INFERENCES FROM INDIFFERENCE-ZONE SELECTION PROCEDURES

by

E. Jack Chen and W. David Kelton

BASF Corporation

University of Cincinnati

Outline:

- **Introduction**
- **Statistical Analysis**
- **Experimental Results**
- **Conclusions**

Indifference-Zone Selection:

Let μ_{i_l} be the l^{th} smallest of the μ_i's, so that $\mu_{i_1} \leq \mu_{i_2} \leq \ldots \leq \mu_{i_k}$.

Let P(CS) denote the probability of correct selection, i.e., design i_1 is selected.

Want P(CS) $\geq P^*$ provided that $\mu_{i_2} - \mu_{i_1} \geq d^*$, where the minimal CS probability P^* and the "indifference" amount d^* are both specified by the user.

Indifference to designs whose $\mu_i - \mu_{i_1} < d^*$.

The Two-Stage Rinott Procedure:

Compute the first-stage sample means

$$\bar{X}_i(n_0) = \sum_{j=1}^{n_0} X_{ij}/n_0,$$

and marginal sample variances

$$S_i^2(n_0) = \frac{\sum_{j=1}^{n_0}(X_{ij} - \bar{X}_i(n_0))^2}{n_0 - 1}.$$

The number of additional simulation replications for each design in the second stage is $N_i - n_0$, where

$$N_i = \max(n_0, \lceil (hS_i(n_0)/d^*)^2 \rceil), \qquad (1)$$

where h (depends on k, P^*, and n_0) is a constant that solves Rinott's integral.

The Two-Stage Rinott Procedure:

Overall sample means

$$\bar{X}_i(N_i) = \sum_{j=1}^{N_i} X_{ij}/N_i.$$

Select the design with the smallest $\bar{X}_i(N_i)$ **as the best one.**

Based on the *least favorable configurations* (LFC):

$$\mu_{i_l} = \mu_{i_1} + d^* \textbf{ for } l = 2, 3, \ldots, k.$$

The number of replications is allocated proportionally to the estimated sample variances.

Statistical Analysis:

- **Multiple Comparisons with a Control.**
- **Multiple Comparisons with the Best.**
- **Techniques to improve the efficiency of R&S Procedures.**

Multiple Comparison with a Control:

For $l = 2, 3, \ldots, k$**, let**

$$d_{i_l} = \max(d^*, \mu_{i_l} - \mu_{i_1}),$$

$$Z_{i_l} = \frac{\bar{X}_{i_1} - (\bar{X}_{i_l} - d_{i_l})}{\sqrt{\sigma^2_{i_l}/N_{i_l} + \sigma^2_{i_1}/N_{i_1}}},$$

$$Q_{i_l} = \frac{h}{\sqrt{\sigma^2_{i_l}/S^2_{i_l}(n_0) + \sigma^2_{i_1}/S^2_{i_1}(n_0)}},$$

where $N_i \geq (h/d_i)^2 S_i^2(n_0)$**.**

$$\begin{aligned}
\mathbf{P(CS)} &= \mathbf{P}[\bar{X}_{i_1} < \bar{X}_{i_l},\ \textbf{for } l = 2, 3, \ldots, k] \\
&\geq \prod_{l=2}^{k} \mathbf{P}[\bar{X}_{i_1} < \bar{X}_{i_l}] \\
&\geq \prod_{l=2}^{k} \mathbf{P}[Z_{i_l} < Q_{i_l}] \\
&= E(\prod_{l=2}^{k} \mathbf{P}[Z_{i_l} < Q_{i_l} | S_1, S_2, \ldots, S_k]) \\
&= E(\Phi^{k-1}(Q_{i_l})) \\
&= P^*.
\end{aligned}$$

Multiple Comparison with a Control:

One tailed $1-\alpha$ CI half-width

$$w_{i_l,i_1} = z_{1-\alpha}\sqrt{\sigma_{i_l}^2/N_{i_l} + \sigma_{i_1}^2/N_{i_1}}.$$

Rinott's procedure is derived based on

$$\frac{d^*}{\sqrt{\sigma_{i_l}^2/N_{i_l} + \sigma_{i_1}^2/N_{i_1}}} \geq Q_{i_l},$$

i.e.,

$$d^* \geq Q_{i_l}\sqrt{\sigma_{i_l}^2/N_{i_l} + \sigma_{i_1}^2/N_{i_1}}.$$

Since the critical constant h ensures $E(\Phi^{k-1}(Q_{i_l})) = P^*$. After the constant h is assigned a numeric value, $\Phi(Q_{i_l}) = (P^*)^{1/(k-1)} = 1-\alpha$, thus, $Q_{i_l} = z_{1-\alpha}$. We then have the result

$$d^* \geq w_{i_l,i_1}.$$

Multiple Comparison with a Control:

