Tricks of the trade: Normalization integral of a Gaussian

The integral of a Gaussian over a finite interval is called the error function

$$\mathrm{Erf}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{\pi}} e^{-y^2}\, dy.$$

It cannot be expressed in terms of simple functions and must be calculated numerically. However, the integral from minus infinity to infinity can be calculated. It is evaluated by the folllowing trick: the square of the integral is evaluated instead of the integral itself.

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2}\, dx \right)^2 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2}\, dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2}\, dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi} e^{-x^2} e^{-y^2}\, dx\, dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi} e^{-(x^2+y^2)}\, dx\, dy \\
&= \frac{1}{\pi} \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2}\, r dr\, d\theta \\
&= \frac{2\pi}{\pi} \int_{0}^{\infty} e^{-r^2}\, r dr \\
&= \int_{0}^{\infty} e^{-r^2}\, d(r^2) \\
&= \int_{0}^{\infty} e^{-\alpha}\, d\alpha \\
&= 1
\end{aligned}$$

$$\longrightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2}\, dx = \sqrt{1} = 1$$

The change in the first line can occur because x is just a place holder. The step from line 3 to line 4 is the change from Cartesian to polar coordinates. Line 4 to line 5 is the integation over θ.

All of the even moments, $\left\langle x^{2n} \right\rangle$, can be evaluated the same way. The odd moments are all zero because the Gaussian is symmetric.