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ALGEBRAIC TOPOLOGY HOMEWORK PROBLEMS
WINTER QUARTER 2011

Please provide plenty of details! Pix are definitely kewl ($\ddot\smile$).

There are a few warm-up problems on stuff covered last Autumn—you might even recognize some of them.

(1) Let $(X,\mathcal{T})$ be a compact Hausdorff space. Let $\mathcal{S},\mathcal{U}$ be topologies on X. Corroborate the following:

(a) $\mathcal{S}$ strictly coarser than $\mathcal{T}$ implies that $(X,\mathcal{S})$ is not Hausdorff.

(b) $\mathcal{U}$ strictly finer than $\mathcal{T}$ implies that $(X,\mathcal{U})$ is not compact.

(Hint: What can you say about the identity map?)

(2) (a) Find an embedding of $\mathsf{S}^1\times\mathsf{I}$ into S^2. Explain why this also gives an embedding of $\mathsf{S}^1\times(0,1)$ into S^2. Give an example of an embedding of $\mathsf{S}^1\times(0,1)$ into S^2 that does not come from an embedding of $\mathsf{S}^1\times\mathsf{I}$ into S^2.

(b) Explain why the following spaces are all the 'same' (i.e., all homeomorphic):

$$\mathsf{S}^1\times(0,1),\quad \mathsf{S}^1\times\mathsf{R},\quad \mathsf{S}^2\setminus\{a,b\},\quad \mathsf{R}^2\setminus\{0\},\quad \{z\in\mathsf{R}^2 : r<|z|<s\};$$

here a,b are any two distinct points on S^2 and $0\le r<s\le\infty$.

(c) Explain why the spaces $\mathsf{B}^2,(-1,1)^2,\mathsf{R}^2$ are homeomorphic.

(d) What about the R^3 subspaces $\mathsf{S}^2,\partial\mathsf{I}^3,(\mathsf{S}^1\times\mathsf{I})\cup(\mathsf{B}^2\times\{0,1\})$?

(3) Recall that a set in R^n is *convex* provided the line segment joining any two points of the set also lies in the set.

(a) Prove that every convex polygon in R^2 is homeomorphic to D^2.

(b) What about an arbitrary convex compact subset of R^2?

(c) What about an arbitrary convex compact subset of R^n?

(4) Let K be a compact subset of $\bigvee_{\lambda\in\Lambda}X_\lambda$ for some collection $\{X_\lambda\mid\lambda\in\Lambda\}$ of topological spaces. Explain why there must be a *finite* set $\Phi\subset\Lambda$ such that $K\subset\bigvee_{\lambda\in\Phi}X_\lambda$.

(5) Define an equivalence relation on R^2 by writing $(x,y)\sim(x',y')$ if $x'-x\in\mathsf{Z}$. Prove that $\mathsf{R}^2/\!\sim$ is a surface. What surface is it?

(6) Define an equivalence relation on $Q:=\{(x,y)\in\mathsf{R}^2\mid x\ge0,y\ge0\}$ by setting $(x,0)\sim(0,7x)$. Prove that $Q/\!\sim$ is a surface. What surface is it? Give an explicit description of a Euclidean disk about the "origin".

Suppose the equivalence relation is given by $(x,0)\sim(0,\sinh x)$. Do we still get a surface? If so, what changes must be made in your proof?

Date: March 31, 2011.

(7) Define an equivalence relation on the set T of points $(x,y,z) \in \mathsf{R}^3$ that satisfy

$$x \geq 0\,, y \geq 0\,, z = 0 \quad \text{or} \quad y \leq 0 \leq z\,, x = 0 \quad \text{or} \quad x \leq 0\,, z \leq 0\,, y = 0$$

by setting $(x,0,0) \sim (0,-3x,0)$ and $(0,0,z) \sim (0,0,-2z)$ for all $x, z \in \mathsf{R}$. Prove that $T/\sim$ is a surface. What surface is it? Give an explicit description of a Euclidean disk about the "origin".

Suppose the identifications of the "boundary" points are given via some non-linear homeomorphisms. Do we still get a surface? What changes occur in your proof?

(8) Define an equivalence relation on R^2 by writing $(x,y) \sim (x',y')$ if $n := x' - x \in \mathsf{Z}$ and $y' = (-1)^n y$. Prove that $\mathsf{R}^2/\sim$ is a surface. What surface is it?

(9) (a) Explain why the Möbius band MB is not a surface. It is a (connected) 2-manifold *with boundary*. What is the "boundary" of MB? (Note that for any topological space X, we have $\partial X = \emptyset$, right? So, the "boundary" of MB is not really it's topological boundary, unless we think of MB as being embedded in some larger ambient space such as R^3. We could call this, e.g., the "edge" of MB, but ... what is the "boundary" of an n-manifold with boundary? It's not really an "edge", is it?)

(b) Demonstrate that the Möbius band with its *"boundary" removed* is homeomorphic to the quotient space

$$([0,1] \times (0,1))/\sim \quad \text{where } (0,y) \sim (1, 1-y)\,.$$

Do you recognize this space from some above problem? Let's call this the ***Möbius plane*** and denote it by MP.

(c) Prove that MP is a surface. Can you see/prove that $\mathsf{MP} \not\approx \mathsf{S}^1 \times \mathsf{R}$?

(d) Let $\mathsf{D}^2 \supset \partial \mathsf{D}^2 = \mathsf{S}^1 \xrightarrow{\varphi} \mathsf{MB}$ be a homeomorphism with image the "boundary" of MB. What is the adjunction space $\mathsf{MB} \sqcup_\varphi \mathsf{D}^2$?

(e) Let ψ be a homeomorphism between the two "boundaries" of two Möbius bands $\mathsf{MB}_1, \mathsf{MB}_2$. What is the adjunction space $\mathsf{MB}_2 \sqcup_\psi \mathsf{MB}_1$?

(10) There are four different ways of describing the ***2-dimensional torus*** T^2.

- as the product space $\mathsf{T}^2 := \mathsf{S}^1 \times \mathsf{S}^1$, a subspace of $\mathsf{R}^2 \times \mathsf{R}^2$;
- as the ***tire tube surface*** TT, a subspace of R^3, obtained by rotating the circle $\{(x,y,z) \in \mathsf{R}^3 : (y-2)^2 + z^2 = 1, x = 0\}$ about the z-axis; or, more simply, $$\mathsf{TT} := \{(x,y,z) : (2 - \sqrt{x^2+y^2})^2 + z^2 = 1\}\,;$$
- as the quotient space $\mathsf{I}^2/\sim$ where $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$, which is called the ***flat torus***;
- as the ***orbit space*** R^2/Γ where Γ is the group of all horizontal and vertical translations $(x,y) \mapsto (x+m, y+n)$ with $m,n \in \mathsf{N}$; equivalently, $\mathsf{R}^2/\sim$ where $(x+1,y) \sim (x,y) \sim (x,y+1)$.

Demonstrate that these four spaces are homeomorphic.

(11) (a) Describe the quotient of the torus T^2 modulo its *longitudinal circle* $\mathsf{S}^1 \times \{1\}$.

