1. Differential Equations

1.1 Definition of Terms

A differential equation is an equation which includes variables, functions of variables, and any number of derivatives of the functions. Examples include:

1. Object moving with constant velocity:

$$\frac{dv(t)}{dt} = 0, \tag{1.1}$$

2. Equation for a driven RLC circuit:

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{Q(t)}{C} = V_0 \sin \omega t, \text{ and} \tag{1.2}$$

3. Equation for the drag on an object being proportional to the velocity squared:

$$m\frac{d^2x(t)}{dt^2} + b\left(\frac{dx(t)}{dt}\right)^2 = 0. \tag{1.3}$$

Solving a differential equation means find the function which satisfies it ($v(t)$ in (1.1), $Q(t)$ in (1.2), and $x(t)$ in (1.3)) and is consistent with the conditions placed on the function (such as the initial positions, velocities, charges, *etc.*).

"Homogeneous" differential equations are those in which every term in the equation contains the function to be solved for or one of its deriviatives. Equations (1.1) and (1.3) are homogeneous equations; equation (1.2) is not because the term $V_0 \sin \omega t$ does not contain the function $Q(t)$.

Homogeneous equations can be *linear* or *non-linear*. A linear differential equation is one in which if two different functions $f_1(t)$ and $f_2(t)$ both are solutions to the equation, then $\alpha f_1(t) + \beta f_2(t)$ is also a solution for all α and β. Equation (1.1) is an example of a linear differential equation; (1.3) is non-linear. Inhomogeneous equations like (1.2) can be broken up into a homogeneous part (the left side of (1.2)) that contains all the terms with the function in it and an inhomogeneous part (the right side of (1.2)) that contains everything else. An inhomogeneous equation is said to be linear if its homogeneous part is linear. As we will see later, finding the solution to the homogeneous part of an inhomogeneous differential equation is the first step in finding the complete solution.

While non-linear equations describe systems that are very important in the real world, they are usually difficult to solve and there is no generalized method for solving them.* In first year physics, we will see only linear differential equations.

The *order* of differential equation is given by the highest number of derivatives in any term (*i.e.* Eq. (1.1) is first order, Eqs. (1.2) and (1.3) are second order, *etc.*). A general feature of the solution of a differential equation is that it will contain a certain number (equal to its order) of "arbitrary" constants that will have to be matched to the conditions placed on the problem. For example, a problem which specifies the force on a particle must also specify two other aspects about the particle's trajectory (*e.g.* its inital position and velocity) in order to be solved.

1.2 Table of Solutions

As with integrals, one way to "solve" differential equations is to memorize their solutions. This is possible in freshman physics because the number of different differential equations is pretty small. All of the equations with their solutions are tabluated on the next page.

1.3 Exact Differential Equations – Solution by Integration

The solution to some first order differential equations can be found by direct integration. If the equation can be written in the form

$$g(f(t))\frac{df(t)}{dt} = h(t) \tag{1.4}$$

then the solution is:

$$\int_{f_0}^{f(t)} g(f)\ df = \int_0^t h(t)\ dt. \tag{1.5}$$

While Eq. (1.4) is hardly the most general form of a first order differential equation, a surprising number of the differential equations encountered in freshman physics can be solved in this manner.

Example 1: Free fall

⋆ Amazing, however, Eq. (1.3) can be solved. The solution is the pecular looking function $x(t) = \frac{m}{b}\ln\left(1 + \frac{b}{m}v_0 t\right) + x_0$. Try it!

