moreover, this theorem asserts that

\[
u'(r) < 0, \quad \text{for all} \quad 0 < r < 1.
\]

(2.3) It follows that \( u(0) \) gives the maximum value of the solution \( u(r) \). The only assumption of this remarkable theorem is a slight smoothness assumption on \( f(u) \), which is considerably weaker than the standing assumption of this paper:

\[
f(u) \in C^2(\bar{\mathbb{R}}_+).
\]

(2.4) Of course, radial solutions were studied prior to [5], in particular in a classical paper of D.D. Joseph and T.S. Lundgren [8], but the theorem of B. Gidas, W.-M. Ni and L. Nirenberg [5] showed that radial solutions represent all solutions of the PDE (2.1), which stimulated the interest in radial solutions.
We now discuss the connection of the problem (2.1) with Schrödinger’s equation. Let \( v(x, t) \) be a complex-valued solution of a nonlinear Schrödinger’s equation \((x \in \mathbb{R}^n, t > 0)\)

\[ iv_t + \Delta v + v|v|^{p-1} = 0. \]

Here \( p > 1 \) is a constant, and \( |v| \) denotes the complex modulus of \( v \). Looking for the standing waves, one substitutes \( v(x, t) = e^{imt}u(x) \), with a real valued \( u(x) \), and a constant \( m > 0 \). Then \( u(x) \) satisfies

\[ \Delta u - mu + u|u|^{p-1} = 0. \]

For a more general equation, where \( v(x, t) \) is a complex-valued solution of

\[ iv_t + \Delta v + f(v) = 0 \]

a similar reduction works for any complex valued function \( f(v) \), satisfying

\[ f(e^{imt}u) = e^{imt}f(u), \] for any real \( m \) and \( u \),

and it leads to the equation

\[ \Delta u - mu + f(u) = 0. \]

At first glance the equation (2.2) appears to be singular at \( r = 0 \). In reality there is no singularity at \( r = 0 \), after all this difficulty is introduced only by the spherical coordinates. In fact, one can easily compute all of the derivatives \( u^{(k)}(0) \), see P. Korman [10], then write down the Maclaurin series of solution, and show that it converges for small \( r \), provided that \( f(u) \) is analytic, see [10]. Since in this paper we only assume that \( f(u) \in C^2(\bar{\mathbb{R}}_+) \), let us recall the following result.

**Lemma 2.1** For any \( \alpha > 0 \), consider the initial value problem

\[ u''(r) + \frac{n-1}{r}u'(r) + \lambda f(u(r)) = 0, \]

\[ u(0) = \alpha, \ u'(0) = 0. \]

Then one can find an \( \epsilon > 0 \), so that this problem has a unique classical solution on the interval \([0, \epsilon)\).

The proof can be found in L.A. Peletier and J. Serrin [22], see also J.A. Iaia [7], or P. Quittner and P. Souplet [23].

We show next that the maximum value of solution \( u(0) = \alpha \) can be used as a global parameter, i.e., it is impossible for two solutions of (2.2) to share the same value of \( \alpha \). This result has been known for a while, see e.g., E.N. Dancer [4].
Lemma 2.2 The value of \( u(0) = \alpha \) uniquely identifies the solution pair \((\lambda, u(r))\) of (2.2) (i.e., there is at most one \( \lambda \), with at most one solution \( u(r) \), so that \( u(0) = \alpha \)).

Proof: Assume, on the contrary, that we have two solution pairs \((\lambda, u(r))\) and \((\mu, v(r))\), with \( u(0) = v(0) = \alpha \). Clearly, \( \lambda \neq \mu \), since otherwise we have a contradiction with uniqueness of initial value problems guaranteed by Lemma 2.1. (Recall that \( u'(0) = v'(0) = 0 \).) The change of variables \( r = \frac{1}{\sqrt{\lambda}}t \) takes (2.5) into

\[
\begin{align*}
\frac{d^2u}{dt^2} + \frac{n-1}{t} \frac{du}{dt} + f(u) &= 0, \quad u(0) = \alpha, \quad u'(0) = 0, \\
\end{align*}
\]

The change of variables \( r = \frac{1}{\sqrt{\mu}}t \) takes the equation for \( v(r) \), at \( \mu \), also into (2.6). By the preceding lemma, \( u(t) \equiv v(t) \), but that is impossible, since \( u(t) \) has its first root at \( t = \sqrt{\lambda} \), while the first root of \( v(t) \) is \( t = \sqrt{\mu} \). ♦

We see that the value of \( u(0) \) gives a global parameter on solution curves. In addition to its theoretical significance, this lemma is the key to numerical computation of bifurcation diagrams. For a sequence of values \( \alpha = \alpha_i \) we compute the corresponding first root \( r = r_i \) for the solution of (2.6). (The initial value problem (2.6) is easy to solve numerically.) Then \( \lambda_i = r_i^2 \) is the corresponding value of \( \lambda \) in (2.2). We then plot all of the points \((\lambda_i, \alpha_i)\), obtaining the solution curve. According to the Lemma 2.2 above, these two-dimensional curves give us a faithful picture of the solution set of the nonlinear PDE (2.2).

3 The shoot-and-scale method

Good analytical understanding of a problem goes hand in hand with efficient numerical calculation of its solutions. We know that for positive solutions the maximum value \( u(0) = \alpha \) uniquely determines the solution pair \((\lambda, u(r))\) of the problem

\[
\begin{align*}
\frac{d^2u}{dr^2} + \frac{n-1}{r} \frac{du}{dr} + \lambda f(u) &= 0, \quad r \in (0, 1), \quad u'(0) = 0, \quad u(1) = 0, \\
\end{align*}
\]

see Lemma 2.2 above. We also know that the parameter \( \lambda \) in (3.1) can be “scaled out” i.e., \( v(r) \equiv u(\frac{1}{\sqrt{\lambda}}r) \) solves the equation

\[
\begin{align*}
\frac{d^2v}{dr^2} + \frac{n-1}{r} \frac{dv}{dr} + f(v) &= 0, \\
\end{align*}
\]

7
This program solves $Au + \lambda f(u) = 0$ for $r<1$, $u=0$ for $r=1$, in $n$ dimensions. Draws the bifurcation curve $\{\lambda, u(0)\}$.