(b) What is the quotient space of T^2 modulo $\mathsf{S}^1 \vee \mathsf{S}^1 = (\mathsf{S}^1 \times \{1\}) \cup (\{1\} \times \mathsf{S}^1)$ (the wedge of its longitudinal and latitudinal circles) ? It is a common surface! (Suggestion: Look at the square I^2 with its identifications.)

(12) The ***Klein bottle*** KB is the quotient space obtained from the square I^2 via the boundary identifications $(0,y) \sim (1,1-y)$ and $(x,0) \sim (x,1)$. Prove that KB is a surface.

(13) Let A be a non-degenerate closed annulus in the plane and define an equivalence relation on A by identifying antipodal points on the outer circle and also identifying antipodal points on the inner circle. Show that the resulting quotient space is (homeomorphic to) the Klein bottle KB.

(14) An object of interest to algebraic geometers is n***-dimensional projective space*** which is the quotient space defined by $\mathsf{P}^n := \mathsf{S}^n/\sim$ where $x \sim -x$; i.e., we identify *antipodal points* x and $-x$. (Actually, this is *real* projective space; there is also a complex projective space. Moreover, algebraic geometers use a somewhat different description for projective space.)

(a) Prove that P^n is homeomorphic to the quotient space $\mathsf{D}^n/\sim$ where $x \sim y$ if $x = y$ or $x, y \in \partial\mathsf{D}^n = \mathsf{S}^{n-1}$ and $x = -y$ (i.e., we identify antipodal boundary points).

(b) Prove that P^n is an n-manifold.

(c) Demonstrate that $\mathsf{P}^1 \approx \mathsf{S}^1$. Can you see/prove that $\mathsf{P}^2 \not\approx \mathsf{S}^2$?

(d) Let $\mathsf{D}^2 \xrightarrow{q} \mathsf{P}^2$ be the quotient map that identifies antipodal points of the unit circle $\mathsf{S}^1 = \partial\mathsf{D}^2$. Let $A := \mathsf{D}^2(0;1/2) \cup \{(x,y) \in \mathsf{D}^2 \,:\, |y-x| < 1/10\} \subset \mathsf{D}^2$. Explain why $q|_A$ is an identification map. Draw a picture of $B := q(A)$; it is a "basket" with a certain type of "handle".

(e) What is the space $\mathsf{P}^2 \setminus \mathsf{B}^2$ where B^2 is any (open) regular Euclidean disk in P^2?

(f) Find a space X with the property that $X = U \cup B$ where $U \neq \emptyset$ is open in X and B is homeomorphic to $\mathsf{S}^1 = \partial\mathsf{D}^2$, via some homeomorphism φ, and is such that the adjunction space $\mathsf{D}^2 \sqcup_\varphi X$ is (homeomorphic to) P^2.

(15) Sketch pictures for each of the spaces $\mathsf{S}^2 \setminus \mathsf{B}^2, \mathsf{T}^2 \setminus \mathsf{B}^2, \mathsf{P}^2 \setminus \mathsf{B}^2, \mathsf{KB}^2 \setminus \mathsf{B}^2$; in each of these, B^2 denotes any (open) regular Euclidean disk in the ambient surface. When possible, identify the space. (E.g., $\mathsf{S}^2 \setminus \mathsf{B}^2 \approx \mathsf{D}^2$, right?)

(16) Review the definition of a polygon diagram and the answer the following.

(a) What happens if some color is used for exactly one edge?

(b) What if three edges have the same color?

(17) Find polygon diagrams that provide flattenings for $\mathsf{S}^2, \mathsf{P}^2$. Be sure to use polygons!

(18) Provide a formal statement that describes the following claim: When we create the geometric realization of a polygon diagram, we may either make all of the identifications at once (as per our definition), or, we can make the identifications one color at a time; both processes produce the same space. Be sure to provide a proof of your assertion.

(19) Suppose that a is a "vertex point" in the geometric realization of some polygon surface diagram with the property that the identification pre-image of a consists of exactly one vertex of the polygon. Give an explicit description of the regular Euclidean disks centered at a.

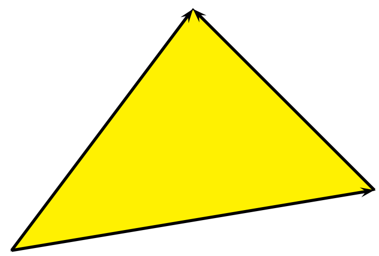


FIGURE 1. A triangle diagram for the Dunce Cap

(20) Give a list of all possible triangle diagrams and their associated geometric realizations. When possible, describe the geometric realization in "concrete" terms. For example, you can describe some of these by starting with a "wire frame skeleton" and attaching a disk via certain maps. (One such diagram is pictured in Figure 1; it's geometric realization DC is called the ***dunce cap space***. Is DC a manifold?)

Here is some terminology for problems #(21)-#(25) below. Let's call a topological space

- a *topological disk* if it is homeomorphic to D^2 ,
- an *arc* if it is homeomorphic to I .

For the (closed) unit disk D^2, viewed as a subspace of R^2, we have $\partial\mathsf{D}^2 = \mathsf{S}^1$ where here ∂ means with respect to to the ambient R^3 topology. (Of course, for a topological space X, $\partial X = \emptyset$, right?). For a topological disk $D = h(\mathsf{D}^2)$ with h some homeomorphism, we write $\partial D := h(\partial\mathsf{D}^2)$. We shall see, eventually, that this notion is well-defined (i.e., it does not depend on the homeomorphism h).

Let D be a topological disk. Let $v_0, v_1, \dots, v_{n-1}, v_n = v_0$ be n successive points along ∂D, with $n \geq 2$. Let A_i be the (closed) subarc of ∂D from v_{i-1} to v_i. Thus $\partial D = A_1 \cup \cdots \cup A_n$ and the subarcs only intersect at their endpoints. Let Λ be a word consisting of n letters (or their "inverses"). We call $(D; \Lambda)$ a ***disk diagram***.

Exactly as we did for polygon diagrams, use Λ to label the subarcs $A_1, \dots, A_n$: each subarc gets both a "color" and a "direction". Let $\sim_\Lambda$ denote the equivalence relation induced by this labeling. Then the ***geometric realization*** of the disk diagram $(D; \Lambda)$ is the topological space $D/\Lambda := D/\sim_\Lambda$.

An equivalent way to define this geometric realization is to view it as the adjunction space $D/\Lambda := K \sqcup_\varphi D$ where $K := \partial D/\Lambda$ is the one-skeleton obtained from the boundary of the disk D and the *attaching homeomorphism* $\varphi : \partial D \to K$ is just the restriction of the quotient map $D \to D/\Lambda$. Thus the geometric realization D/Λ is the space obtained by attaching the disk D to the one-skeleton K as described by the boundary label Λ.

(21) Prove that the geometric realization of a disk diagram is a compact connected space. When will it be a manifold? When a surface?

(22) Let $\omega := e^{2\pi i/n} \in \mathsf{S}^1$; often we call ω an *n-th root of unity* because $\omega^n = 1$. The ***pseudo-projective plane of order*** n is the quotient space $\mathsf{PP}_n := \mathsf{D}^2/\sim$ where the equivalence relation is defined by identifying points $z, w \in \partial\mathsf{D}^2 = \mathsf{S}^1$ that satisfy $w = \omega\, z$.

