1. Complex Numbers

For real numbers, the square root of -1 is undefined. We can, however, set up a number system (called the complex numbers) that associates a value to $\sqrt{-1}$. These numbers will behave, in a way, like two component vectors with the first component being the "real" part of the number (and behaving like real numbers) and the second component being the "imaginary" portion and being proportional to $\sqrt{-1}$. $\sqrt{-1}$ is usually written as i, so

$$i^2 = -1.$$

A complex number is written as

$$\mathbf{A} = a_r + ia_i$$

where a_r is the real part of the number and ia_i is the imaginary part. a_r and a_i are both real numbers.

The rule for adding two complex numbers is

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= (a_r + ia_i) + (b_r + ib_i)\\ &= (a_r + b_r) + i(a_i + b_i)\end{aligned}$$

i.e. the real part is the sum of the real parts and the imaginary part is the sum of the imaginary parts. This operation has all of the properties associated with addition of real numbers. It is associative ($(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$), is commutative ($\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$), has an identity element ($\mathbf{0} = 0 + i0$), and has an inverse ($-\mathbf{A} = -a_r - ia_i$).

Multiplication of two complex numbers is defined as follows:

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= (a_r + ia_i) \cdot (b_r + ib_i)\\ &= a_r b_r + ia_r b_i + ia_i b_r + (i)^2 a_i b_i\\ &= (a_r b_r - a_i b_i) + i(a_r b_i + a_i b_r).\end{aligned}$$

Multiplication is also associative ($(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$), commutative ($\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$), has an identity element ($\mathbf{1} = 1 + i0$), and has an inverse for all elements except the additive identity:

$$\begin{aligned}
\mathbf{A}^{-1} &= \frac{1}{a_r + ia_i} \\
&= \frac{1}{a_r + ia_i} \cdot \frac{a_r - ia_i}{a_r - ia_i} \\
&= \frac{a_r - ia_i}{a_r^2 - i^2 a_i^2} \\
&= \frac{a_r - ia_i}{a_r^2 + a_i^2} \\
&= \frac{a_r}{a_r^2 + a_i^2} - i\frac{a_i}{a_r^2 + a_i^2}.
\end{aligned}$$

Addition and multiplication are also distributive ($\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$).

Notice that, with these definitions of multiplication and addition, the real numbers are the subset of the complex numbers with zero imaginary part.

Complex functions are mappings from complex numbers to complex numbers, $f(z) = f_r(z) + f_i(z)$ where z is a complex number. Most real functions have complex analogs. The complex analog of a real function that can be expanded in a Taylor expansionaround zero is merely the function with the same expansion except with the complex parameter z replacing the real parameter x. Therefore, if the real function has the Taylor expansion

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

then the analogous complex function is given by the expansion

$$f(z) = \sum_{i=0}^{\infty} a_i z^i.$$

A complex function of special note is the exponential function $e^{(a_r + ia_i)} = e^{a_r} \cdot e^{ia_i}$. The power series expansion works out such that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This is true whether θ is real or complex. The complex sine and cosine functions still have all the usual trigometric identities associated with them. Of special use are $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. Combining this with the exponential of an imaginary number gives

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

An n^{th} order polynomial

$$c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$$

will have exactly n complex roots (numbers which satisfy the equation of the polynomial equals zero). If all of the coefficients, c_n, are real, then the roots that have imaginary parts come in pairs. In this case, if $x_r + ix_i$ is a root, then $x_r - ix_i$ is also a root.