Clear["*"]

\[f[u_] = u + 0.5 u \sin[u];\]

\[n = 3;\]

\[u00 = 0;\]

\[delu = 0.07;\]

\[nsteps = 400;\]

\[tend = 1000;\]

\[h = \text{MachineEpsilon};\]

\[dirichlet = \{};\]

\[\text{For}[k = 1, k \leq nsteps, k++,\]

\[u0 = u00 + k \cdot \text{delu};\]

\[\text{sol} = \text{NDSolve}\left[\left\{x''[t] = -\left(n-1\right) \cdot x'[t] / t - f[x[t]], x[0] = u0, x'[0] = 0, \text{WhenEvent}\left[\right.x[t] \cdot x'[t] = 0, \text{If}[-Abs[x[t]] < 10^{-8}, \text{AppendTo}[\text{dirichlet}, (t^2, u0)]];\right\}, x, \{t, h, tend\}\right];\]

\[\text{ListPlot}[\text{dirichlet}, \text{Joined} \to \text{True}, \text{AxesLabel} \to \{"\lambda", \"u(0)\"\}, \text{PlotRange} \to \text{All}, \text{PlotStyle} \to \text{Thickness}[0.007], \text{PlotLabel} \to \"Bifurcation Diagram for Dirichlet problem\"]\]

Figure 1: The program to compute the global solution curve for the problem (3.1), with $f(u) = u + \frac{1}{2}u \sin u$, in space dimension $n = 3$.

while $v(0) = u(0) \equiv \alpha$, and $v'(0) = u'(0) = 0$. The first root of $v(r)$ is $r = \sqrt{\lambda}$. We, therefore, solve the initial value problem

\[v'' + \frac{n-1}{r}v' + f(v) = 0, \quad v(0) = \alpha, \quad v'(0) = 0,\]

and compute its first positive root $r$. Then $\lambda = r^2$, by the above remarks. This way for each $\alpha$ we can find the corresponding $\lambda$. After we choose sufficiently many $\alpha$'s and compute the corresponding $\lambda$'s, we plot the pairs $(\lambda_n, \alpha_n)$, obtaining a bifurcation diagram in $(\lambda, \alpha)$ plane. If one needs to compute the actual solution $u(r)$ of (3.1) at some point $(\lambda, u(0) = \alpha)$, this can be easily done by using the NDSolve command in Mathematica. The program for producing a bifurcation diagram of (3.1), consists of essentially one short loop. It can be found on one of the authors web-page: http://homepages.uc.edu/~kormanp/, or in the Figure 1.

The shortness of this program may come as a surprise to some readers, considering that it produces the global solution curve for the problem (3.1),
with \( f(u) = u + \frac{1}{2} u \sin u \). This bifurcation diagram is presented in Figure 2. By the way, this computation illustrates the phenomenon of “oscillatory bifurcation from infinity” first described in P. Korman [11]. Along the solution curve, \( u(0) \to \infty \), \( \lambda \to \lambda_1 \) and \( \lambda - \lambda_1 \) changes sign infinitely many times, and moreover, \( u(r)/u(0) \) tends to \( \varphi_1(r) \). Here \( (\lambda_1, \varphi_1(r)) \) is the principal eigenpair of the Dirichlet Laplacian on the unit ball, with \( \varphi_1(0) = 1 \). What makes the situation different from the classical “bifurcation from infinity” case is that \( g(u) = \frac{1}{2} u \sin u \) term does not satisfy the condition \( \lim_{u \to \infty} \frac{g(u)}{u} = 0 \).

We now describe the program. Beginning with \( u(0) \equiv u_{00} = 0 \), we choose the step size \( \text{del}u = 0.07 \), and compute the \( \lambda \)'s corresponding to \( u(0) = u_{00} + n \text{del}u \), for the steps \( n = 1, 2, \ldots, n_{\text{steps}} = 400 \). We implemented the shoot-and-scale method, using a highly sophisticated and accurate Mathematica’s command NDSolve to solve the initial value problem (3.2). Formally, the equation has a singularity at \( r = 0 \). We know that \( u(0) \) is finite and given, and that \( u'(0) = 0 \). Instead of starting the integration at zero and getting a zero by zero divide exception, we start at a value called \$MachineEpsilon, which is a value near 0 and is approximately \( 10^{-16} \). We integrate our equation (3.2) until the first root of either \( v(r) \) or \( v'(r) \) is achieved. In case it is the first root of \( v(r) \), the square of this root (\( = \lambda \)) and the corresponding value of \( v(0) = u(0) \) are stored in the file named “dirichlet”, which is later plotted. Mathematica requires us to specify the upper limit of integration, so we choose a large number \( \text{tend} = 1000 \), which is never achieved in practice.

A simple adjustment is needed if the solution curve has disjoint branches. Replace the option \( \text{Joined} \to \text{True} \) by \( \text{Joined} \to \text{False} \) in the ListPlot command, to avoid the branches being joined. A nicer picture can be obtained by first plotting each branch separately, and then plotting them jointly (using Mathematica’s command Show).

### 4 The shoot-and-scale method in case of \( p \)-Laplacian

We now apply the shoot-and-scale method to compute curves of positive solutions for the Dirichlet problem in case of the \( p \)-Laplacian

\[
(4.1) \quad \varphi(u')' + \frac{n-1}{r} \varphi(u') + \lambda f(u) = 0, \quad u'(0) = u(1) = 0, 
\]

where \( \varphi(v) = v|v|^{p-2}, \quad p > 1 \). As in the Laplacian case \( p = 2 \), the value of \( u(0) \) uniquely identifies the solution pair \( (\lambda, u(r)) \), so that the solution curves can be drawn in the \( (\lambda, u(0)) \) plane. The proof is exactly the same
Figure 2: The global solution curve for the problem (3.1), with \( f(u) = u + \frac{1}{2} u \sin u \), in space dimension \( n = 3 \)

as in case \( p = 2 \), using the following existence and uniqueness result from P. Korman [10].

**Lemma 4.1** Assume that \( f(u) \) is Lipschitz continuous. Then one can find an \( \epsilon > 0 \) so that the problem

\[
\varphi(u')' + \frac{n-1}{r} \varphi(u') + \lambda f(u) = 0, \quad u(0) = \alpha, \quad u'(0) = 0
\]

has a unique solution \( u(r) \in C^1[0,\epsilon] \cap C^2(0,\epsilon) \), for any \( \alpha > 0 \).

We have \( \varphi(u')' = \varphi'(u')u'' = (p-1)|u'|^{p-2}u'' \), using that \( \varphi'(v) = (p-1)|v|^p \). Dividing (4.1) by \( (p-1)|u'|^{p-2} \), we have

\[
(4.2) \quad u'' + \frac{n-1}{(p-1)r} u' + \lambda \frac{f(u)}{(p-1)|u'|^{p-2}} = 0, \quad u'(0) = u(1) = 0.
\]

Similarly to the case of the Laplacian \( p = 2 \), one may approach (4.2) by solving the initial value problem

\[
(4.3) \quad u'' + \frac{n-1}{(p-1)r} u' + \frac{f(u)}{(p-1)|u'|^{p-2}} = 0, \quad u(0) = \alpha > 0, \quad u'(0) = 0
\]

until its solution \( u(r) \) reaches the first root at some \( \xi \). (The form (4.3) of our equation is convenient for Mathematica’s NDSolve command to solve.)
Then by scaling we obtain a solution of (4.2) at \( \lambda = \xi^p \), and a point \((\lambda, \alpha)\) on the solution curve in the \((\lambda, u(0))\) plane. After computing a sufficient number of such points, one plots the solution curve. Observe that for \( p > 2 \) we have a true singularity at \( r = 0 \) in the last term of (4.3) (in case \( p = 2 \) there is only an apparent singularity at \( r = 0 \)). It was proved in P. Korman [15] that the solution of (4.3) satisfies

\[
(4.4) \quad u(r) \approx \alpha + a_1 r^{\frac{p}{p-1}}, \quad \text{for } r \text{ small},
\]

where the constant \( a_1 < 0 \) is computed using the formula (3.3) of that paper. A simple-minded approach involves choosing a small \( h \), say \( h = 0.00001 \), then approximating \( u(h) \) and \( u'(h) \) using (4.4), and use these values as the initial conditions while integrating (4.3) for \( r > h \). Remarkably, this approach works reasonably well, even for fast growing \( f(u) \) like \( f(u) = e^u \), even though the last term in (4.3) is huge at \( r = h \). This is due to the high accuracy of the Mathematica’s NDSolve command.

