Momentum Transfer Tensor Integration

To calculate the differential decay width of a muon, we must integrate over the momenta of the outgoing neutrinos. This integral can be considered as the following tensor:

$$I_{\alpha\beta} = \int \frac{d^3k_1}{2k_1(2\pi)^3}\frac{d^3k_2}{2k_2(2\pi)^3} k_{1\alpha}k_{2\beta}(2\pi)^4\delta^4(q_\gamma - k_{1\gamma} - k_{2\gamma}).$$

This integral can only be a function of $q_\gamma = k_{1\gamma} + k_{2\gamma}$. Since it is a Lorentz tensor, it must be of the following form:

$$I_{\alpha\beta} = Aq^2 g_{\alpha\beta} + Bq_\alpha q_\beta.$$

Evaluating the first integral explicitly is difficult. However, if we can find the values of A and B, we have the solution to the integration. To do this, we evaluate not $I_{\alpha\beta}$ but rather $g^{\alpha\beta}I_{\alpha\beta}$ and $q^\alpha q^\beta I_{\alpha\beta}$. This allows us to evaluate Lorentz scalers rather than tensors and, because of Lorentz invariance, allows us to evalute them in a frame that is convienent to us. First:

$$\begin{aligned} g^{\alpha\beta}I_{\alpha\beta} &= Aq^2 g^{\alpha\beta}g_{\alpha\beta} + Bg^{\alpha\beta}q_\alpha q_\beta \\ &= 4Aq^2 + Bq^2 \end{aligned}$$

$$\begin{aligned} q^\alpha q^\beta I_{\alpha\beta} &= Aq^2 q\cdot q + Bq\cdot qq\cdot q \\ &= A(q^2)^2 + B(q^2)^2 \end{aligned}$$

The integral itself we can evaluate in the center of mass frame for the two neutrinos. In this frame $k_1 = (k, 0, 0, k)$ and $k_2 = (k, 0, 0, -k)$. Thus the integral becomes:

$$\begin{aligned} I_{\alpha\beta} &= \frac{1}{(2\pi)^2}\int \frac{k^2dkd\Omega}{4k^2} k_{1\alpha}k_{2\beta}\delta(q-2k) \\ &= \int \frac{dk}{4\pi} k_{1\alpha}k_{2\beta}\delta(q-2k) \end{aligned}$$

In this frame

$$g^{\alpha\beta} k_{1\alpha} k_{2\beta} = k_1 \cdot k_2 = 2k^2$$

and

$$q^\alpha q^\beta k_{1\alpha} k_{2\beta} = q \cdot k_1 q \cdot k_2 = (k_1 \cdot k_2)^2 = 4k^4$$

Thus

$$\begin{aligned} g^{\alpha\beta} I_{\alpha\beta} &= \int \frac{dk}{4\pi} 2k^2 \delta(q - 2k) \\ &= \left. \frac{2k^2}{2 \cdot 4\pi} \right|_{k=q/2} \\ &= \frac{q^2}{16\pi} \\ &= 4Aq^2 + Bq^2 \end{aligned}$$

Likewise:

$$\begin{aligned} q^\alpha q^\beta I_{\alpha\beta} &= \int \frac{dk}{4\pi} 4k^4 \delta(q - 2k) \\ &= \left. \frac{4dkk^4}{2 \cdot 4\pi} \right|_{k=q/2} \\ &= \frac{q^4}{32\pi} \\ &= Aq^4 + Bq^4 \end{aligned}$$

Solving these two equations for A and B gives $A = \frac{1}{96\pi}$ and $B = \frac{1}{48\pi}$ or

$$I_{\alpha\beta} = \frac{1}{48\pi} \left(\frac{g_{\alpha\beta}}{2} + q_\alpha q_\beta \right).$$