The simultaneous upper one-tailed confidence intervals,

$$\mathbf{P}[\mu_i - \mu_{i_1} \in [0, \bar{X}_i - \bar{X}_{i_1} + d^*], \forall i] \geq P^*.$$

Let $\bar{X}_b = \min_{i=1}^{k} \bar{X}_i$.

$$\mathbf{P}[\mu_i - \mu_{i_1} \in [0, \bar{X}_i - \bar{X}_b + d^*]] \geq \mathbf{P}[\mu_i - \mu_{i_1} \in [0, \bar{X}_i - \bar{X}_{i_1} + d^*]].$$

$$\mathbf{P}[\mu_i - \mu_{i_1} \in [0, \bar{X}_i - \bar{X}_b + d^*], \forall i] \geq P^*.$$

Multiple Comparison with the Best:

Define the events

$$E = \{\mu_i - \mu_{i_1} \leq \bar{X}_i - \bar{X}_{i_1} + d^*, \forall i \neq i_1\},$$

$$E_L = \{\mu_i - \min_{j \neq i} \mu_j \geq (\bar{X}_i - \min_{j \neq i} \bar{X}_j - d^*)^-, \forall i\},$$

$$E_U = \{\mu_i - \min_{j \neq i} \mu_j \leq (\bar{X}_i - \min_{j \neq i} \bar{X}_j + d^*)^+, \forall i\},$$

$$E_T = \{\mu_i - \min_{j \neq i} \mu_j \in [(\bar{X}_i - \min_{j \neq i} \bar{X}_j - d^*)^-, (\bar{X}_i - \min_{j \neq i} \bar{X}_j + d^*)^+], \forall i\}.$$

$(z)^- = \min(0, z)$ **and** $(z)^+ = \max(0, z)$**.**

Edwards and Hsu (1983) show that if $\mathbf{P}[E] \geq P^*$ **and** $E \subset E_L \cap E_U$**, then** $\mathbf{P}[E_T] \geq P^*$.

These MCB CIs are the same as those established in Nelson and Matejcik (1995).

The Adjusted ETSS Procedure:

- **Under the LFC** $\mu_i - \mu_{i_1} = d^* \geq w_{ii_1}$ $\forall i \neq i_1$**, and**

  $\mathbf{P}[\bar{X}_i - \bar{X}_{i_1} \geq \mu_i - \mu_{i_1} - d^*] \geq (P^*)^{1/(k-1)}.$

- **ETSS attempts to obtain** $\mu_i - \mu_{i_1} = d_i \geq w_{ii_1}$ **and**

  $\mathbf{P}[\bar{X}_i - \bar{X}_{i_1} \geq \mu_i - \mu_{i_1} - d_i] \geq (P^*)^{1/(k-1)}.$

- **If** $\mu_i - \mu_{i_1} > d^*$**, then**

$$\mu_i - \mu_{i_1} - d^* > \mu_i - \mu_{i_1} - d_i = 0.$$

- **Let** $\bar{X}_b(n_0) = \min_{i=1}^k \bar{X}_i(n_0)$ **and** $U(\bar{X}_b(n_0))$ **be the one-tailed upper** P^* **confidence limit of** μ_b**.**

- **Let**

$$\hat{d}_i = \max(d^*, \bar{X}_i(n_0) - U(\bar{X}_b(n_0))).$$

- **For** $i = 1, 2, \ldots, k,$ **let**

$$N_i = \max(n_0, \lceil (hS_i(n_0)/\hat{d}_i)^2 \rceil). \qquad (2)$$

The Adjusted ETSS Procedure:

The simultaneous upper one-tailed confidence intervals,

$$\mathbf{P}[\mu_i-\mu_{i_1} \in [0, \max_{j\neq i}(\bar{X}_i-\bar{X}_j+w_{i,j})^+], \forall i] \geq P^*.$$

The simultaneous MCB CIs,

$$\mathbf{P}[\mu_i - \min_{j\neq i} \mu_j \in$$
$$[\max_{j\neq i}(\bar{X}_i - \bar{X}_j - w_{i,j})^-, \max_{j\neq i}(\bar{X}_i - \bar{X}_j + w_{i,j})^+], \forall i]$$
$$\geq P^*.$$

If we replace $w_{i,j}$ **by** $\max(d_i, d_j)$**, these CIs will still hold since** $w_{i,j} \leq \max(d_i, d_j)$**.**