A better approach is to use the regularizing transformation \( r \rightarrow z \), given by

\[
(4.5) \quad z = r^{\frac{p}{p(p-1)}},
\]

which was introduced in P. Korman [15]. We present next a simplified proof of that result.

**Lemma 4.2** Denote

\[
(4.6) \quad \beta = \frac{p}{2(p-1)}, \quad a = \beta^p, \quad A = \beta^{p-1}(\beta - 1) + \frac{n-1}{p-1}\beta^{p-1}.
\]

Then, for \( p > 2 \), the change of variables (4.5) transforms (4.3) into

\[
(4.7) \quad a u''(z) + \frac{A}{z} u'(z) + \frac{1}{p-1)[u'(z)]^{p-2} f(u) = 0,
\]

\[
 u(0) = \alpha, \quad u'(0) = 0.
\]

**Proof:** Consider the transformation \( z = r^\beta \) in (4.3), with the constant \( \beta \) to be specified. Using that \( u_r = \beta r^{\beta-1} u_z \) and \( u_{rr} = \beta^2 r^{2\beta-2} u_{zz} + \beta(\beta - 1) r^{\beta-2} u_z \) obtain from (4.3)

\[
\beta^2 r^{2\beta-2} u_{zz} + \left( \beta(\beta - 1) + \beta \frac{n-1}{p-1} \right) r^{\beta-2} u_z + \frac{f(u)}{(p-1)\beta^{p-2}r^{(\beta-1)(p-2)}|u_z|^{p-2}} = 0.
\]

Divide this equation by \( r^{2\beta-2} \), and multiply by \( \beta^{p-2} \) to obtain

\[
au''(z) + \frac{A}{z} u'(z) + \frac{1}{(p-1)\beta^{(\beta-1)p}|u'(z)|^{p-2}} f(u) = 0.
\]

11
This equation becomes (4.7), provided we choose \( \beta \) so that

\[
\frac{1}{r^{(\beta-1)p}} = z^{p-2} = r^{\beta(p-2)} ,
\]

which occurs for \( \beta = \bar{\beta} = \frac{p}{2(p-1)} \).

The equation in (4.7) has no singularity at \( z = 0 \), in fact the solution is of class \( C^2 \) for continuous \( f(u) \) (all of the derivatives \( u^{(n)}(0) \) can be computed if \( f(u) \in C^\infty \)). It follows that solutions of (4.1) are of the form \( u(r) = w \left( r^{\frac{p}{2(p-1)}} \right) \), where the function \( w(z) \) is twice differentiable at \( z = 0, w(z) \in C^2[0,1] \). However, for the Mathematica’s NDSolve command the equation (4.7) is still considered to be singular. Therefore we use the above mentioned approximation from P. Korman [15]

(4.8) \[ u(z) \approx \alpha + a_1 z^2 + a_2 z^4 , \]

to approximate \( u(h) \) and \( u'(h) \), and then perform the numerical calculation for \( z > h \). Here \( a_1 < 0 \) is given by the formula (3.3) of [15]:

\[
a_1 = - \left[ \frac{1}{(p-1)2^{p-2}B_1} f(\alpha) \right]^{\frac{1}{p-1}} ,
\]

where \( B_1 = 2a + \frac{2A}{p-1} \). We now calculate \( a_2 \) using the formula (3.4) of [15]:

\[
a_2 = - \frac{1}{B_2 C_2} \lim_{z \to 0} \frac{z^{p-2}}{(p-1)(2a_1 z)^{p-1}} \frac{f(\alpha + a_1 z^2) + 2a_1 a + 2a_1^2}{z^2} .
\]

This limit does not change if we replace \( f(\alpha + a_1 z^2) \) by \( f(\alpha) + f'(\alpha) a_1 z^2 \), which leads to

\[
a_2 = - \frac{f'(\alpha) a_1}{B_2 C_2 (p-1)(2a_1)^{p-2}} .
\]

If \( z_0 \) denotes the first root of \( u(z) \), while \( \xi \) is the first root of \( u(r) \), then in view of (4.5) the value of \( \lambda \) corresponding to \( u(0) = \alpha \) is

\[
\lambda = \xi^p = z_0^{2(p-1)} .
\]

After calculating sufficiently many points \( (\lambda_n, \alpha_n) \), we plot the solution curve for the problem (4.1). An example is presented in Figure 3.
Figure 3: The global solution curve for the $p$-Laplace problem (4.1), with $f(u) = e^u$, $p = 4$ and $n = 5$

5  Bifurcation diagrams for Neumann boundary conditions

A small modification of the above program produces positive decreasing solutions of the Neumann problem

\[ u'' + \frac{n-1}{r}u' + \lambda f(u) = 0, \quad r \in (0, 1), \quad u'(0) = u'(1) = 0 \]
\[ u(r) > 0, \quad u'(r) < 0, \quad r \in (0, 1). \]

There is no analog of B. Gidas, W.-M. Ni and L. Nirenberg [5] result for Neumann problem, but radial solutions on a ball are still of interest.

In Figure 4 we present the program to compute the bifurcation diagrams for (5.1), in case $f(u) = u(u - 1)(7 - u)$ (electronically available at http://homepages.uc.edu/~kormanp/). For this $f(u)$ the problem (5.1) has a trivial solution $u(r) = 1$ for all $\lambda$. Our computations show that the situation is different for the cases $n = 1$ and $n \geq 2$. In the case $n = 1$ there are two half-branches bifurcating from $u = 1$ at some $\lambda_0 \approx 1.65$ that continue for all $\lambda > \lambda_0$ without any turns, see Figure 5. That both half-branches bifurcating from $u = 1$ do not turn was proved rigorously by R. Schaaf [24], in case $n = 1$, for a class of nonlinearities $f(u)$ which includes polynomials with simple roots. In Figure 6 we present the bifurcation picture for $n = 5$, which is typical for all other dimensions $n \geq 2$ that we tried. This time two half-branches bifurcate from $u = 1$ at some $\lambda_1 \approx 5.5$. The lower branch continues without any turns for all $\lambda > \lambda_1$, while the upper branch continues for decreasing $\lambda$, and makes exactly one turn. Proving these facts rigorously is a very challenging open problem. The bifurcation methods as in e.g., P. Korman [12] or T. Ouyang and J. Shi [21] appear to be inapplicable here.
(*Finds the decreasing positive solutions of radial Neumann problem

\[ u_{rr} + \frac{n-1}{r} u_r + \lambda f(u) = 0 \]
for \( r < 1 \), \( u'(0) = u'(1) = 0 \),
in \( n \) dimensions. Draws the bifurcation curve \( \{ \lambda, u(0) \} \).*)