(a) Describe PP_n as the geometric realization of some disk diagram.

(b) Prove that PP_2 is a surface. Do you recognize it?

(c) Is PP_n a manifold if $n > 2$?

(23) (a) Let I, J be arcs in $\mathsf{S}^1 = \partial\mathsf{D}^2$. Find a homeomorphism $\mathsf{D}^2 \xrightarrow{\Psi} \mathsf{D}^2$ with $\Psi(I) = J$.

(b) Suppose $I \xrightarrow{\psi} J$ is a homeomorphism with $\psi(I) = J$ for some arcs I, J in $\mathsf{S}^1 = \partial\mathsf{D}^2$. Find a homeomorphism $\mathsf{D}^2 \xrightarrow{\Psi} \mathsf{D}^2$ with $\Psi|_I \equiv \psi$.
(Hints: First do this with S^1 replaced by R, and then replaced by I.)

(24) Suppose that, for $i = 1, 2$, D_i are topological disks and $A_i \subsetneq \partial D_i$ are arcs. Let $\psi : A_1 \to A_2$ be any homeomorphism. Demonstrate that $D_2 \sqcup_\psi D_1 \approx \mathsf{D}^2$. What elementary transformation of polygonal presentations does this correspond to? What happens if $A_i = \partial D_i$?

(25) Suppose that D is a topological disk and $A, B \subset \partial D$ are non-overlapping adjacent arcs with $A \cap B = \{z\}$ a single point. Let $\psi : A \to B$ be an 'orientation reversing' homeomorphism; i.e., $\psi(z) = z$. Prove that $D/(x \sim \psi(x)) \approx \mathsf{D}^2$. What elementary transformation of polygonal presentations does this correspond to?

(26) Let $\mathcal{P}$ be a polygonal presentation. Suppose that the geometric realization $|\mathcal{P}|$ is a surface. Prove that by pasting together appropriated edges, $\mathcal{P}$ can be 'reduced' to a polygon diagram.

(27) Identify the geometric realization of each of the following polygonal presentations.

(a) $\mathcal{P} :=< abacb^{-1}c^{-1} >$

(b) $\mathcal{P} :=< abca^{-1}b^{-1}c^{-1} >$

(c) $\mathcal{P} :=< abc, bde, c^{-1}df, e^{-1}fa >$

(d) $\mathcal{P} :=< abc, bde, dfg, fhi, haj, c^{-1}kl, e^{-1}mn, g^{-1}ok^{-1}, i^{-1}l^{-1}m^{-1}, j^{-1}n^{-1}o^{-1} >$

We recall the connected sum operation whose "definition is pictured" in Figure 2. We use this notion to describe the operations of ***attaching a cap, attaching a handle, attaching a cross-cap,*** or ***attaching a cross-handle*** to a given surface. If we start with a given surface M, then these actions applied to M produce the surfaces $M\#\mathsf{S}^2$, $M\#\mathsf{T}^2$, $M\#\mathsf{P}^2$, $M\#\mathsf{KB}$ respectively.

Below we examine alternative descriptions for these connected sum operations. Before doing this we consider several 'warm-up' problems.

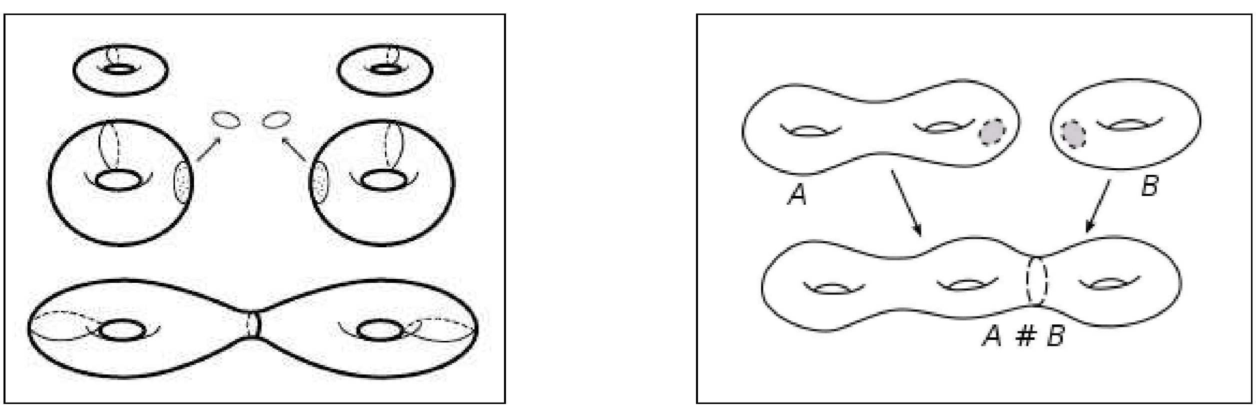


FIGURE 2. Two connected sums of tori: $\mathsf{T}^2\#\mathsf{T}^2 \approx \mathsf{T}^2_2$ and $\mathsf{T}^2_2\#\mathsf{T}^2 \approx \mathsf{T}^2_3$

(28) Prove that for any n-manifold M, $M\#\mathsf{S}^n \approx M$. (Figure 7 illustrates a special case.)

We define an operation on surfaces called ***tube connection***. We start with two surfaces M_0, M_1. Let B_i be an open regular Euclidean disk in M_i with $S_i := \partial B_i \approx \mathsf{S}^1$. Consider any embedding

$$\mathsf{S}^1 \times \mathsf{I} \supset \mathsf{S}^1 \times \{0,1\} \xrightarrow{h} (M_0 \setminus B_0) \cup (M_1 \setminus B_1)$$

with $\mathsf{S}^1 \times \{i\} \xrightarrow{h|} S_i \subset M_i \setminus B_i$ a homeomorphism for each $i \in \{0,1\}$. We use the map h to attach the *tube* $\mathsf{S}^1 \times \mathsf{I}$ to each of the punctured surfaces $(M_0 \setminus B_0), (M_1 \setminus B_1)$, the 'bottom' of the tube being attached to S_0 and the 'top' of the tube to S_1. Formally, the ***tube connection*** of M_0 and M_1 is the adjunction space

$$M_0 \#_{\mathrm{t}} M_1 := [(M_0 \setminus B_0) \cup (M_1 \setminus B_1)] \sqcup_h \left[\mathsf{S}^1 \times \mathsf{I}\right] .$$

(29) Let M_0, M_1 be surfaces. Prove that $M_0 \#_{\mathrm{t}} M_1 \approx M_0 \# M_1$.

Next we describe an alternative way to view the operations of ***attaching a handle*** or ***attaching a cross-handle*** to a given surface. Note that in contrast to the connected sum operation, these are operations on a single surface. In fact, we do this in somewhat more generality. So, let X be a topological space (you can think of a surface if you wish). Assume that D is a closed Euclidean disk in X; this just means that $D \approx \mathsf{D}^2 \subset \mathsf{R}^2$. Let D_0, D_1 be two disjoint closed Euclidean disks in the interior of D. (You might as well picture this in D^2.) Let B_i denote the interior of D_i and $S_i := \partial B_i = \partial D_i$. Now mark each S_i with an orientation arrow. If both orientation arrows go in the same direction (both clockwise or both counterclockwise, in the R^2 picture), then we say that the orientations are *twisted*; otherwise—when the orientation arrows go in opposite directions—we say that the orientations are *non-twisted*. In the first picture in Figure 3 the orientations are non-twisted; in Figure 4 they are twisted. (Below I explain this terminology; I use it to follow Lee!)