Using CRNs:

$$\begin{aligned}
\mathbf{P(CS)} &= \mathbf{P}[\bar{X}_{i_1} < \bar{X}_{i_l}, \ \textbf{for } l = 2, 3, \ldots, k] \\
&\geq \prod_{l=2}^{k} \mathbf{P}[\bar{X}_{i_1} < \bar{X}_{i_l}] \\
&\geq \prod_{l=2}^{k} \mathbf{P}[Z_{i_l} < Q_{i_l}] \\
&= E(\prod_{l=2}^{k} \mathbf{P}[Z_{i_l} < Q_{i_l} | S_1, S_2, \ldots, S_k]) \\
&= E(\prod_{l=2}^{k} \Phi(Q_{i_l})) \\
&= E(\Phi^{k-1}(Q_{i_l})) \\
&= P^*.
\end{aligned}$$

The first inequality no longer holds when CRNs are used.

Using CRNs:

- **By the *Bonferroni* inequality**
$$\mathbf{P}(\mathbf{CS}) = \mathbf{P}[\bar{X}_{i_1} < \bar{X}_{i_l}, \text{ for } l = 2, 3, \ldots, k]$$
$$\geq 1 - \sum_{l=2}^{k}(1 - \mathbf{P}[\bar{X}_{i_1} < \bar{X}_{i_l}]).$$
- **If we find the constant h so that**
$$\mathbf{P}[\bar{X}_{i_1} < \bar{X}_{i_l}] \geq P = (1 - \frac{1 - P^*}{k - 1}),$$
then $\mathbf{P}(\mathbf{CS}) \geq P^*$.
- **CRNs can be used to increase $\mathbf{P}[\bar{X}_{i_1} < \bar{X}_{i_l}]$ and $\mathbf{P}(\mathbf{CS})$ without any further assumptions.**
- **For example, if $k = 10$ and $P^* = 0.95$, use**
$$(1 - \frac{1 - P^*}{k - 1})^{k-1} = 0.951097$$
to find the constant h.
- **If h is obtained with $k = 10$ and $P^* = 0.95$, we state that $\mathbf{P}(\mathbf{CS}) \geq 0.948852$, i.e., $1 - (k - 1)(1 - (P^*)^{1/(k-1)})$.**

Proposition:

Perform all pairwise comparisons to eliminate inferior designs.

Let

$$P = 1 - (1 - P^*)/(k - 1)$$

and

$$w_{ij} = t_{P,r-1}\sqrt{S_i(r)^2/r + S_j(r)^2/r}.$$

If

$$\bar{X}_i > \bar{X}_j + w_{ij},$$

then we don't reject the null hypothesis that $\mu_i > \mu_j$ at confidence level P.

Probability of incorrectly eliminate design i_1 is no more than $1 - P$.

Sequentialize the selection procedure to avoid relying heavily on the first-stage information.

The Sequentialized ETSS Procedure:

1. **Initialize the set** I **to include all** k **designs. Simulate** $r = n_0$ **replications or batches for each design** $i \in I$**. Set the iteration number** $j = 0$**, and** $N_{1,j} = N_{2,j} = \ldots = N_{k,j} = n_0$**, where** $N_{i,j}$ **is the sample size allocated for design** i **at the** j^{th} **iteration. Let** $\bar{X}_{i,j}$ **denote the sample mean of design** i **at the** j^{th} **iteration.**
2. **Let** $\bar{X}_{b,j} = \min_{i \in I} \bar{X}_{i,j}$**. For all** $i \in I$**, compute** $\hat{d}_{i,j} = \max(d^*, \bar{X}_{i,j} - U(\bar{X}_{b,j}))$**, where** $U(\bar{X}_{b,j})$ **is the upper one-tailed** P^* **confidence limit of** μ_b **at the** j^{th} **iteration, and compute**
$$\delta_{i,j+1} = \lceil ((hS_i(r))/\hat{d}_{i,j})^2 - r)^+ \rceil.$$
3. **Set** $j = j + 1$ **and the incremental sample size at the** j^{th} **iteration** $\delta_j = \min_{i \in I}\{\delta_{i,j} | \delta_{i,j} > 0\}$**.**

4. **If $i \neq b$ and $\delta_{i,j} = 0$, delete design i from I.**
5. **Perform all pairwise comparisons and delete inferior design i from I.**
6. **For all $i \in I$, simulate additional δ_j samples and set $r = r + \delta_j$. If there is more than one element in I, go to step 2.**
7. **Return the values b and $\bar{X}_b$, where $\bar{X}_b = \min \bar{X}_i$, $1 \leq i \leq k$ and i was not eliminated by all pairwise comparisons.**

Experimental Results:

Increasing mean with equal variances. $X_{ij} \sim \mathcal{N}(i, 6^2)$**,** $i = 1, 2, \ldots, 10.$