Clear["\*""]
f[u_] = u (u - 1) (7 - u);
n = 3;
u00 = 0;
delu = 0.1;
nsteps = 70;
tend = 1000;
h = $MachineEpsilon;
neumann = {};
For[k = 1, k \leq nsteps, k++,
  u0 = u00 + k * delu;
  sol = NDSolve[\{x''[t] = -(n-1)*x'[t]/t - f[x[t]], x[0] == u0, x'[0] == 0, WhenEvent[\{x[t]*x'[t] == 0\}, If[Abs[x'[t]] < 10^-8, AppendTo[neumann, \{t^2, u0\}]; "StopIntegration"]\}, x, \{t, h, tend\}];
]
ListPlot[neumann, Joined -> True, AxesLabel -> \{"\lambda", "u(0)"\},
PlotRange -> All, PlotStyle -> Thickness[0.008],
PlotLabel -> "Bifurcation Diagram for Neumann problem"]

Figure 4: The program to compute the global Neumann solution curve for the problem (5.1), with \( f(u) = u(u - 1)(7 - u) \)

Figure 5: The global Neumann solution curves for the problem (5.1), with \( f(u) = u(u - 1)(7 - u) \), \( n = 1 \)
Figure 6: The global Neumann solution curves for the problem (5.1), with $f(u) = u(u - 1)(7 - u)$, $n = 5$

We now describe the program presented in Figure 4. We integrate the initial value problem (3.2), similarly to the Dirichlet case. Integrations are stopped if either the root of $v(r)$ or of $v'(r)$ is reached. We record these values of $r$ and the corresponding $v(0)$ in two separate files, and then plot the file involving $v'(r) = 0$, obtaining the curve of positive decreasing solutions of the Neumann problem (5.1).

6 Solution curves for non-autonomous problems

We wish to compute the curves of positive solutions for non-autonomous problems

$$\Delta u + \lambda f(|x|, u) = 0 \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B,$$

where $B$ is the unit ball $|x| < 1$ in $\mathbb{R}^n$. We shall assume that the function $f(r, u) \in C^2(B \times \mathbb{R}_+)$ satisfies

$$f(r, u) > 0, \quad \text{for } 0 < r < 1 \text{ and } u > 0,$$

$$f_r(r, u) \leq 0, \quad \text{for } 0 < r < 1 \text{ and } u > 0.$$  

By B. Gidas, W.-M. Ni and L. Nirenberg [5], any positive solution of (6.1) is radially symmetric, i.e., $u = u(r)$, $r = |x|$, and hence it satisfies

$$u'' + \frac{n - 1}{r} u' + \lambda f(r, u) = 0, \quad u'(0) = u(1) = 0.$$

The shoot-and-scale method does not apply for non-autonomous problems (one can still shoot, but the scaling fails.)
The standard approach to numerical computation of solutions involves curve following, i.e., continuation in $\lambda$ by using the predictor-corrector type methods, see e.g., E.L. Allgower and K. Georg [2]. These methods are well developed, but not easy to implement, as the solution curve $u = u(x, \lambda)$ may consist of several parts, each having multiple turns. Here $\lambda$ is a local parameter, but usually it is not a global one, because of the possible turning points. We wish to perform the continuation using a global parameter, similarly to the shoot-and-scale method.

The quantity $\alpha = u(0)$ gives the maximum value of any positive solution. In case $n = 1$, the value of $\alpha = u(0)$ is known to be a global parameter, i.e., the value of $\alpha$ uniquely identifies the solution pair $(\lambda, u(r))$, see P. Korman [13] or P. Korman and J. Shi [19]. In case $n > 1$, we shall still do continuation in $\alpha$, computing the solution curve of (6.4) in the form $\lambda = \lambda(\alpha)$, although now there is no guarantee that all positive solutions of (6.4) are obtained. We begin with a simple lemma.

**Lemma 6.1** The solution of the linear problem

$$u'' + \frac{n-1}{r} u' + g(r) = 0 \quad \text{for } 0 < r < 1, \quad u(0) = \alpha, \quad u'(0) = 0$$

can be represented in the form

$$u(r) = \alpha + \frac{1}{n-2} r^{n+2} \int_0^r \left( z^{n-2} - r^{n-2} \right) z g(z) \, dz, \quad \text{for } n \neq 2,$$

$$u(r) = \alpha + \int_0^r (\ln z - \ln r) \, z g(z) \, dz, \quad \text{for } n = 2.$$

**Proof:** Integrating

$$\left( r^{n-1} u'(r) \right)' = -r^{n-1} g(r)$$

over the interval $(0, z)$, we express

$$u'(z) = -\frac{1}{z^{n-1}} \int_0^z t^{n-1} g(t) \, dt.$$

Integrating over the interval $(0, r)$, we have

$$u(r) = \alpha - \int_0^r \frac{1}{z^{n-1}} \int_0^z t^{n-1} g(t) \, dt \, dz.$$

Integrating by parts in the last integral (with $u = \int_0^z t^{n-1} g(t) \, dt$, $dv = \frac{1}{z^{n-1}} \, dz$), we conclude the proof.

$\diamondsuit$
If we solve the initial value problem

\[(6.5)\quad u'' + \frac{n-1}{r} u' + \lambda f(r, u) = 0, \quad u(0) = \alpha, \quad u'(0) = 0,\]

then we need to find \(\lambda\), so that \(u(1) = 0\), in order to obtain a solution of (6.4). By Lemma 6.1, we rewrite the equation (6.4) in the integral form (for \(n \neq 2\))

\[u(r) = \alpha + \frac{\lambda}{n-2} r^{1-n} \int_0^r (z^{n-2} - r^{n-2}) zf(z, u(z)) \, dz, \quad \text{for} \ n \neq 2,\]

and then the equation for \(\lambda\) is

\[(6.6)\quad F(\lambda) \equiv u(1) = \alpha + \frac{\lambda}{n-2} \int_0^1 (z^{n-2} - 1) zf(z, u(z)) \, dz = 0.\]

We solve this equation by using Newton’s method

\[\lambda_{n+1} = \lambda_n - \frac{F(\lambda_n)}{F'(\lambda_n)}, \quad n = 0, 1, 2, \ldots.\]

The iterations begin at \(\lambda_0\), which we choose to be the value of \(\lambda\) corresponding to the preceding value of \(\alpha = u(0)\). Calculate

\[F(\lambda_n) = \alpha + \frac{\lambda_n}{n-2} \int_0^1 (z^{n-2} - 1) zf(z, u(z)) \, dz,\]

\[F'(\lambda_n) = \frac{1}{n-2} \int_0^1 (z^{n-2} - 1) zf(z, u(z)) \, dz\]

\[+ \frac{\lambda_n}{n-2} \int_0^1 (z^{n-2} - 1) z f_u(z, u(z)) u_\lambda \, dz,\]

where \(u = u(r, \lambda_n)\) and \(u_\lambda = u_\lambda(r, \lambda_n)\) are respectively the solutions of

\[(6.7)\quad u'' + \frac{n-1}{r} u' + \lambda_n f(r, u) = 0, \quad u(0) = \alpha, \quad u'(0) = 0,\]

and

\[(6.8)\quad u_\lambda'' + \frac{n-1}{r} u_\lambda' + \lambda_n f_u(r, u(r, \lambda_n)) u_\lambda + f(r, u(r, \lambda_n)) = 0\]

\[u_\lambda(0) = 0, \quad u_\lambda'(0) = 0.\]

(As we vary \(\lambda\), when solving (6.6), we keep \(u(0) = \alpha\) fixed, and that is the reason why \(u_\lambda(0) = 0\).) This method is very easy to implement. It requires
repeated solutions of the initial value problems (6.7) and (6.8) (using the NDSolve command in Mathematica).