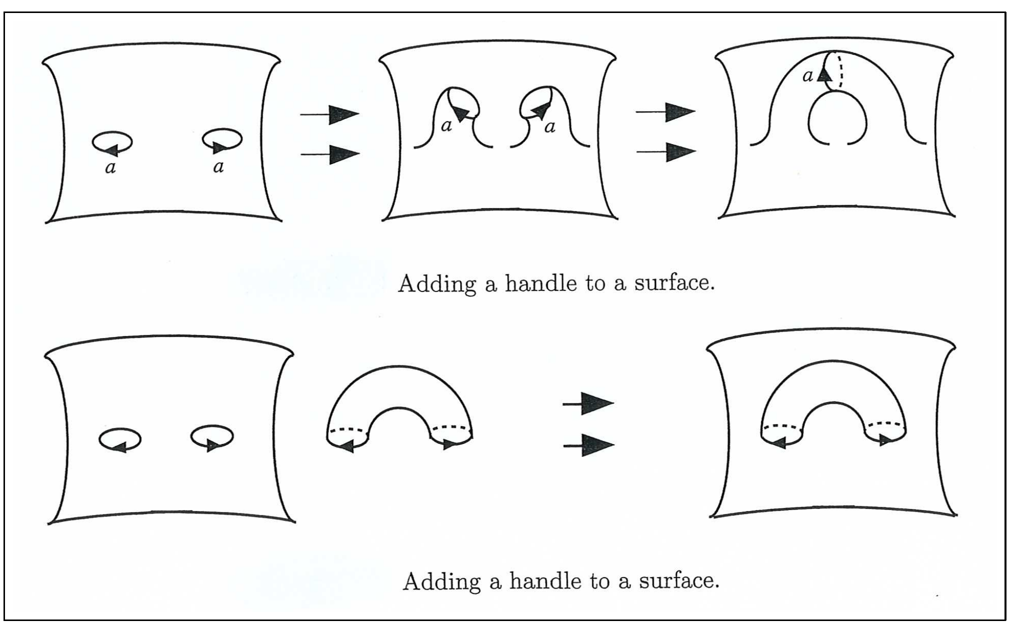
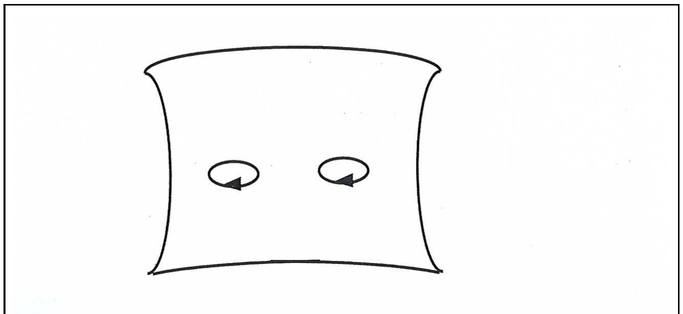


FIGURE 3. Attaching a handle to a surface



Preparing a surface for the addition of a cross-handle.

FIGURE 4. Attaching a cross-handle to a surface

Let $S_0 \overset{h}{\to} S_1$ be any homeomorphism induced by the orientations on each of S_0, S_1: as the point z traces out the circle S_0 going in S_0's direction, $h(z)$ traces out the circle S_1 going in S_1's direction. Starting with the twice-punctured space $X \setminus (B_0 \cup B_1)$, we paste together the circles S_0, S_1 by identifying the points $z \sim h(z)$ for $z \in S_0$. We thus obtain the quotient space

$$[X \setminus (B_0 \cup B_1)] \,/\, (z \sim h(z)) \ .$$

This new space depends on whether the orientations of the circles S_0, S_1 are twisted or non-twisted. In the latter case, we say that a ***handle*** has been attached to X and we denote this space by

$$X \#_{\mathrm{a}} \mathsf{H} := [X \setminus (B_0 \cup B_1)] \,/\, (z \sim h(z)) \ ;$$

in the twisted case, a ***cross-handle*** has been attached to X, denoted

$$X \#_{\mathrm{a}} \mathsf{CH} := [X \setminus (B_0 \cup B_1)] \,/\, (z \sim h(z)) \ .$$

See the pictures in Figures 3, 4, 5.

(30) Let M be a surface. Prove that: (a) $M\#_{\mathrm{a}}\mathsf{H} \approx M\#\mathsf{T}^2$ and (b) $M\#_{\mathrm{a}}\mathsf{CH} \approx M\#\mathsf{KB}$.

Hints: Figures 3 and 5 illustrate the idea behind (a). To give a proper/correct/formal proof, I suggest that you first prove that $\mathsf{T}^2 \setminus \mathsf{B}^2 \approx \mathsf{D}^2\#_{\mathrm{a}}\mathsf{H}$.

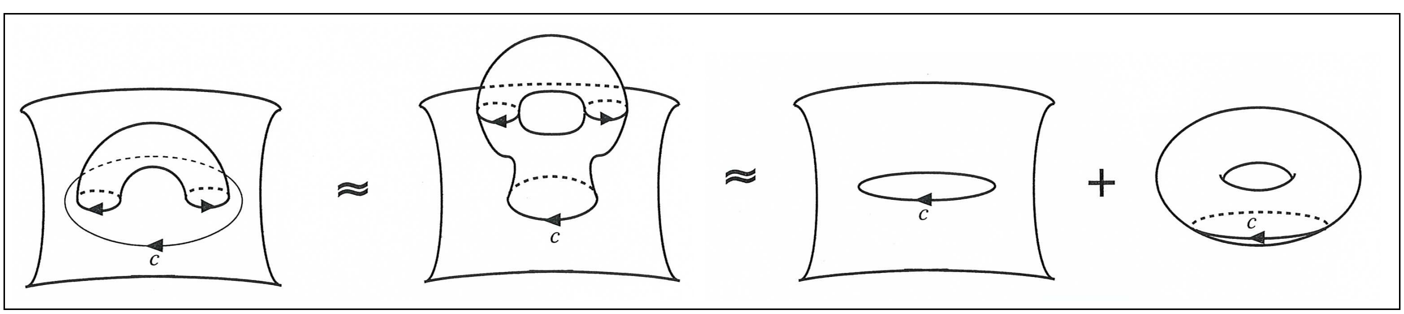


FIGURE 5. Attaching a handle to a surface produces its connected sum with T^2

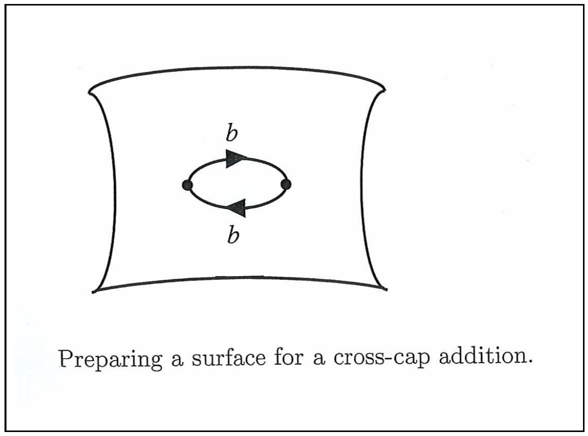


FIGURE 6. Attaching a cross-cap

In a similar way, we can interpret the operations of ***attaching a cap*** or ***attaching a cross-cap*** as pasting together the boundary semicircles of a once punctured space $X \setminus B$. See Figures 6 and 7.

