Particle Physics

Lagrangians and Groups

Problem Set 3

Due: Thursday, February 24,2000

Problem 1 – Show that the Lagrangian for two **real** scaler fields ϕ_1 and ϕ_2

$$\mathcal{L} = \partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2 - m^2(\phi_1^2 + \phi_2^2)$$

is invariant under the transformation:

$$\begin{aligned} \phi_1 &\longrightarrow \cos\theta\phi_1 + \sin\theta\phi_2 \\ \phi_2 &\longrightarrow \cos\theta\phi_2 - \sin\theta\phi_1 \end{aligned}$$

if θ is independent of position. Calculate the conserved current that is associated with this transformation. What are the equations of motion for ϕ_1 and ϕ_2? What is the stress-energy tensor for this field? How is this Lagrangian related to a single complex scaler field?

Problem 2 – Show that the Lagrangian for a complex scaler field:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

is invariant under an infinitesimal rotation, $x_\mu \to x_\mu + R_\mu^\nu x_\nu$ where

$$R_\mu^\nu = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & 0 & \theta_3 & -\theta_2 \\ \beta_2 & -\theta_3 & 0 & \theta_1 \\ \beta_2 & \theta_2 & -\theta_1 & 0 \end{pmatrix}$$

and where "infinitesimal" means that terms second order in the matrix parameters can be ignored. Find the "angular momentum" tensor corresponding to the conserved currents from this transformation.

Problem 3 – The Lagrangian for a charged, spin 1/2, Dirac particle (and it's antiparticle) is given by:

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi$$

where γ^{μ} are the four Dirac matrices, $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ with each of the ψ_i being a complex number, and $\bar{\psi} = (\psi^*)^T\gamma^0 = (\,\psi_1^* \quad \psi_2^* \quad -\psi_3^* \quad -\psi_4^*\,)$ (this funny combination is needed for Lorentz invariance). What are the equations of motion for ψ and $\bar{\psi}$? (Note: The two equations of motion are independent. They can be transformed into the equations of motion for the real and imaginary parts of ψ.) Show that the Lagrangian is invariant under the transform

$$\begin{aligned} \psi &\longrightarrow e^{i\theta}\psi \\ \bar{\psi} &\longrightarrow e^{-i\theta}\bar{\psi} \end{aligned}$$

where θ is independent of position. What is the conserved current associated with this transformation?

Problem 4 – Show that a gauge field must be added to the Lagrangain given in problem 3 to make it invariant under the transformation given in problem 3 if θ is a function of position.

Problem 5 – Consider a 4 dimensional representation of SU(2) [that means that the SU(2) matrices are 4x4 matrices and that the eigenvectors have four elements]. Write the three matrix operators that correspond to I_z with eigenvalues $2/3, 1/2, -1/2, -3/2$ and the raising and lowering operators I_+ and I_- (properly normalize them). Construct I_x and I_y from these. Show that $I^2 = I_x^2 + I_y^2 + I_z^2 = 15/4 = I(I+1)$