Equation	Solution
$\frac{df}{dt} = c$	$f(t) = f_0 + ct$
$\frac{d^2 f}{dt^2} = c$	$f(t) = f_0 + f_0' t + \frac{1}{2} c t^2$
$\frac{df}{dt} + \frac{f}{\tau} = c$	$f(t) = \tau c + (f_0 - \tau c) e^{-t/\tau}$
$\frac{d^2 f}{dt^2} + \omega^2 f = c$	$f(t) = \left(f_0 - \frac{c}{\omega^2}\right) \cos \omega t + \frac{f_0'}{\omega} \sin \omega t + \frac{c}{\omega^2}$
$\frac{d^2 f_H}{dt^2} + \Gamma \frac{df_H}{dt} + \omega^2 f_H = 0$	$f_H(t) = \begin{cases} (A \sin \omega' t + B \cos \omega' t)\, e^{-\frac{\Gamma t}{2}}, & \text{if } 4\omega^2 > \Gamma^2 \\ \quad \text{where } \omega' = \sqrt{\omega^2 - \frac{\Gamma^2}{4}} & \\ (A + Bt)\, e^{-\frac{\Gamma t}{2}}, & \text{if } 4\omega^2 = \Gamma^2 \\ \left(A e^{\gamma t} + B e^{-\gamma t}\right) e^{-\frac{\Gamma t}{2}}, & \text{if } 4\omega^2 < \Gamma^2 \\ \quad \text{where } \gamma = \sqrt{\frac{\Gamma^2}{4} - \omega^2} & \end{cases}$
$\frac{d^2 f}{dt^2} + \Gamma \frac{df}{dt} + \omega^2 f = c$	$f(t) = f_H(t) + \frac{c}{\omega^2}$
$\frac{d^2 f}{dt^2} + \Gamma \frac{df}{dt} + \omega^2 f = F \sin \omega_D t$	$f(t) = f_H(t) - \frac{F}{\omega_D Z} \cos(\omega_D t - \phi)$ where $Z = \sqrt{\Gamma^2 + \omega_D^2 \left[\frac{\omega^2}{\omega_D^2} - 1\right]^2}$ and $\phi = \frac{\omega^2 - \omega_D^2}{\Gamma \omega_D}$.

The equation for free-fall can be written as

$$\frac{dv(t)}{dt} = g \tag{1.6}$$

which is of the form given in equation (1.4) [if we take $g(v) = 1$ and $h(t) = g$], and, therefore, can be solved by direct integration.

$$\begin{aligned} \int_0^t \frac{dv}{dt} dt &= \int_{v_0}^{v(t)} dv = v(t) - v_0 \\ &= \int_0^t g \ dt = gt \end{aligned}$$

Therefore

$$v(t) = v_0 + gt.$$

The position can also be found by direct integration. Since $\frac{dx}{dt} = v(t) = v_0 + gt$,

$$\begin{aligned} \int_0^t \frac{dx}{dt} dt &= \int_{x_0}^{x(t)} dx = x(t) - x_0 \\ &= \int_0^t v_0 + gt \ dt = v_0 t + \frac{1}{2} g t^2 \end{aligned}$$

giving

$$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2.$$

Example 2: The RC Circuit

The equation for an RC circuit

$$R \frac{dQ}{dt} + \frac{Q}{C} = 0$$

can be rewritten as

$$\frac{1}{Q(t)} \frac{dQ}{dt} = -\frac{1}{RC}.$$

This can be directly integrated.

$$\int_0^t \frac{1}{Q}\frac{dQ}{dt}\,dt = \int_{Q_0}^{Q(t)} \frac{dQ}{Q} = \ln\frac{Q(t)}{Q_0}$$
$$= \int_0^t -\frac{1}{RC}\,dt = -\frac{t}{RC}.$$

Therefore,

$$\ln\frac{Q(t)}{Q_0} = -\frac{t}{RC}$$

and

$$Q(t) = Q_0 e^{-\frac{t}{RC}}.$$

Example 3: Solving Equation (1.3)

Non-linear equation (1.3) was solved by direct integration. That equation can be re-written in terms of velocity instead of position as

$$\frac{1}{v^2}\frac{dv}{dt} = -\frac{b}{m}.$$

This can then be integrated as

$$\int_0^t \frac{1}{v_2}\frac{dv}{dt}\,dt = \int_{v_0}^{v(t)} \frac{1}{v^2}\,dv = -\frac{1}{v}\bigg|_{v_0}^{v(t)} = \frac{1}{v_0} - \frac{1}{v(t)}$$
$$= -\int_0^t \frac{b}{m}\,dt = -\frac{bt}{m}.$$

Therefore

$$\frac{1}{v(t)} = \frac{1}{v_0} + \frac{bt}{m}$$

and

$$v(t) = \frac{v_0}{1 + \frac{b}{m}v_0 t}.$$

The function for $x(t)$ given in the footnote at the beginning of this chapter again comes from straight integration of the function for $v(t)$.