Finally, we connect the two ends of a tube to just one surface. See the bottom pictures in Figure 4. Again, we do this in somewhat more generality. So, let X be a topological space (you can think of a surface if you wish). Assume that D is a closed Euclidean disk in X, that D_0, D_1 are two disjoint closed Euclidean disks in the interior of D, that B_i is the interior of D_i, and $S_i := \partial B_i = \partial D_i$. Mark each S_i with an orientation arrow. Again, the orientations are *twisted* if both orientation arrows go in the same direction and *non-twisted* otherwise.
Consider any embedding

$$\mathsf{S}^1 \times \mathsf{I} \supset \mathsf{S}^1 \times \{0,1\} \xrightarrow{h} X \setminus (B_0 \cup B_1)$$

with $\mathsf{S}^1 \times \{i\} \xrightarrow{h|} S_i \subset X \setminus (B_0 \cup B_1)$ a homeomorphism for each $i \in \{0,1\}$ that respects the orientations of S_0, S_1: as $z \in \mathsf{S}^1 \times \{i\}$ traces out $\mathsf{S}^1 \times \{i\}$ in its 'positive' direction, the point $h(z) \in S_i$ traces out the circle S_i in the appropriate direction. We

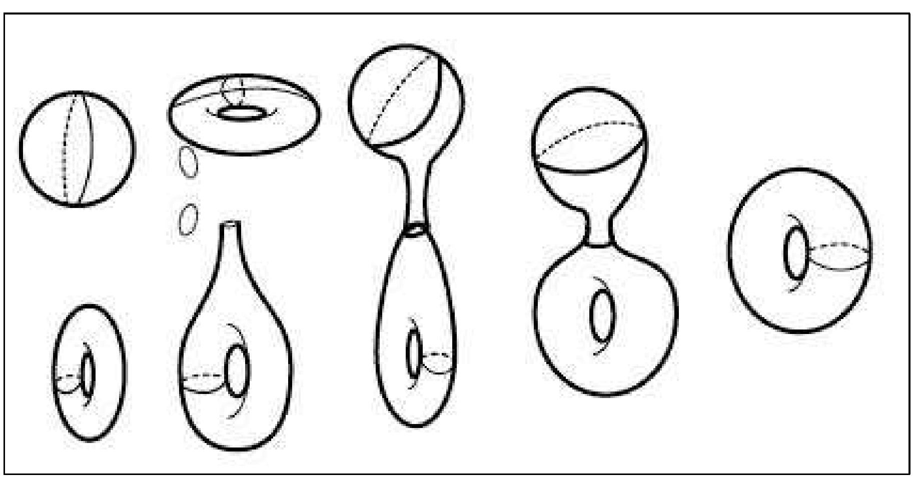


FIGURE 7. Attaching a cap to T^2 gives $\mathsf{T}^2 \# \mathsf{S}^2 \approx \mathsf{T}^2$

use the map h to attach the *tube* $\mathsf{S}^1 \times \mathsf{I}$ to the twice-punctured space $X \setminus (B_0 \cup B_1)$, the 'bottom' of the tube being attached to S_0 and the 'top' of the tube to S_1. Formally, we consider the adjunction space

$$[(X \setminus (B_0 \cup B_1)] \sqcup_h \left[\mathsf{S}^1 \times \mathsf{I}\right] .$$

(31) Let X be a surface. Form the space $[(X \setminus (B_0 \cup B_1)] \sqcup_h [\mathsf{S}^1 \times \mathsf{I}]$ described in the preceding two paragraphs. Prove that:
 (a) when the orientations are non-twisted, this space is homeomorphic to
 $X \#_{\mathrm{a}} \mathsf{H} \approx X \# \mathsf{T}^2$,
 (b) when the orientations are twisted, this space is homeomorphic to
 $X \#_{\mathrm{a}} \mathsf{CH} \approx X \# \mathsf{KB}$.

 In (a) above the tube is attached to $X \setminus (B_0 \cup B_1)$ with no twisting (see the bottom pictures in Figure 4) whereas in (b) we must "twist" the tube so that the orientations will align; to visualize this "twisting" pass the tube "across" the surface of the space and attach the two ends of the tube to "opposite" sides of the space.

(32) Let $A \overset{j}{\hookrightarrow} X$ be an injective map from a set A into a space X and give A the subspace topology; typically A is an actual subset of X, but it need not be such (right?).

 We say that A is a ***retract*** of X if there is a continuous map $X \xrightarrow{r} A$ with $r \circ j = 1_A$; in this case, the map r is called a ***retraction*** of X onto the subspace A.

 (a) Let X be Hausdorff. Suppose $A \subset X$ is a retract of X. Verify that A is closed.

 (b) Prove that if X is connected, or compact, then so is any retract of X.

 (c) Prove that a retraction is an identification map.

(33) Prove that there exists a retraction of X onto a subspace A if and only if for every space Y and every continuous map $f : A \to Y$ there is a continuous map $F : X \to Y$ with $f = j \circ F$ (where $j : A \hookrightarrow X$).

(34) Prove that two constant maps $X \xrightarrow{h,k} Y$ are homotopic if and only if $h(X)$ and $k(X)$ lie in the same path component of Y.

(35) Let $f, g : X \to \mathsf{R}^n_*$ be any two maps from a topological space X into the punctured Euclidean space $\mathsf{R}^n_* := \mathsf{R}^n \setminus \{0\}$. Assume that for all points $x \in X$,

$$|f(x) - g(x)| \le |f(x)|.$$

 Corroborate that $f \simeq g$. (Does this remind you of a theorem in *Complex Analysis*?)

(36) Let V be a vector space. We say that $S \subset V$ is ***star-shaped*** *with respect to* $v \in V$ provided for each $x \in X$, the line segment $[v, x]$ lies in S. Prove that any two maps $f, g : X \to S$ from a topological space X into a star-shaped subset of a vector space are homotopic.

(37) Let Z be a locally compact Hausdorff space. Assume $X \xrightarrow{p} Y$ is an identification map. Prove that the product map

$$X \times Z \xrightarrow{p \times 1_Z} Y \times Z$$

 is also an identification map. (Here $Z \xrightarrow{1_Z} Z$ denotes the identity map on Z.)

(38) Let $X \xrightarrow{p} Y$ be an identification map. Suppose $X \times \mathsf{I} \xrightarrow{H} Z$ is a homotopy that respects the identifications of p in the sense that

$$\forall\, x, x' \in X : \quad p(x) = p(x') \implies \forall\, t \in \mathsf{I}\,, H(x,t) = H(x',t)\,.$$

Demonstrate that H induces a homotopy $Y \times \mathsf{I} \xrightarrow{\tilde{H}} Z$ which has the property that $\tilde{H} \circ (p \times 1_{\mathsf{I}}) = H$.