In case $n = 2$, we have

$$F(\lambda) = \alpha + \lambda \int_0^1 z \ln zf(z, u(z)) \, dz,$$

$$F'(\lambda) = \int_0^1 z \ln zf(z, u(z)) \, dz + \lambda \int_0^1 z \ln zf_u(z, u(z)) u_\lambda(z) \, dz,$$

and the rest is as before.

**Example** We solved the problem

(6.9) \quad u'' + \frac{n-1}{r} u' + \lambda (1 - 1.1r^2) e^u = 0, \quad \text{for } 0 < r < 1,

$$u'(0) = u(1) = 0,$$

for the Gelfand’s equation of combustion theory, with a sign-changing potential $1 - 1.1r^2$. The global curve of positive solutions, for $n = 3$, is presented in Figure 7. For any point $(\lambda, \alpha)$ on this curve ($\alpha = u(0)$), the actual solution $u(r)$ is easily computed by shooting (using the NDSolve command in Mathematica), i.e., by solving (6.5). In Figure 8 we present the solution $u(r)$ for $\lambda \approx 2.59566$, when $u(0) = 9.1$. (This solution lies on our solution curve in Figure 7, after the second turn.) The picture in Figure 7 suggests that solution curve for the problem (6.9) has infinitely many turns, similarly to the case of the constant potential, according to the classical result of D.D. Joseph and T.S. Lundgren [8].
7 Solution curves for the elastic beam equation

We now compute curves of positive solutions for the equation modeling the displacements of an elastic beam, which is clamped at both end points,

\begin{align}
    u''''(x) &= \lambda f(u(x)), \text{ for } x \in (-1, 1) \\
    u(-1) &= u'(-1) = u(1) = u'(1) = 0, 
\end{align}

obtaining the solution curves in $(\lambda, u(0))$ plane, similarly to the second order equations. Under the assumption that

\begin{align}
    f(u) > 0, \text{ and } f'(u) > 0 \text{ for } u > 0 
\end{align}

it was shown in P. Korman [9] that any positive solution of (7.1) is an even function taking its global maximum at $x = 0$. Moreover, the value of $u(0)$ is a global parameter, uniquely identifying the solution pair $(\lambda, u(x))$ (see also P. Korman and J. Shi [18] for a generalization). That $u(0)$ is a global parameter is quite a surprising result, because unlike the second order equations, the knowledge of $u(0) = \alpha$ and $u'(0) = 0$ does not identify the solutions of the initial value problem corresponding to the equation in (7.1) (we have $u'''(0) = 0$ by the symmetry of solutions, but one also needs to know the value of $u''(0) = \beta$, in order to “shoot”).

As we just mentioned, the initial conditions corresponding to the positive symmetric solutions of (7.1) are

\begin{align}
    u(0) &= \alpha > 0, \ u'(0) = 0, \ u''(0) = \beta \leq 0, \ u'''(0) = 0. 
\end{align}

We record the following simple observation.
Lemma 7.1 For a given continuous function \( g(x) \) the solution of

\[
    u^{iv} = g(x),
\]

together with the initial conditions (7.3) is

\[
    u(x) = \int_0^x \frac{(x-t)^3}{3!} g(t) \, dt + \alpha + \frac{\beta x^2}{2}. \tag{7.4}
\]

We know that the value of \( u(0) = \alpha \) uniquely identifies the solution pair \((\lambda, u(x))\) (and plotting of \( \lambda = \lambda(\alpha) \) gives the bifurcation diagram). However, we do not know the corresponding value of \( u''(0) = \beta \), which makes it impossible to shoot and scale. We now describe an algorithm to compute both \((\beta, \lambda)\), given \( \alpha \).

We begin by converting the equation in (7.1), together with the initial conditions (7.3) into an integral equation

\[
    u(x) = \lambda \int_0^x \frac{(x-t)^3}{3!} f(u(t)) \, dt + \alpha + \frac{\beta x^2}{2},
\]

by using (7.4). To obtain the solution of our problem (7.1) corresponding to \( u(0) = \alpha \), we need to find \((\beta, \lambda)\) so that

\[
    u(1) = \lambda \int_0^1 \frac{(1-t)^3}{3!} f(u(t)) \, dt + \alpha + \frac{\beta}{2} = 0, \tag{7.5}
\]

\[
    u'(1) = \lambda \int_0^1 \frac{(1-t)^2}{2} f(u(t)) \, dt + \beta = 0.
\]

Denoting

\[
    F(\lambda, \beta) = \lambda \int_0^1 \frac{(1-t)^3}{3!} f(u(t)) \, dt + \alpha + \frac{\beta}{2}, \quad G(\lambda, \beta) = \lambda \int_0^1 \frac{(1-t)^2}{2} f(u(t)) \, dt + \beta,
\]

we recast the equations (7.5) as

\[
    F(\lambda, \beta) = 0 \tag{7.6}
\]

\[
    G(\lambda, \beta) = 0.
\]

We use Newton’s method to solve the system (7.6). Assuming that the \( n \)-th iterate \((\lambda_n, \beta_n)\) is already computed, we linearize (7.6) at this point

\[
    F_\lambda(\lambda_n, \beta_n)(\lambda - \lambda_n) + F_\beta(\lambda_n, \beta_n)(\beta - \beta_n) = 0
\]

\[
    G_\lambda(\lambda_n, \beta_n)(\lambda - \lambda_n) + G_\beta(\lambda_n, \beta_n)(\beta - \beta_n) = 0,
\]

20
then solve this system for \((\lambda, \beta)\), and declare the solution to be our next iterate \((\lambda_{n+1}, \beta_{n+1})\). Observe that the solution \(u(t)\) of (7.1) and (7.3), which was used in the definitions of \(F(\lambda, \beta)\) and \(G(\lambda, \beta)\) is not known precisely, so we shall replace it by the best approximation available at each iteration step. Namely, define \(u_n(x)\) to be the solution of the following initial value problem

\[
\begin{align*}
  u''''(x) &= \lambda_n f(u(x)) \\
  u(0) &= \alpha, \quad u'(0) = 0, \quad u''(0) = \beta_n, \quad u'''(0) = 0.
\end{align*}
\]

