1. Approximations for Small Numbers

1.1 The Taylor Expansion

If you know both the value of a function and that of its deriviatives at some point x, then the value of the function at a nearby point, $x + \delta x$, is given by the Taylor expansion:

$$f(x+\delta x) = f(x) + \left.\frac{df}{dx}\right|_x \delta x + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_x (\delta x)^2 + \cdots + \frac{1}{n!}\left.\frac{d^n f}{dx^n}\right|_x (\delta x)^n + \cdots \quad (1.1)$$

assuming that the function and all the deriviatives are continuous between the two points. In (1.1) $\left.\frac{d^n f}{dx^n}\right|_x$ is the n^{th} deriviative of the function $f(x)$ evaluated at the point x. If δx is small enough, then a good approximation for $f(x+\delta x)$ is to just use the first two terms of (1.1)

$$f(x+\delta x) \approx f(x) + \left.\frac{df}{dx}\right|_x \delta x$$

This is called the "First Order Approximation" meaning that terms up to first order in δx are kept. [The "Second Order Approximation" keeps terms up to δx^2, *etc.*].

"Small enough" is a difficult term to define quatitatively. No matter what rule we devise, we can always find some function that will disprove it. Therefore, what we say here works for your "average" function (ones that you will normally encounter) but not for all (*i.e.* the wierd ones that the mathematician think up) functions. First of all, small is relative. When we say "small enough" we have to know what is being compared. A millimeter is small compared to the length of a football field but not compared to the diameter of an atom. Therefore, a good rule is to make your expansion parameters unitless quatities, and then, small is always compared to 1. Therefore, the Taylor expansion given above should be written (for our purposes) as:

$$f(\delta) = f(0) + \left.\frac{df}{d\delta}\right|_{\delta=0} \delta + \frac{1}{2!}\left.\frac{d^2 f}{d\delta^2}\right|_{\delta=0} \delta^2 + \cdots + \frac{1}{n!}\left.\frac{d^n f}{d\delta^n}\right|_{\delta=0} \delta^n + \cdots$$

with $\delta = \frac{\delta x}{x}$. The first order expansion is

$$f(\delta) = f(0) + \left.\frac{df}{d\delta}\right|_{\delta=0} \delta. \quad (1.2)$$

Using approximation (1.2)

$$
\begin{aligned}
&\sin\theta \approx \theta \\
&\tan\theta \approx \theta \\
&\ln(1+x) \approx x \\
&(1\pm x)^n \approx 1 \pm nx \\
&e^{\pm x} \approx 1 \pm x \\
&\sqrt{1\pm x} \approx 1 \pm \frac{1}{2}x \\
&\frac{1}{1\pm x} \approx 1 \mp x \\
&\frac{1}{\sqrt{1\pm x}} \approx 1 \mp \frac{1}{2}x
\end{aligned}
$$

These are valid for $\theta, x \ll 1$. Notice that in these approximations, x is taken to be a unitless quantity and θ is measured in radian, which is also unitless.

There are times when you have to go to second order. The following functions are expanded to second order

$$
\begin{aligned}
&\cos\theta \approx 1 - \frac{1}{2}\theta^2 \\
&\sqrt{1\pm x^2} \approx 1 \pm \frac{1}{2}x^2 \\
&\frac{1}{\sqrt{1\pm x^2}} \approx 1 \mp \frac{1}{2}x^2
\end{aligned}
$$

In each of these functions, the first order term is zero because $\left.\frac{df}{dx}\right|_{x=0} = 0$.

Another time when you have to include the second order terms is when you are adding a group of functions and all of the first order terms cancel out. [See, for example, the quadrapole calculation given below.]