(39) Prove that any retract of a contractible space is contractible.

(40) Recall that $[X, Y]$ denotes the set of homotopy classes of maps from X to Y.

(a) Prove that for any space X, $[X, \mathsf{I}]$ contains a single element.

(b) Show that when Y is path connected, $[\mathsf{I}, Y]$ contains a single element.

(c) Confirm that a contractible space is path connected.

(d) Prove that if X is contractible and Y is path connected, then $[X, Y]$ contains a single element.

(41) Demonstrate that for any space X, the following are equivalent.

(a) X has the homotopy type of a point.

(b) 1_X is null-homotopic.

(c) For all spaces Z and all maps $f, g : Z \to X$, $f \simeq g$.

(d) For all spaces Z, $[Z, X]$ consists of a single element.

(42) A ***deformation*** of X into A is a homotopy $X \times \mathsf{I} \xrightarrow{H} X$ of 1_X to $j \circ d$ where $X \xrightarrow{d} A$ is any continuous map (and $j : A \hookrightarrow X$ is an injective map). When such a deformation exists, we say that X is ***deformable*** to A.

Suppose X is deformable to A and $A \xrightarrow{f,g} Y$ are homotopic maps. Suppose $X \xrightarrow{F,G} Y$ are any extensions of f, g (meaning $f = j \circ F$ and $g = j \circ G$). Prove that $F \simeq G$.

(43) A ***deformation retraction*** of X into A is a homotopy $X \times \mathsf{I} \xrightarrow{H} X$ of 1_X to $j \circ r$ where $r : X \to A$ is a retraction from X onto A (and $j : A \hookrightarrow X$ an injective map). If such a deformation retraction exists, we call A a ***deformation retract*** of X.

Sometimes the above notion is called a ***weak*** *deformation retraction* to contrast it with a ***strong*** *deformation retraction*; the latter has the additional requirement that the homotopy from 1_X to $j \circ r$ must be *relative* to A. This means that every point of A is kept fixed: for all $a \in A$ and all $t \in \mathsf{I}$, $H(a, t) = a$.

Note that for a (weak) deformation retract, the inclusion and retraction are inverse homotopy equivalences.

Prove that a subspace A of a space X is a (weak) deformation retract of X if and only if A is a retract of X and X is deformable into A.

(44) Prove that the center circle of a Möbius band is a strong deformation retract of the Möbius band.

(45) (a) Prove that the circle $\{(x, y) \in \mathsf{R}^2 : x^2 + y^2 = 1\}$ is a strong deformation retract of the once punctured plane $\mathsf{R}^2_* := \mathsf{R}^2 \setminus \{0\}$.

(b) Demonstrate that the figure eight space $\mathsf{FE} := \{(x, y) \in \mathsf{R}^2 : (x \pm 1)^2 + y^2 = 1\}$ is a strong deformation retract of the twice punctured plane $\mathsf{R}^2_{**} := \mathsf{R}^2 \setminus \{(\pm 1, 0)\}$.

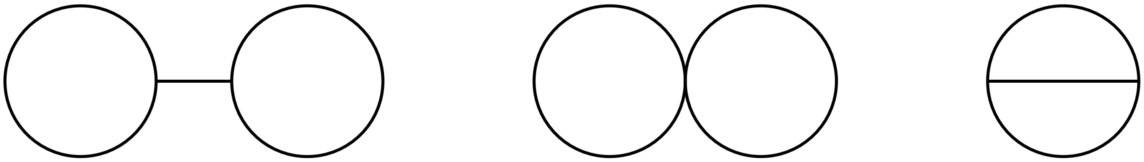


FIGURE 8. Dumb Bell, Figure Eight, and Theta Spaces

(46) Provide rigorous proofs for the following facts concerning the dumb bell, figure eight, and theta spaces DB, FE, Θ that are pictured in Figure 8.

(a) Each of these is a deformation retract of a twice punctured plane.

(b) These spaces are all homotopically equivalent.

(c) No two of these spaces are homeomorphic.

(47) Recall that the ***natural comb*** NC, ***harmonic comb*** HC, and ***doubled harmonic comb*** DHC are the subspaces of R^2 defined by

$$\mathsf{NC} := ([0,\infty)\times\{0\}) \cup \bigcup_{n=0}^{\infty} (\{n\}\times[0,1])\ ,$$

$$\mathsf{HC} := ([0,1]\times\{0\}) \cup (\{0\}\times[0,1]) \cup \bigcup_{n=1}^{\infty} \{1/n\}\times[0,1]\,,$$

$$\mathsf{DHC} := (\{0\}\times[-1,1]) \cup ([0,1]\times\{1\}) \cup \bigcup_{n=1}^{\infty} (\{1/n\}\times[0,1])$$
$$\cup \bigcup_{n=1}^{\infty} (\{-1/n\}\times[-1,0]) \cup ([-1,0]\times\{-1\})\ .$$

(a) Which of these spaces is (or is not) contractible?

(b) For each of these spaces X let $p(x,y) := (0,y)$ and put $A := p(X)$.

(i) Is A a retract of X? If so, is p a retraction of X onto A?

(ii) Is A a (strong or weak) deformation retract of X? If so, is p a retraction of X onto A satisfying the requirement that $j\circ p \simeq 1_X$? (Here $j : A \hookrightarrow X$ is the inclusion map and 1_X the identity map on X.)

(iii) Does p define a homotopy equivalence between X, A?

(48) Recall that the 2-torus is $\mathsf{T}^2 := \mathsf{S}^1\times\mathsf{S}^1$. Confirm that $\mathsf{T}^2\setminus\{pt\}$ has the homotopy type of the figure eight space FE.

(49) Prove that the torus $\mathsf{T}^2 := \mathsf{S}^1\times\mathsf{S}^1$ is a deformation retract of $(\mathsf{S}^1\times\mathsf{D}^2)\setminus(\mathsf{S}^1\times\{0\})$.

(50) Recall that when Z is a closed subspace of X and $Z \overset{\varphi}{\to} Y$ is continuous, the space $Y\sqcup_\varphi X$, called the ***adjunction*** *of X to Y via φ*, is constructed by attaching X to Y using φ as follows:

$$Y\sqcup_\varphi X \quad \text{is the quotient space} \quad (X\sqcup Y)/\,(z\sim\varphi(z))\ ;$$

a more precise description of the equivalence relation is that $u\sim v$ if either (i) $u=v$ or (ii) $u,v\in Z$ and $\varphi(u)=\varphi(v)$ or (iii) $u\in Z$ and $v=\varphi(u)\in Y$.

(a) Suppose $X \supset A \supset Z \overset{\varphi}{\to} Y$ with A, Z closed and φ continuous. Prove that $Y \sqcup_\varphi A$ is a closed subspace of $Y \sqcup_\varphi X$.

(b) Assume $X \supset Z \overset{\varphi}{\to} Y$ with Z closed and φ continuous. Suppose Z is a strong deformation retract of X. Demonstrate that $Y \sqcup_\varphi Z$ is a strong deformation retract of $Y \sqcup_\varphi X$.

