Electric Field Transformation

When we started looking at special relativity, we looked at the force between two oppositely charge particles. In their rest frame, the force was (from Coulomb's Law):

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}$$

However, in a frame moving with a velocity v with respect to the rest frame, the force is a combination of Coulomb's Law and Ampere's Law:

$$\begin{aligned} F =& \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2} - \frac{\mu_0}{4\pi}\frac{(qv)^2}{r^2} \\ =& \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}\left(1 - \frac{v^2}{c^2}\right) \end{aligned}$$

Up until now, we have not explicitly looked at forces. Since $F = \frac{dp}{dt}$, transforming a force from one frame to another will be just as complicated as transforming a velocity. Furthermore the frame at rest with respect to the particle is not an inertial frame; it is accelerating with the particle. The best we can do is to pick two frames, one initially at rest with the particle and one in the laboratory, and ask that each consistently calculate the change in energy of the test particle.

Rather than look at two particles attacting each other, I am instead going to look at a particle going through an electric field generated by a parallel plate capacitor. This will simplify things because the electric field is at least constant between the plates. First let's look at the case when the field is perpendicular to the motion of the particle. I'm going to assume that the field is weak enough such that I can treat the change in perpendicular momentum non-relativistically. In that case, the impulse, $\triangle p_\perp \approx F\triangle t = q\mathcal{E}_\perp\frac{\ell}{\beta c}$ where $\mathcal{E}_\perp$ is the electric field between the plates ($F = q\mathcal{E}_\perp$) and ℓ is the length of the capacitor. When I transform to the frame in which the initial electron was at rest, then the final $p_\perp$ should remain the same (that is the component of momentum perpendicular to the direction of motion which does not change with Lorentz transformations). However, the length over which the field was present was reduced in the electron's rest frame by a factor of γ. Thus the field must increase by a factor of γ to keep the impulse the same.*

Thus, getting back to the example of the two moving charges, if we we compare impulses, in the frame where they are stationary, we get:

$$\triangle p_{\perp} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}\,\triangle t$$

In the moving system the this becomes:

$$\triangle p_{\perp} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}\,\gamma \times \frac{1}{\gamma^2} \times \gamma\triangle t$$

The first γ comes from the increase in the electric field strength, the $1/\gamma^2$ term from the addition of the magnetic force to the electric force in the moving frame, and the final γ from the time dilation factor. All of the γ's cancel out and the impulse is the same in both frames.

While the component of the electric field perpendicular to the direction of motion increases by a factor of γ, the component along the direction of motion doesn't change. Again, I use a capacitor to illustrate this.

Assume that there are two capacitor plates that are a distance d apart and that differ in potential by V.

An electron enters the region from the left with a total energy E_i. It gains $\triangle E = eV/d$ of energy (the particle is accelerated) going through the gap and exits

* Another way of arguing this is to say that the total charge of a system is relativistically invariant (this is necessary for charge to be conserved). Then, to satisfy Gauss's law, number of electric field lines is the same for all inertia frames. However, because of length contraction, the line density and hence the electric field must change.

with $E_f = E_i + eV$. Now:

$$
\begin{aligned}
E_i^2 =& p_i^2 c^2 + m_e^2 c^4 \\
E_f^2 = (E_i + \triangle E)^2 =& p_f^2 + m_e^2 c^4 = (p_i + \triangle p)^2 + m_e^2 c^4
\end{aligned}
$$

If the added energy is small compared to the total incoming energy, then:

$$
\begin{aligned}
\rightarrow 2E_i \triangle E \approx& 2p_i \triangle p \\
\triangle p \approx& \frac{p_i}{E_i} \triangle E = \beta \triangle E
\end{aligned}
$$

Transforming this back into the rest frame of the electron ($\beta = p_i/E_i$ and $\gamma = E_i/m_e$), we get

$$
\begin{aligned}
E_f^{rest} =& \gamma \left(E_f - \beta p_f c \right) \\
=& \gamma \left(E_i + \triangle E - \beta \left(p_i + \triangle p_i \right) \right) \\
=& \gamma \left(E_i - \beta p_i \right) + \gamma \left(\triangle E - \beta \triangle p \right) \\
\approx& E_i^{rest} + \gamma \left(1 - \beta^2 \right) \triangle E
\end{aligned}
$$

Thus

$$
\triangle E^{rest} = \frac{\triangle E^{lab}}{\gamma}
$$

In the lab, the electric field

$$
\mathcal{E}_{\|}^{lab} = \triangle E^{lab} / e d^{lab}.
$$

In the electron's rest frame, the electric field must be

$$
\mathcal{E}_{\|}^{rest} = \triangle E^{rest} / e d^{rest}.
$$

Because of length contraction, $d^{rest} = d^{lab}/\gamma$, and

$$
\mathcal{E}_{\|}^{rest} = \mathcal{E}_{\|}^{lab}.
$$

The electric field in the direction of motion does not change.