Clearly, the solution \(u(x)\) of (7.1) and (7.3) depends on both \(\lambda\) and \(\beta\). We denote by \(u_\lambda(x)\) the derivative of \(u(x)\) with respect to \(\lambda\), and by \(u_\beta(x)\) the derivative of \(u(x)\) with respect to \(\beta\). Then we compute the quantities in (7.7) as follows

\[
\begin{align*}
  F_\lambda(\lambda_n, \beta_n) &= \int_0^1 \frac{(1-t)^3}{3!} f(u_n(t)) \, dt + \lambda_n \int_0^1 \frac{(1-t)^3}{3!} f'(u_n(t)) u_\lambda(t) \, dt, \\
  F_\beta(\lambda_n, \beta_n) &= \lambda_n \int_0^1 \frac{(1-t)^3}{3!} f'(u_n(t)) u_\beta(t) \, dt + \frac{1}{2}, \\
  G_\lambda(\lambda_n, \beta_n) &= \int_0^1 \frac{(1-t)^2}{2} f(u_n(t)) \, dt + \lambda_n \int_0^1 \frac{(1-t)^2}{2} f'(u_n(t)) u_\lambda(t) \, dt, \\
  G_\beta(\lambda_n, \beta_n) &= \lambda_n \int_0^1 \frac{(1-t)^2}{2} f'(u_n(t)) u_\beta(t) \, dt + 1.
\end{align*}
\]

To compute \(u_\lambda(x)\) one needs to solve the following linear initial value problem

\[
\begin{align*}
  u_\lambda'''(x) &= f(u_n(x)) + \lambda_n f'(u_n(x)) u_\lambda(x) \\
  u_\lambda(0) &= 0, \quad u'_\lambda(0) = 0, \quad u''_\lambda(0) = 0, \quad u'''_\lambda(0) = 0.
\end{align*}
\]

obtained by differentiating (7.1) and (7.3) in \(\lambda\). (We have \(u_\lambda(0) = 0\), because \(u(0)\) is kept fixed at \(\alpha\), and \(u''_\lambda(0) = 0\), since only \(\lambda\) is varied, and \(\beta\) is kept fixed.) To compute \(u_\beta(x)\) one needs to solve the following linear initial value problem

\[
\begin{align*}
  u_\beta'''(x) &= \lambda_n f'(u_n(x)) u_\beta(x) \\
  u_\beta(0) &= 0, \quad u'_\beta(0) = 0, \quad u''_\beta(0) = 1, \quad u'''_\beta(0) = 0.
\end{align*}
\]

obtained by differentiation of (7.1) and (7.3) in \(\beta\).
We now review the computation of the global curve \( \lambda = \lambda(\alpha) \) of positive solutions for the problem (7.1). We choose \( \alpha_0 \), the initial value for \( \alpha \), the step-size called \textit{step}, and the number of steps \( N \), then for \( \alpha_m = \alpha_0 + m \text{ step} \), \( m = 1, 2, \ldots N \), we calculate the corresponding values of \( \lambda \) by obtaining \( (\lambda, \beta) \) as a limit of Newton’s iterates \( (\lambda_n, \beta_n) \). Given \( (\lambda_n, \beta_n) \), the next Newton’s iterate \( (\lambda_{n+1}, \beta_{n+1}) \) is produced as follows.

1. Solve (7.8) to calculate \( u_n(x) \),
2. Solve (7.9) to calculate \( u_\lambda(x) \),
3. Solve (7.10) to calculate \( u_\beta(x) \),
4. Calculate the numbers \( F_\lambda(\lambda_n, \beta_n) \), \( F_\beta(\lambda_n, \beta_n) \), \( G_\lambda(\lambda_n, \beta_n) \) and \( G_\beta(\lambda_n, \beta_n) \),
5. Solve the linear system (7.7) for \( (\lambda, \beta) \) which is the desired \( (\lambda_{n+1}, \beta_{n+1}) \).

Newton’s method is started at \( (\lambda_0, \beta_0) \), for which we use the values of \( \lambda \) and \( \beta \) at the preceding value of \( \alpha \), namely \( \alpha = \alpha_{m-1} \).

**Example** We used Mathematica to compute the solutions of the problem

\begin{equation}
\tag{7.11}
\begin{aligned}
    &u''''(x) = \lambda e^{ax}, &\text{for } x \in (-1, 1) \\
    &u(-1) = u'(-1) = u(1) = u'(1) = 0,
\end{aligned}
\end{equation}

with \( a = 5 \). The \( S \)-shaped global curve of positive solutions is presented in Figure 9. For any point \( (\lambda, \alpha) \) on this curve, the actual solution \( u(x) \) is easily computed by shooting (using the NDSolve command in Mathematica), i.e., by solving the equation in (7.1) together with the initial conditions (7.3), using the function \( \beta = \beta(\alpha) \) which was calculated while producing the above bifurcation diagram. For second order equations, the function \( f(u) = e^{ax+u} \) is prominent in connection to the “perturbed Gelfand equation” of combustion theory, and the solution curve is also \( S \)-shaped, see e.g., [12] for the references.

**Remark** Our algorithm also works for non-autonomous problems

\begin{equation}
\begin{aligned}
    &u''''(x) = \lambda f(x, u(x)), &\text{for } x \in (-1, 1) \\
    &u(-1) = u'(-1) = u(1) = u'(1) = 0,
\end{aligned}
\end{equation}

although it is not known if the value of \( u(0) \) is a global parameter here. In case \( u(0) \) is not a global parameter, there might be solutions not lying on the computed solution curve.
We describe numerical computation of solutions for the problem

\( u'' + f(u) = \mu \sin x + e(x), \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0 \).

(8.1)

On the interval \((0, \pi)\), we have \( \lambda_1 = 1, \varphi_1(x) = \sin x, \lambda_2 = 4, \varphi_2(x) = \sin 2x \), etc. Decompose \( u(x) = \xi \sin x + U(x) \), with \( \int_0^\pi U(x) \sin x \, dx = 0 \). It was shown in P. Korman [14] that \( \xi \) is a global parameter, uniquely identifying the solution pair \((u(x), \mu)\) of (1.4), provided that

\( f'(u) < \lambda_2 = 4, \quad \text{for all } u \in R. \)

(8.2)

In case the condition (8.2) fails, the value of \( \xi \) is not a global parameter in general, as we shall see later in this section. However, we shall still compute the solution curve of (8.1): \((u(\xi), \mu(\xi))\), although if the condition (8.2) fails, there might be solutions not lying on the computed solution curve. We use Newton’s method to perform continuation in \( \xi \).