(51) Classify the letters in the alphabet, as pictured, up to homeomorhpism type, and then up to homotopy type.

ABCDEFGHIJKLMNOPQRSTUVWXYZ

(52) Determine the homotopy type of the complement of the coordinate axes in R^3. (Hint: This space has the same homotopy type as a wedge of some circles.)

(53) Let X be the quotient space obtained by identifying the north and south poles in S^2. Let $Y := \mathsf{R}^3 \setminus S$ where $S := \mathsf{S}^1 \times \{0\} = \{(x,y,0) \in \mathsf{R}^3 : x^2+y^2 = 1\}$. Let $Z := \mathsf{TT} \cup D$ where TT is the tire tube space $\mathsf{TT} := \{(x,y,z) \in \mathsf{R}^3 : (2 - \sqrt{x^2+y^2})^2 + z^2 = 1\}$ and $D := \mathsf{D}^2 \times \{0\} = \{(x,y,0) \in \mathsf{R}^3 : x^2 + y^2 \le 1\}$.

Prove that X, Y, Z all have the same homotopy type.

(Suggestion: the space $\mathsf{PC} := \{(x,y,z) \in \mathsf{R}^3 : (r - \sqrt{x^2+y^2})^2 + z^2 = r^2\}$, with $r = 1$ or $r = 1/2$, might prove helpful.)

(54) Suppose that for all $\lambda \in \Lambda$, X_λ and Y_λ have the same homotopy type. Confirm that the product spaces $\bigtimes X_\lambda$ and $\bigtimes Y_\lambda$ have the same homotopy type.

(55) Let $m, n \in \mathsf{Z}$ with $n > m \ge 0$. Demonstrate that $\mathsf{R}^n \setminus \mathsf{R}^m$ has the same homotopy type as S^{n-m-1}. (Here we view $\mathsf{R}^m \approx \mathsf{R}^m \times \{0\} \subset \mathsf{R}^n$; and $\mathsf{R}^0 := \{0\}$.)

(56) Let $m, n \in \mathsf{N}$. With <u>one</u> exception, each of the following spaces has the homotopy type of a wedge of finitely many circles.

$$\mathsf{R}^2 \setminus \{x_1, \dots, x_m\} \quad , \quad \mathsf{S}^2 \setminus \{z_1, \dots, z_m\} \quad , \quad \mathsf{T}^2_n \setminus \{t\} \quad , \quad \mathsf{P}^2_n \setminus \{p\}\,.$$

Prove these assertions.

(57) Find a space X with a point $a \in X$ such that the inclusion map $\{a\} \hookrightarrow X$ is a homotopy equivalence, but $\{a\}$ is not a (strong) deformation retract of X. Can you find a homotopy equivalence that is not a (weak) deformation retraction?

(58) The ***cone*** over X is the quotient space $\mathsf{Cone}(X) := (X \times [0,1]) \,/\, (X \times \{1\})$.

(a) Prove that the cone over a compact $X \subset \mathsf{R}^n$ is homeomorphic to the ***geometric cone***

$$\mathsf{GC}(X) := \{((1-t)x, t) \in \mathsf{R}^n \times \mathsf{R} = \mathsf{R}^{n+1} : 0 \le t \le 1\}$$

formed by taking the union of all line segments joining points of $X \times \{0\}$ (a subspace of R^{n+1}) to the point $(0, \dots, 0, 1) \in \mathsf{R}^{n+1}$.

(b) Is $\mathsf{Cone}(X)$ homeomorphic to $\mathsf{GC}(X)$ for every closed set $X \subset \mathsf{R}^n$?

(c) Verify that $\mathsf{Cone}(X)$ is compact whenever it is locally compact.

(d) Prove that for *any* space X, $\mathsf{Cone}(X)$ is contractible.

(e) Demonstrate that a continuous map $X \to Y$ is null-homotopic if and only if it extends to a continuous map $\mathsf{Cone}(X) \to Y$.

(59) The ***mapping cylinder*** associated with a continuous map $X \xrightarrow{f} Y$ is defined by

$$\mathsf{MC}_f := ((X \times \mathsf{I}) \sqcup Y)/[(x,0) \sim f(x)]\,.$$

Thus, $\mathsf{MC}_f = Y \sqcup_F (X \times \mathsf{I})$ where $F : X \times \{0\} \to Y$ is $F(x,0) := f(x)$. For example, when $Y = \{pt\}$ is a singleton (so f is a constant map), MC_f is homeomorphic to the cone $\mathsf{Cone}(X)$.

(a) Show that there are natural 'inclusion' maps $X \overset{i}{\hookrightarrow} \mathsf{MC}_f$ and $Y \overset{j}{\hookrightarrow} \mathsf{MC}_f$ that are embeddings of X, Y onto closed subspaces $\tilde{X}, \tilde{Y}$ of MC_f.

(b) Verify that $X \xrightarrow{i} \mathsf{MC}_f$ is homotopic to $X \xrightarrow{j \circ f} \mathsf{MC}_f$.

(c) Show that the projection $X \times \mathsf{I} \to X \times \{0\}$ induces a natural retraction $\mathsf{MC}_f \xrightarrow{r} Y$ with the property that $f = r \circ i$.

(d) Confirm that $\mathsf{MC}_f \simeq Y$ by demonstrating that Y is a strong deformation retract of MC_f. More precisely, construct a strong deformation retraction $R : \mathsf{MC}_f \times \mathsf{I} \to \mathsf{MC}_f$ which illustrates that $1_{\mathsf{MC}_f} \simeq j \circ r$ (rel $\tilde{Y}$).

Thanks to transitivity of $\simeq$, the above tells us that $X \simeq Y$ if and only if $X \simeq \mathsf{MC}_f$. The following further elucidates this phenomena.

(e) Prove that X is a retract of MC_f if and only if f has a left homotopy inverse.

(f) Prove that MC_f is deformable to X if and only if f has a right homotopy inverse.

(60) Prove that X is contractible if and only if, for any constant map $X \xrightarrow{k} \{a\}$, X is a retract of MC_k .

(61) Demonstrate that two spaces X, Y have the same homotopy type if and only if there is a space Z containing both X and Y as deformation retracts.

(62) Recall that $\Pi(X) := \mathcal{P}(X)/\!\sim$ (the set of equivalence classes of paths modulo path homotopy). Define $\Theta : \Pi(X) \to X \times X$ by $\Theta([\alpha]) = (\alpha(0), \alpha(1))$. Demonstrate that:

(a) Θ is surjective if and only if X is path connected.

(b) Θ is bijective if and only if X is simply connected.

(63) Let $x \in X$ and $y \in Y$. Find a natural isomorphism between $\pi_1(X \times Y, (x,y))$ and $\pi_1(X,x) \times \pi_1(Y,y)$.

(64) Let $X \xrightarrow{r} A$ be a retraction of X onto a subspace A. Prove that for each $a \in A$,

$$\pi_1(X,a) \xrightarrow{r_*} \pi_1(A,a)$$

is an epimorphism (i.e., a surjective homomorphism).