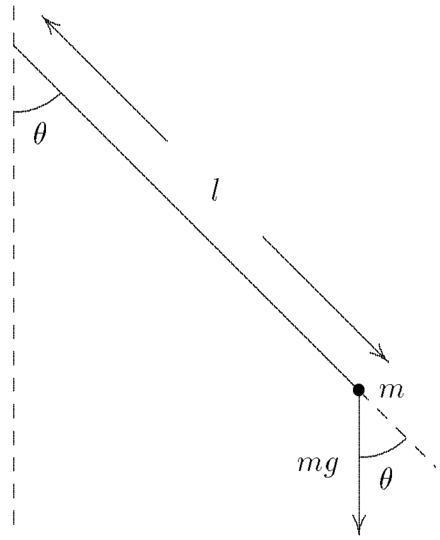
1.2 Commonly Used Expansions

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \\
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \\
(1 \pm x)^n &= 1 \pm nx + \frac{n(n-1)x^2}{2!} \pm \frac{n(n-1)(n-2)x^3}{3!} + \cdots \\
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \\
\frac{1}{1 \pm x} &= 1 \mp x + x^2 \mp x^3 + x^4 \mp \cdots \\
\frac{1}{(1 \pm x)^2} &= 1 \mp 2x + 3x^2 \mp 4x^3 + \cdots
\end{aligned}
$$

1.3 Examples of Small Number Approximations

1.3.1 Pendulum

The torque on a pendulum, $\vec{\tau} = I\frac{d^2\vec{\theta}}{dt^2}$, is $\vec{\tau} = \vec{l} \times \vec{F}$. For the pendulum pictured below



the magnitude of the torque is $\tau = -lmg\sin\theta^{\star}$ and the moment of inertia is $I = ml^2$. Then

$$ml^2\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta.$$

There is no closed form solution to this differential equation. However, if the angle is small, then we can use the first order expansion of the sine function, $\sin\theta \approx \theta$ and the equation becomes that of the simple harmonic oscillator

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta.$$

This is very typical for a set of problems in which the angles are small. For these problems, the following approximations can be made:

$$\sin\theta \approx \theta \approx \tan\theta, \text{ and}$$

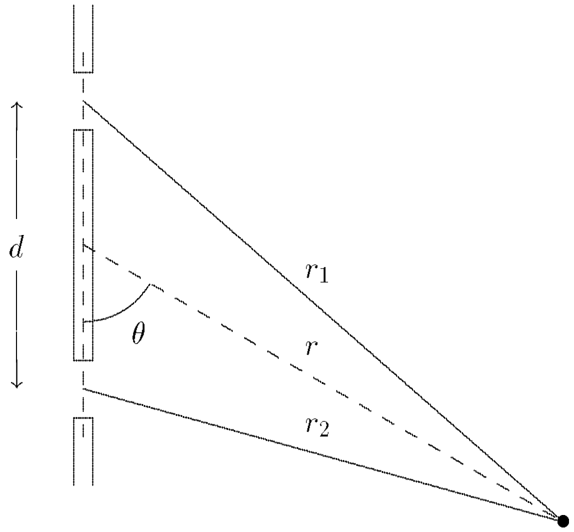
$$\cos\theta \approx 1.$$

1.3.2 Double Slit Interference

To find the interference pattern from a double slit, we must find the difference

⋆ The minus sign comes from the direction of the angle vector, $\vec{\theta}$, with respect to the torque vector, $\vec{\tau}$. If we define the positive z axis out of the paper and the x axis is down in the above picture, then the angle pictured is positive (with the right hand, fingers pointing down and curling around the direction of the angle gives the thumb coming out of the paper) and the torque is negative (right hand fingers along l curling into F gives the tumb into the paper).

in length of the optical paths coming from each slit.