1.4 Linear Homogeneous Equations with Constant Coefficients

Each term in a linear differential equation will have one and only one dependence on the function. If we group the functions in order of the number of deriviatives we will get a series that looks something like:

$$\alpha_n(t)\frac{d^n f(t)}{dt^n} + \alpha_{n-1}(t)\frac{d^{n-1} f(t)}{dt^{n-1}} + \cdots + \alpha_1(t)\frac{df(t)}{dt} + \alpha_0(t)f(t) = 0 \tag{1.7}$$

Usually we are dealing with constant coefficients ($\alpha_i(t) = \alpha_i$). The general solution that follows assumes that we are. If I replace each $\frac{d}{dt}$ by a D and make the definition that $D^n = \frac{d^n}{dt^n}$, I can re-write (1.7) as

$$\alpha_n D^n f + \alpha_{n-1} D^{n-1} f + \cdots + \alpha_1 D f + \alpha_0 f = 0 \tag{1.8}$$

or

$$(\alpha_n D^n + \alpha_{n-1} D^{n-1} + \cdots + \alpha_1 D + \alpha_0) f = 0.$$

Here the D's are operators, operating on (*i.e.* taking the derivatives of) what comes after the paranthesis. The characteristic polynomial of (1.7) is defined as

$$\alpha_n D^n + \alpha_{n-1} D^{n-1} + \cdots + \alpha_1 D + \alpha_0 = 0 \tag{1.9}$$

where the D's are now just algebraic variables. Since the characteristic equation is an n^{th} order polynomial, there are n roots to it. For the roots that are real and distinct (λ), a solution to the differential equation is

$$f(t) = Ae^{\lambda t}. \tag{1.10}$$

The complex roots for (1.9) come in pairs, $\lambda_+ = \lambda_r + i\lambda_i$ and $\lambda_- = \lambda_r - i\lambda_i$. If you allow complex functions as the solutions to the differential equations, then (1.10) is the solution for the complex roots as well. If you require real functions as the solutions, then each complex pair of roots gives a solution to the differential equation of the form

$$f(t) = Ae^{\lambda_r t}\sin\lambda_i t + Be^{\lambda_r t}\cos\lambda_i t.$$

Finally, if the roots of (1.9) are not distinct, (*i.e.* the charastic polynomial has a factor of $(D-\lambda)^n$ in it where $n > 1$), then there is a solution of the form

$$f(t) = \left(\sum_{i=0}^{n-1} R_i t^i\right) e^{\lambda t}.$$

Therefore, the most general solution to (1.8) (assuming no redundant complex roots) is

$$f(t) = \sum_{\substack{i=\text{distinct}\\ \text{real}\\ \text{roots}}} C_i e^{\lambda_i t} + \sum_{\substack{i=\text{complex}\\ \text{pairs}}} [A_i \sin \lambda_i^i t + B_i \cos \lambda_i^i t] e^{\lambda_r^i t} +$$

$$\sum_{\substack{i=\text{redundant}\\ \text{roots of}\\ \text{order n}}} \left(\sum_{j=0}^{n-1} R_{ij} t^j \right) e^{\lambda_i t}$$

where the coefficients A_i, B_i, C_i, and R_{ij} are arbitrary constants and have to be determined by the boundary conditions or the initial conditions of the problem. The fact that equation (1.7) is an n^{th} order differential equation implies that there must be n arbitrary constants in the solution. In the solution given above, there is one arbitrary constant per root and therefore n constants total.

Example 4: Newton's First Law – Velocities

Newton's first law, "What is in motion stays in motion; what is at rest, stays at rest unless acted on by an external force," can be expressed for velocities in the absence of an external force as

$$\frac{dv(t)}{dt} = 0. \tag{1.11}$$

The characteristic polynomial for (1.11) is $D = 0$ with the root being $\lambda = 0$. Therefore the solution is

$$v(t) = Ae^{0t} = A$$

or that v is a constant. If the initial velocity is given, the $A = v_0$ and $v(t) = v_0$.

Example 5: An RC circuit – Again

The equation for the charge in an RC circuit is

$$R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

which has the characteristic equation is $RD + \frac{1}{C} = 0$ with the root of $\lambda = -\frac{1}{RC}$. Therefore the solution to the equation is

$$Q(t) = Q_0 e^{-\frac{t}{RC}}$$

where Q_0 is determined by the initial charge on the capacitor.