We begin by implementing the “linear solver”, i.e., the numerical solution of the following problem: given any \( \xi \in R \), and any continuous functions \( a(x) \) and \( g(x) \), find \( u(x) \) and \( \mu \in R \) solving

\[ u'' + a(x)u = \mu \sin x + g(x), \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0, \quad \int_0^\pi u(x) \sin x \, dx = \xi. \]

(8.3)

Here \( g(x) \) is any continuous function, not necessarily orthogonal to \( \sin x \). The general solution of the differential equation in (8.3) is

\[ u(x) = Y(x) + c_1u_1(x) + c_2u_2(x), \]

(8.4)
where \( Y(x) \) is any particular solution of that equation, and \( u_1(x), u_2(x) \) are two linearly independent solutions of the corresponding homogeneous equation

\[(8.5) \quad u'' + a(x)u = 0, \quad 0 < x < \pi.\]

We shall calculate the particular solution in the form \( Y(x) = \mu Y_1(x) + Y_2(x) \), where \( Y_1(x) \) solves

\[u'' + a(x)u = \sin x, \quad u(0) = 0, \quad u'(0) = 1,\]

and \( Y_2(x) \) is the solution of

\[u'' + a(x)u = g(x), \quad u(0) = 0, \quad u'(0) = 1.\]

Let \( u_1(x) \) be the solution of the equation (8.5) together with the initial conditions \( u(0) = 0, \quad u'(0) = 1 \), and let \( u_2(x) \) be any solution of (8.5) with \( u_2(0) \neq 0 \). The condition \( u(0) = 0 \) in (8.3) implies that \( c_2 = 0 \) in (8.4), so that there is no need to compute \( u_2(x) \), and (8.4) takes the form

\[(8.6) \quad u(x) = \mu Y_1(x) + Y_2(x) + c_1 u_1(x).\]

The condition \( u(\pi) = 0 \) and the last line in (8.3) imply that

\[\mu Y_1(\pi) + c_1 u_1(\pi) = -Y_2(\pi),\]

\[\mu \int_0^\pi Y_1(x) \sin x \, dx + c_1 \int_0^\pi u_1(x) \sin x \, dx = \xi - \int_0^\pi Y_2(x) \sin x \, dx,\]

Solving this system for \( \mu \) and \( c_1 \), and using \( c_1 \) in (8.6), gives the solution \((u(x), \mu)\) of (8.3).

Turning to the problem (8.1), we begin with an initial value of \( \xi = \xi_0 \), and using a step size \( \Delta \xi \), on a mesh \( \xi_i = \xi_0 + i \Delta \xi, \quad i = 1, 2, \ldots, \text{nsteps} \), we compute the solution \((u(x), \mu)\) of (8.1), satisfying \( \int_0^\pi u(x) \sin x \, dx = \xi_i \), by using Newton’s method. Namely, assuming that the iterate \((u_n(x), \mu_n)\) is already computed, we linearize the equation (8.1) at \( u_n(x) \):

\[u_{n+1}'' + f(u_n) + f'(u_n)(u_{n+1} - u_n) = \mu \sin x + g(x),\]

and calculate \((u_{n+1}(x), \mu_{n+1})\) by using the method described above for the problem (8.3) with \( a(x) = f'(u_n(x)) \), \( g(x) = -f(u_n(x)) + f'(u_n(x))u_n(x) + e(x) \), and \( \xi = \xi_i \). After several iterations, we compute \((u(\xi_i), \mu(\xi_i))\). We found that three iterations of Newton’s method, coupled with \( \Delta \xi \) not too large (say, \( \Delta \xi = 0.2 \)), were sufficient for accurate computation of the solution.
curves. To start Newton’s iterations, we used the solution \( u(x) \) computed at the preceding step, i.e., \( u_0(x) = u(x)|_{\xi=\xi_{i-1}} \).

**Example**  We used *Mathematica* to solve

\[
(8.7) \quad u'' + \sin u = \mu \sin x + x - \frac{\pi}{2}, \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0.
\]

Observe that \( \int_0^\pi (x - \frac{\pi}{2}) \sin x \, dx = 0 \). The solution curve \( \mu = \mu(\xi) \) is presented in Figure 10. The condition (8.2) applies here, and hence this curve exhausts the solution set of (8.7).

![Figure 10: The global solution curve for the problem (8.7)](image)

Let us look at the points of intersection of the solution curve with the \( \xi \)-axis, where \( \mu = 0 \). Figure 10 strongly suggests that the problem

\[
u'' + \sin u = x - \frac{\pi}{2}, \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0
\]

has exactly three solutions, one of which has zero first harmonic.

We now discuss computation of multiple solutions for the problem

\[
(8.8) \quad u'' + f(u) = 0, \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0.
\]

Of course, solutions of (8.8) can be computed by shooting, but we describe a more general approach which is also applicable to PDE’s. Embed (8.8) into a family of problems

\[
(8.9) \quad u'' + f(u) = \mu_1 \sin x, \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0.
\]

Decompose its solution as \( u(x) = \xi_1 \sin x + U(x) \), with \( \int_0^\pi U(x) \sin x \, dx = 0 \). As above, we can compute the solution pair \((u(x), \mu_1)\) as a function of \( \xi_1 \),
and draw the curve \( \mu_1 = \mu_1(\xi_1) \). At the points of intersection of this curve with the \( \xi_1 \) axis, where \( \mu_1 = 0 \), we obtain solutions of (8.8). Similarly, we can embed (8.8) into another family of problems

(8.10) \[ u'' + f(u) = \mu_2 \sin 2x, \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0. \]

Decompose its solution as \( u(x) = \xi_2 \sin 2x + U(x) \), with \( \int_0^\pi U(x) \sin 2x \, dx = 0 \). We perform the continuation in the second harmonic \( \xi_2 \) in a completely similar way to the computation in the first harmonic, described above. We compute the solution pair \((u(x), \mu_2)\) as a function of \( \xi_2 \), and draw the curve \( \mu_2 = \mu_2(\xi_2) \). At the points of intersection of this curve with the \( \xi_2 \) axis, where \( \mu_2 = 0 \), we obtain solutions of (8.8). Our computations indicate that these solutions of (8.8) can be made different from the ones obtained by the continuation in the first harmonic \( \xi_1 \), if \( f'(0) > \lambda_2 = 4 \).

In a recent paper A. Castro et al [3] considered the problem (8.8) (they also considered the corresponding PDE problem on a general domain). Assuming that \( f(0) = 0, f'(0) > \lambda_2, f'(0) \neq \lambda_k \) for any \( k \geq 3 \), and that there exists \( \gamma < \lambda_1 \) and \( \rho > 0 \) such that \( \frac{f(u)}{u} \leq \gamma \) for \( |u| \geq \rho \), they proved that the problem (8.8) has at least four non-trivial solutions, in addition to the trivial one. One of these solutions is positive, and one is negative. We now illustrate their result for the following example:

(8.11) \[ u'' + \frac{6u}{1 + u + 2u^2} = 0, \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0. \]

Here \( f'(0) = 6 \) lies between \( \lambda_2 = 4 \) and \( \lambda_3 = 9 \). We calculated the solution curve \( \mu_1 = \mu_1(\xi_1) \) for the problem

(8.12) \[ u'' + \frac{6u}{1 + u + 2u^2} = \mu_1 \sin x, \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0. \]

The result is presented in Figure 11.

The function \( \mu_1 = \mu_1(\xi_1) \) has a root at \( \xi_1 \approx -3.8 \), which implies that the problem (8.11) has a solution with the first harmonic \( \xi_1 \approx -3.8 \). This solution, which we denote by \( u_1(x) \), is negative on \((0, \pi)\). We computed \( u'_1(0) \approx -3.1968 \), which together with \( u_1(0) = 0 \) is sufficient to identify this solution. The root \( \xi_1 = 0 \) corresponds to the trivial solution of (8.11). The root at \( \xi_1 \approx 2.7 \) gives us a second non-trivial solution of (8.11), \( u_2(x) \), which is positive. From the solution \( u_2(x) \) we find \( u'_2(0) \approx 2.0606 \).

