Problem Set 3
Problem 1

$$\mathcal{L} = \partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2 - m^2(\phi_1^2 + \phi_2^2)$$

$$\begin{aligned}\phi_1 &\longrightarrow \cos\theta\phi_1 + \sin\theta\phi_2 \\ \phi_2 &\longrightarrow \cos\theta\phi_2 - \sin\theta\phi_1\end{aligned}$$

$$\begin{aligned}\mathcal{L} =& (\cos\theta\partial_\mu\phi_1 + \sin\theta\partial_\mu\phi_2)(\cos\theta\partial^\mu\phi_1 + \sin\theta\partial^\mu\phi_2) \\ &+ (\cos\theta\partial_\mu\phi_2 - \sin\theta\partial_\mu\phi_1)(\cos\theta\partial^\mu\phi_2 - \sin\theta\partial^\mu\phi_1) \\ &- m^2((\cos\theta\phi_1 + \sin\theta\phi_2)^2 + (\cos\theta\phi_2 - \sin\theta\phi_1)^2) \\ =&(\sin^2\theta + \cos^2\theta)\partial_\mu\phi_1\partial^\mu\phi_1 \\ &+ (\sin^2\theta + \cos^2\theta)\partial_\mu\phi_2\partial^\mu\phi_2 \\ &- (\sin^2\theta + \cos^2\theta)m^2(\phi_1^2 + \phi_2^2) \\ =&\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2 - m^2(\phi_1^2 + \phi_2^2)\end{aligned}$$

For the infinitismal transform:

$$\begin{aligned}\phi_1 &\longrightarrow \phi_1 + \theta\phi_2 \\ \phi_2 &\longrightarrow \phi_2 - \theta\phi_1\end{aligned}$$

and

$$\Psi = \begin{pmatrix}\phi_2 \\ \phi_1\end{pmatrix}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)} = \partial^\mu\phi_1$$

and likewise for ϕ_2. Therefore the conserved current is:

$$j^\mu = \phi_2 \partial^\mu \phi_1 - \phi_1 \partial^\mu \phi_2$$

The equation of motion for ϕ_1 is:

$$\partial_\mu \partial^\mu \phi_1 - m^2 \phi_1 = 0$$

and likewise for ϕ_2. The stress energy tensor is:

$$\partial^\nu \phi_1 \partial_\mu \phi_1 + \partial^\nu \phi_2 \partial_\mu \phi_2 - (\partial^\alpha \phi_1 \partial_\alpha \phi_1 + \partial^\alpha \phi_2 \partial_\alpha \phi_2 - m^2(\phi_1^2 + \phi_2^2))\delta^\nu{}_\mu$$

Substituting $\phi_1 = \frac{\phi + \phi^*}{2}$ and $\phi_2 = \frac{\phi - \phi^*}{2i}$ yields the Lagrangian for the complex scaler field. This justifies our assertion that ϕ and ϕ^* can be considered independent scaler fields.