Example 6: Newton's First Law for Position – Redundant Roots

In the absence of an external force, the equation for the position of a particle is

$$\frac{d^2x}{dt^2} = 0.$$

This is a simple equation with redundant roots. The characteristic equation is $D^2 = 0$ which has a second order root of 0. Therefore the solution is

$$x(t) = (A_0 + A_1 t)\, e^{0t} = A_0 + A_1 t.$$

We can determine A_0 from the initial position, x_0, and A_1 from the initial velocity, v_0, giving

$$x(t) = x_0 + v_0 t.$$

Example 7: The Harmonic Oscillator or LC Circuit

The equation of motion for a harmonic oscillator is

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{1.12}$$

The equation for an LC circuit is the same except with different coefficients (for the LC circuit, the equation is $L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$). The characteristic equation for (1.12) is

$$mD^2 + k = 0$$

which has two purely imaginary roots, $\lambda = \pm i\sqrt{\frac{k}{m}}$. Since $\lambda_r = 0$ for these roots, the solution to the equation is

$$\begin{aligned} x(t) &= \left[A \sin\sqrt{\frac{k}{m}}t + B\cos\sqrt{\frac{k}{m}}t\right] e^{0t} \\ &= A\sin\sqrt{\frac{k}{m}}t + B\cos\sqrt{\frac{k}{m}}t \end{aligned}$$

where A and B are determined from the conditions placed on the problem.

Example 8: Damped Harmonic Oscillator or RLC Circuit

The equation for the damped harmonic oscillator is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{1.13}$$

and for the RLC circuit is

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0.$$

The characteristic equation for (1.13) is

$$mD^2 + bD + k = 0,$$

(the equation for the RLC circuit is the same with different coefficients) which has roots determined by the quadratic formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

There are three types of solutions depending on the value of $b^2 - 4mk$. If $b^2 - 4mk > 0$, the roots are real and

$$x(t) = Ae^{\left(\frac{-b+\sqrt{b^2-4mk}}{2m}\right)t} + Be^{\left(\frac{-b-\sqrt{b^2-4mk}}{2m}\right)t}.$$

This is the over damped case. If $b^2 - 4mk = 0$, then the roots are redundant and

$$x(t) = (A + Bt)\, e^{-\frac{b}{2m}t}.$$

This case is called the critically damped case. Finally, for $b^2 - 4mk < 0$, the system oscillates and the amplitudes of the oscillation are slowly damped out. If

$$\omega' = \frac{\sqrt{4mk - b^2}}{2m} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega^2 - \frac{b^2}{4m^2}}$$

(where $\omega = \sqrt{k/m}$ is the angular frequency of the undamped oscillator), the solution is

$$x(t) = \left[A\sin\omega' t + B\cos\omega' t\right] e^{-\frac{b}{2m}t}.$$

1.5 Inhomogeneous Equations

There are three steps in solving inhomogeneous equations: (1) isolate the homogeneous part of the equation and solve it (this step is called "finding the general solution"); (2) find some function (usually by some lucky guess or from your physics intuition about the problem at hand) that satisfies the full inhomogeneous equation (this step is called "finding the particular solution"); and finally (3) adding the two solutions together and satisfying the boundary conditions.

<u>Example 9</u>: Free Fall – Again

The equation for a body falling under the influence of gravity is

$$\frac{d^2x}{dt^2} = g. \tag{1.14}$$

The homogeneous part of this equation ($d^2x/dt^2 = 0$) was solved in Example 6 giving the general solution of $x(t) = x_0 + v_0 t$. Now, we have to guess some function that will satisfy (1.14). Since there is a second deriviative in (1.14), $x_s = \alpha t^2$ might work for some α. Plugging this guess into (1.14) gives $2\alpha = g$. Therefore, if $\alpha = g/2$, we have a solution. Our particular solution is

$$x_s = \frac{g}{2}t^2 = \frac{1}{2}gt^2$$

and the complete solution is the sum of the general solution and the particular solution,

$$x(t) = x_0 + v_0 t + \frac{1}{2}gt^2.$$

<u>Example 10</u>: LR circuit with a Battery

The equation for an LR circuit connected to a DC voltage source is

$$L\frac{dI}{dt} + IR = V.$$

The general solution is the same as that for the RC circuit given in Example 5 except the coefficients are different. The solution is

$$I(t) = I_0 e^{-\frac{R}{L}t}.$$

Our guess for the specific solution comes from our knowledge of what happen in the circuit a long time after the battery has been turned on. We would expect all of the change in the current to die away and that I should go to some constant. Therefore, trying $I = k$ in the equation, we find that, if $k = \frac{V}{R}$, our guess works. Therefore, the total solution to the equation becomes

$$I(t) = I_0 e^{-\frac{R}{L}t} + \frac{V}{R}$$

where I_0 is determined from the initial conditions. [Notice that if at $t = 0$, $I(t) = 0$, then $I_0 = -\frac{V}{R}$ and not zero.]