LFC with equal variances. $X_{1j} \sim \mathcal{N}(0, 6^2)$ $X_{ij} \sim \mathcal{N}(1, 6^2)$**,** $i = 2, 3, \ldots, 10.$

The indifference amount d^* **is set to** 1.0**.**

Table 1: $\hat{P}$(CS) for Experiment 1

	$P^* = 0.90$		$P^* = 0.95$	
n_0	**20**	**30**	**20**	**30**
$\hat{P}$**(CS)**	**0.9866**	**0.9868**	**0.9956**	**0.9939**
PSC	**0.9390**	**0.9355**	**0.9674**	**0.9677**
PC2	**1.0000**	**1.0000**	**1.0000**	**1.0000**
PC3	**0.9886**	**0.9896**	**0.9941**	**0.9954**
PC4	**0.9912**	**0.9901**	**0.9953**	**0.9951**
PC5	**0.9886**	**0.9880**	**0.9942**	**0.9950**
PC6	**0.9892**	**0.9892**	**0.9957**	**0.9949**
PC7	**0.9912**	**0.9883**	**0.9931**	**0.9958**
PC8	**0.9876**	**0.9904**	**0.9948**	**0.9947**
PC9	**0.9886**	**0.9882**	**0.9945**	**0.9935**
PC10	**0.9899**	**0.9872**	**0.9948**	**0.9948**

PSC is the percentage of the simultaneous CIs contain the true value.

PC$l = \mathbf{P}[\mu_{i_l} - \mu_{i_1} \in [0, \bar{X}_{i_l} - \bar{X}_b + d^*]]$**, for** $l = 2, 3, \ldots, k$.

Table 2: $\hat{P}$(CS) for Experiment 2

	$P^* = 0.90$		$P^* = 0.95$	
n_0	**20**	**30**	**20**	**30**
$\hat{P}$**(CS)**	**0.9866**	**0.9868**	**0.9956**	**0.9939**
PP2	**0.9866**	**0.9868**	**0.9956**	**0.9939**
PP3	**1.0000**	**1.0000**	**1.0000**	**1.0000**
PP4	**1.0000**	**1.0000**	**1.0000**	**1.0000**
PP5	**1.0000**	**1.0000**	**1.0000**	**1.0000**
PP6	**1.0000**	**1.0000**	**1.0000**	**1.0000**
PP7	**1.0000**	**1.0000**	**1.0000**	**1.0000**
PP8	**1.0000**	**1.0000**	**1.0000**	**1.0000**
PP9	**1.0000**	**1.0000**	**1.0000**	**1.0000**
PP10	**1.0000**	**1.0000**	**1.0000**	**1.0000**

$\mathbf{PP}l = \mathbf{P}[\bar{X}_{i_l} > \bar{X}_{i_1}]$, **for** $l = 2, 3, \ldots, k$.

Table 3: $\hat{P}$(CS) and Sample Sizes for Experiment 3

Procedure	$P^* = 0.90$ $\hat{P}$(CS)	$\bar{T}$	$P^* = 0.95$ $\hat{P}$(CS)	$\bar{T}$
Rinott(20)	**0.9326**	**4259**	**0.9650**	**5412**
ETSS(20)	**0.6834**	**1820**	**0.7318**	**2317**
ETSS$_a$(20)	**0.8735**	**3347**	**0.9346**	**4640**
SARS(20)	**0.9529**	**3840**	**0.9800**	**5165**
SAMC(20)	**0.9363**	**2731**	**0.9705**	**3820**
Rinott(30)	**0.9320**	**4057**	**0.9655**	**5120**
ETSS(30)	**0.7662**	**2013**	**0.8029**	**2516**
ETSS$_a$(30)	**0.8976**	**3326**	**0.9530**	**4520**
SARS(30)	**0.9475**	**3655**	**0.9773**	**4872**
SAMC(30)	**0.9338**	**2730**	**0.9742**	**3705**

ETSS is the Enhanced Two-Stage Selection Procedure.
ETSS$_a$ is the adjusted ETSS.
SARS is the sequentialized ETSS.
SAMC is SARS with multiple comparisons.

Conclusions:

- **MCC and MCB CIs can be constructed by the outcomes of indifference-zone selection procedures.**
- **The CI half-with constructed by Rinott's procedure is d^*.**
- **The CI half-with constructed by ETSS and its variants is $\max(d_i, d_j)$.**
- **The tight CI half-with obtained by Rinott's procedure comes at a cost.**
- **Improve the efficiency by taking into account sample means.**
- **The sequentialized ETSS improve both efficiency and P(CS).**
- **Using CRNs can improve P(CS).**