In other words, we need to find $r_1 - r_2$ where

$$r_1 = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 + rd\cos\theta}, \text{ and} \tag{1.3}$$

$$r_2 = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 - rd\cos\theta}. \tag{1.4}$$

In these equations, d is the distance between the two slits, r is the distance from the center of the two slits to the point of interest and θ is the angle from the plane of the slits to the direction of r. Equations (1.3) and (1.4) come from the cosine law with the edges of the triangle being r, r_i, and $\frac{d}{2}$. In these equations, I have used the fact that $\cos(180° - \theta) = -\cos\theta$. Factoring a r^2 out from under the radial gives us the following

$$r_1 = r\sqrt{1 + \left(\frac{d}{2r}\right)^2 + \frac{d}{r}\cos\theta}$$

$$r_2 = r\sqrt{1 + \left(\frac{d}{2r}\right)^2 - \frac{d}{r}\cos\theta}$$

Now we can approximate the radical to get

$$r_1 \approx r\left[1 + \frac{1}{2}\frac{d}{r}\cos\theta\right]$$

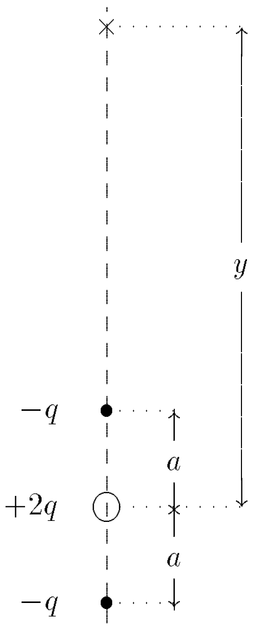
$$r_2 \approx r\left[1 - \frac{1}{2}\frac{d}{r}\cos\theta\right]$$

(notice that the $\left(\frac{d}{2r}\right)^2$ doesn't contribute to the approximation at first order – it will in second order). Finally, the difference is given by

$$r_1 - r_2 \approx d\cos\theta.$$

1.3.3 Electric Quadrapole Field

In this example, the first order terms are going to cancel out and we will have to expand to second order. However, since we don't know that before we begin, we have to go through the first order approximation first.



One form of an electric quadrapole has a charge of $-q$ at $(0, a)$, another $-q$ at $(0, -a)$ and $+2q$ at $(0, 0)$ as shown in the figure above. The electric field along the positive y-axis is given by

$$E_y = \frac{1}{4\pi\epsilon_0}\left[\frac{-q}{(y-a)^2} + \frac{2q}{y^2} + \frac{-q}{(y+a)^2}\right]. \tag{1.5}$$

If we go out far enough such that y is large compared to a (that gives us the hint of how we are going to expand – $a \ll y$ implies that $\frac{a}{y} \ll 1$, and that we should expand in terms of the unitless ratio $\frac{a}{y}$) we dervive a simple approximation to the field. Start by factoring a factor of $\frac{q}{y^2}$ from the brackets in (1.5) giving

$$E_y = \frac{q}{4\pi\epsilon_0 y^2}\left[\frac{-1}{(1-\frac{a}{y})^2} + 2 + \frac{-1}{(1+\frac{a}{y})^2}\right]. \tag{1.6}$$

Now, the first and third terms in the brackets can be expanded to first order in $\frac{a}{y}$ to give

$$\begin{aligned}
E_y &\approx \frac{q}{4\pi\epsilon_0 y^2}\left[-\left(1+2\frac{a}{y}\right) + 2 - \left(1-2\frac{a}{y}\right)\right] \\
&\approx \frac{q}{4\pi\epsilon_0 y^2}\left[-1+2-1-2\left(\frac{a}{y}\right)+2\left(\frac{a}{y}\right)\right] \\
&\approx 0.
\end{aligned}$$

Since all the terms in the first order approximation cancel out, we must include the second order terms to estimate the field. Expanding the first and third terms in (1.6) to second order gives

$$\begin{aligned}
E_y &\approx \frac{q}{4\pi\epsilon_0 y^2}\left[-\left(1+2\frac{a}{y}+3\left(\frac{a}{y}\right)^2\right) + 2 - \left(1-2\frac{a}{y}+3\left(\frac{a}{y}\right)^2\right)\right] \\
&\approx \frac{q}{4\pi\epsilon_0 y^2}\left[-3\left(\frac{a}{y}\right)^2 - 3\left(\frac{a}{y}\right)^2\right] \\
&\approx -\frac{3qa^2}{2\pi\epsilon_0 y^4}.
\end{aligned}$$