To obtain two more solutions of (8.11), we calculated the solution curve \( \mu_2 = \mu_2(\xi_2) \) for the problem

(8.13) \[ u'' + \frac{6u}{1 + u + 2u^2} = \mu_2 \sin 2x, \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0, \]

The function \( \mu_2 = \mu_2(\xi_2) \) has a root at \( \xi_2 \approx -2.8 \), which implies that the problem (8.13) has a solution with the first harmonic \( \xi_2 \approx -2.8 \). This solution, which we denote by \( u_2(x) \), is negative on \((0, \pi)\). We computed \( u'_2(0) \approx -2.81968 \), which together with \( u_2(0) = 0 \) is sufficient to identify this solution. The root \( \xi_2 = 0 \) corresponds to the trivial solution of (8.13). The root at \( \xi_2 \approx 2.7 \) gives us a second non-trivial solution of (8.13), \( u_3(x) \), which is positive. From the solution \( u_3(x) \) we find \( u'_3(0) \approx 2.0606 \).
as explained above. The result is presented in Figure 12. The function $\mu_2 = \mu_2(\xi_2)$ has a root at $\xi_2 \approx -0.85$, which implies that the problem (8.11) has a solution, called $u_3(x)$, with the second harmonic $\xi_2 \approx -0.85$. This solution is sign-changing with exactly one root inside $(0, \pi)$, so that it is different from $u_1(x)$ and $u_2(x)$. From the solution $u_3(x)$ we find $u'_3(0) \approx -1.222$ The root $\xi_2 = 0$ again corresponds to the trivial solution of (8.11). The root at $\xi_2 \approx 0.85$ gives us the fourth non-trivial solution of (8.11), $u_4(x)$. From the solution $u_4(x)$ we find $u'_4(0) \approx 1.222$. In fact, $u_4(x) = u_3(\pi - x)$. (In general, if $u(x)$ is a solution of (8.11), so is $u(\pi - x)$.)

When doing continuation in $\xi_2$, one discovers that there is at least one more solution curve, in addition to the one in Figure 12. It is the curve joining $u_1(x)$ and $u_2(x)$. To “jump” on the solution curve in Figure 12, we began Newton’s iterations at each $\xi_2$ with $u_0(x) = \sin 2x$, a function that resembles the solutions $u_3(x)$ and $u_4(x)$.

As we mentioned in the Introduction, $\xi_1$ is a global parameter for the
problem (here $\int_0^{\pi} e(x) \sin x \, dx = 0$

$$u'' + f(u) = \mu_1 \sin x + e(x), \quad 0 < x < \pi, \quad u(0) = u(\pi) = 0,$$

uniquely determining the solution pair $(\mu_1, u(x))$, provided that $f'(u) < \lambda_2 = 4$, for all $u \in \mathbb{R}$. The problem (8.12) illustrates that this condition cannot be dropped, because (8.12) has at least two solution curves $\mu_1 = \mu_1(\xi_1)$ (since the solutions $u_3(x)$ and $u_4(x)$ do not lie on the solution curve in Figure 11).

9 Some open problems

Our computations raise a number of open questions, both of theoretical and computational nature (some were already mentioned above).

Equations with supercritical nonlinearities present both theoretical and computational challenges, which we explain on the example of the equation due to C.S. Lin and W.-M. Ni [20]

$$(9.14) \quad u'' + \frac{n-1}{r} u' + \lambda \left( u^q + u^{2q-1} \right) = 0, \quad u'(0) = u(1) = 0,$$

with $\frac{n}{n-2} < q < \frac{n+2}{n-2} < 2q - 1$. Here $\frac{n+2}{n-2}$ is the critical exponent for the space dimension $n$. When shooting, solutions of supercritical equations may fail to achieve the root of either $u(r)$ or $u'(r)$ (leading to the ground state solutions), therefore computations were terminated only when the negative values of $u(r)$ were achieved. Moreover, one can have several solution curves involving vastly different scales. We solved numerically the problem (9.14) with $n = 3$ and $q = 4$. The problem has two parabola-like curves opening to the right in the $(\lambda, u(0))$ plane, one above the other, given in Figure 13 and Figure 14, and an even higher third curve given in Figure 15, at much larger values of $\lambda$. All three curves exhibit horizontal asymptotes. The horizontal asymptotes are not possible for subcritical $f(u)$, with $f(u) > 0$, see T. Ouyang and J. Shi [21]. A very challenging open problem is to show that there is exactly one turn on the lower two curves of (9.14).

When computing the curves of positive solutions of non-autonomous problems

$$u'' + \frac{n-1}{r} u' + \lambda f(r, u) = 0, \quad u'(0) = u(1) = 0,$$

it is desirable to know under what conditions the value of $u(0)$ is a global parameter, as we saw above. In case $n = 1$, $u(0)$ is known to be a global
parameter, assuming that the condition $f_r(r, u) \leq 0$ holds, see P. Korman [13], or P. Korman and J. Shi [19]. We conjecture that the same result holds in case $n > 1$ too. It would be desirable to find other conditions for this result to hold, and some counter-examples.

Similarly, for positive solutions of the problem

$$u'' + \frac{n-1}{r} u' + \lambda f(u) + g(u) = 0, \quad u'(0) = u(1) = 0$$

it would be desirable to find conditions guaranteeing that $u(0)$ is a global parameter. Then solution curves can be computed by a method similar to the one we used for non-autonomous problems. In case $f(u)$ and $g(u)$ are pure powers, one can rescale the problem, putting the parameter in front of the nonlinearity, and apply the shoot-and-scale method.

For positive solutions of the elastic beam equation, clamped at the end
points,
\[ u'''(x) = \lambda f(u(x)), \quad \text{for} \quad x \in (-1, 1) \]
\[ u(-1) = u'(-1) = u(1) = u'(1) = 0, \]
it would be interesting to study further under what conditions the value of \( u(0) \), the maximal value of solution, is a global parameter. Can one drop the conditions (7.2)? Are there any counterexamples? It appears to be a very challenging research direction to prove the exact multiplicity results suggested by our computations. For example, is the solution curve for the problem (7.11) exactly \( S \)-shaped? The time map method does not apply here, and the bifurcation approach is not sufficiently developed.

Another case of a global parameter occurs for positive solutions \( u = u(x, y) \) of the problem
\[ u_{xx} + u_{yy} + \lambda f(u) = 0 \quad \text{for} \quad (x, y) \in D, \quad u = 0 \quad \text{on} \quad \partial D, \]
where \( D \) is a class of symmetric domains in the \((x, y)\)-plane, which in particular includes ellipses (see [6], or [12]). By the results of B. Gidas, W.-M. Ni and L. Nirenberg [5] any positive solution is symmetric with respect to \( x \) and \( y \), so that \( u(0, 0) \) gives the maximum value of any positive solution. M. Holzmann and H. Kielhofer [6] showed that \( u(0, 0) \) is in fact a global parameter, uniquely identifying the solution pair \((\lambda, u(x, y))\). It would be interesting to compute the global solution curves in the \((\lambda, u(0, 0))\)-plane.

In a recent paper [17], we provided both theoretical and computational results for first order equations, periodic in time, with the approach similar to that of Section 8. A number of open questions were raised in that paper.
References