Example 11: Driven RLC Circuits

The equation for a driven RLC circuit is

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{Q(t)}{C} = V_0 \sin \omega_D t \tag{1.15}$$

where V_0 is the amplitude of the AC voltage generator and ω_D is the driving angular frequency. Again, the general solution to this equation is given in Example 8. The general solution is damped and will die away with time. Therefore, to get the particular solution we should try functions that we would expect to have the behavior of the circuit a long time after the generator has been turned on. From AC circuit theory, we would expect the current to oscillate with an angular frequency of ω_D just like the driving voltage but, possibly, to be offset in phase from the driving voltage. If the current is oscillating with an angular frequency of ω_D, then the charge must be oscillating with the same angular frequency. We should therefore try a particular solution of the form

$$Q_p = A \sin \omega_D t + B \cos \omega_D t. \tag{1.16}$$

(We could have tried just one of the terms in the above equation, but would have found out that just one term did not give us enough flexibility to find a particular solution.) Plugging (1.16) into (1.15) gives

$$-\omega_D^2 L \left[A \sin \omega_D t + B \cos \omega_D t\right] + \omega_D R \left[A \cos \omega_D t - B \sin \omega_D t\right] + \\ \frac{1}{C}\left[A \sin \omega_D t + B \cos \omega_D t\right] = V_0 \sin \omega_D t.$$

This equation will be satisfied for all t if the coefficients of the sine terms and the coefficients of the cosine terms vanish. This gives us two simultaneous equations for A and B

$$\begin{aligned} -\omega_D^2 LA - \omega_D RB + \frac{A}{C} &= V_0 \\ -\omega_D^2 LB + \omega_D RA + \frac{B}{C} &= 0 \end{aligned}$$

Solving these two equations for A and B gives (try it!)

$$\begin{aligned} A &= -\frac{V_0}{\omega_D} \frac{\left[\frac{1}{\omega_D C} - \omega_D L\right]}{R^2 + \left[\frac{1}{\omega_D C} - \omega_D L\right]^2} \\ B &= -\frac{V_0}{\omega_D} \frac{R}{R^2 + \left[\frac{1}{\omega_D C} - \omega_D L\right]^2} \end{aligned}$$

Again, taking our cue from AC circuit theory, we define

$$\begin{aligned} Z &= \sqrt{R^2 + \left[\frac{1}{\omega_D C} - \omega_D L\right]^2} \text{ and} \\ \tan\varphi &= \frac{\frac{1}{\omega_D C} - \omega_D L}{R} \end{aligned}$$

and we get

$$\begin{aligned} A &= -\frac{V_0}{\omega_D Z} \sin\varphi, \text{ and} \\ B &= -\frac{V_0}{\omega_D Z} \cos\varphi. \end{aligned}$$

Substituting these values back into (1.16) gives

$$\begin{aligned} Q_p &= -\frac{V_0}{\omega_D Z} \sin\varphi \sin\omega_D t - \frac{V_0}{\omega_D Z} \cos\varphi \cos\omega_D t \\ &= -\frac{V_0}{\omega_D Z} \cos(\omega_D t - \varphi) \end{aligned}$$

Differentiating this equation with respect to time gives

$$I_p = \frac{V_0}{Z} \sin(\omega_D t - \varphi)$$

which is the familiar result from AC circuit theory. Finally, the complete solution is the sum of the general solution given in Example 8 and the specific solution given above. For example, if the system is underdamped,

$$Q(t) = \left[A \sin \omega' t + B \cos \omega' t\right] e^{-\frac{R}{2L}t} - \frac{V_0}{\omega_D Z} \cos(\omega_D t - \varphi).$$

This solution allows us to match the intial conditions of the charges and the currents.