(65) Let $a \in A \subset \mathsf{R}^n$ and $(A,a) \xrightarrow{h} (Y,y)$. Suppose h extends to a continuous map $\mathsf{R}^n \to Y$. Prove that h_* is the trivial homomorphism (that maps everything to the identity e in $\pi_1(Y,y)$).

(66) Let $A \subset X$ and write $A \overset{j}{\hookrightarrow} X$ for the inclusion map. Let $X \xrightarrow{f} A$ be continuous. Suppose $H : X \times \mathsf{I} \to X$ is a homotopy between $j \circ f$ and 1_X.

(a) Show that if f is a retraction, then j_* is an isomorphism.

(b) Prove that if $H(A \times \mathsf{I}) \subset A$, then j_* is an isomorphism.

(c) Give an example for which j_* is <u>not</u> an isomorphism.

(67) Let α and β be paths in X from a to b and b to c respectively. Put $\gamma := \alpha * \beta$. Prove that $\Phi_\gamma = \Phi_\beta \circ \Phi_\alpha$. (These are the change of base point homomorphisms.)

(68) Let x, y be two points in a path connected space X.

(a) When will a given pair of paths joining x, y induce the same isomorphism of the fundamental groups $\pi_1(X, x), \pi_1(X, y)$?

(b) When will *all* paths joining x, y induce the same isomorphism of the fundamental groups?

(69) Determine necessary and sufficient conditions for $\pi_1(X, x)$ to be abelian.

(70) Prove that: For a path connected space, the homomorphism of fundamental groups induced by a continuous map is independent of base point, up to isomorphisms of the groups involved.

More precisely, let $X \xrightarrow{f} Y$ be continuous with $f(x_0) = y_0$ and $f(x_1) = y_1$. Let $(f_x)_*$ denote the homomorphism of fundamental groups induced by $f : (X, x) \to (Y, f(x))$. Suppose α is a path in X from x_0 to x_1. Put $\beta := f \circ \alpha$. Prove that

$$\Phi_\beta \circ (f_{x_0})_* = (f_{x_1})_* \circ \Phi_\alpha .$$

In other words, we have the following commutative diagram of group homomorphisms:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(f_{x_0})_*} & \pi_1(Y, y_0) \\ \Big\downarrow \Phi_\alpha & & \Big\downarrow \Phi_\beta \\ \pi_1(X, x_1) & \xrightarrow{(f_{x_1})_*} & \pi_1(Y, y_1) \end{array}$$

(71) Suppose $A \overset{j}{\hookrightarrow} X$ is an injection and $X \xrightarrow{r} A$ a retraction. Let $a \in A$. Assume that $j_*[\pi_1(A, a)]$ is a normal subgroup of $\pi_1(X, a)$. Prove that $\pi_1(X, a)$ is isomorphic to $\mathrm{Im}(j_*) \times \mathrm{Ker}(r_*)$.

(72) Prove that each of the spaces NC, HC, and DHC is simply connected. (These are the natural, harmonic, and doubled harmonic comb spaces as defined in problem #(47).)

(73) Let $X = \mathsf{S}^m \vee \mathsf{S}^n$ with $m, n > 1$. Let x be the 'vertex' of this wedge (i.e., the 'wedge point'). Calculate $\pi_1(X, x)$. (Careful! The union of two simply connected spaces need not be simply connected.)

(74) Suppose $X = U \cup V$ with U, V both open and $U \cap V$ path connected. Consider the inclusion maps $i : U \hookrightarrow X$, $j : V \hookrightarrow X$.

(a) What can you say about the fundamental group of X if:

j_* is the trivial homomorphism? both i_* and j_* are trivial?

(b) Give an example where i_* and j_* are trivial, but neither U nor V is simply connected.

(75) Munkres has two nice problems dealing with special cases of the Seifert-Van Kampen Theorem; see p.433 #'s 1, 2.

(76) Let M, N be connected n-manifolds. Describe the fundamental group of the connected sum $M \# N$.

(77) Determine the fundamental groups of the following spaces:

$$\mathsf{S}^1 \vee \mathsf{S}^1 \quad , \quad \mathsf{S}^2 \vee \mathsf{S}^2 \quad , \quad \mathsf{S}^2 \vee \mathsf{S}^1 \quad , \quad \mathsf{S}^2 \cup I$$
$$\mathsf{T}^2 \vee \mathsf{T}^2 \quad , \quad \mathsf{P}^2 \vee \mathsf{P}^2 \quad , \quad \mathsf{TT} \cup D_1 \quad , \quad \mathsf{TT} \cup D_2$$
$$\mathsf{TT} \cup D_1 \cup D_2 \quad , \quad \mathsf{TT} \cup A \quad , \quad \mathsf{R}^3 \setminus A \quad , \quad \mathsf{R}^4 \setminus L\,.$$

Here S^2 is the unit sphere in R^3, $I := \{(x,y,z) \in \mathsf{R}^3 : x = 0 = y, -1 \le z \le 1\}$, T^2 is the 2-dimensional torus, TT is the usual tire tube space in R^3,

$$D_1 := \{(x,y,z) \in \mathsf{R}^3 : (x-2)^2 + z^2 \le 1, y = 0\},$$
$$D_2 := \mathsf{D}^2 \times \{0\} = \{(x,y,z) \in \mathsf{R}^3 : x^2 + y^2 \le 1, z = 0\},$$

$A := X \cup Y \cup Z$ is the union of the three coordinate axes X, Y, Z in R^3, and $L := \{(0,0,0,w) \in \mathsf{R}^4 : w \in \mathsf{R}\}$ is the 'vertical' axis in R^4.

The space $\mathsf{TT} \cup D_1$ is called a ***torus with a membrane***. What can you say about $\mathsf{TT} \cup D_1$ versus $\mathsf{TT} \cup D_2$? Does it matter whether we add a 'vertical' or a 'horizontal' membrane? (Perhaps the adjectives 'meridianal' and 'latitudinal' are more accurate.) What if we replace the torus with the Klein bottle?

(78) Recall that the ***wedge*** of spaces X_i $(i \in I)$ (with respect to 'base points' $x_i \in X_i$) is defined by

$$Z = \bigvee_{i \in I} X_i := X/A = X/\sim \qquad \text{where} \quad X := \coprod_{i \in I} X_i \ \text{ and } \ A := \{x_i : i \in I\}\,;$$

thus Z is the quotient space of the disjoint union of the spaces X_i where for all $i, j \in I, x_i \sim x_j$. Let $z \in Z$ denote the 'wedge point' (aka, the 'vertex' or 'common point'), i.e., the equivalence class A in Z.

In class we outlined an argument showing that for spaces with non-degenerate base points x_i (meaning that there is an open set $U_i \subset X_i$ such that $\{x_i\}$ is a strong deformation retract of U_i) and a finite index set, say $I = \{1, \dots, N\} \subset \mathsf{N}$,

$$\pi_1(Z,z) \cong \mathop{\ast}_{n=1}^{N} \pi_1(X_n, x_n)\,.$$

Now assume that I is an arbitrary index set. Suppose that for each $i \in I$, $X_i \approx \mathsf{S}^1$. Put $L := \{\ell_i \mid i \in I\}$ where ℓ_i represents the path homotopy class of a loop λ_i in X_i which generates $\pi_1(X_i, x_i)$. Demonstrate that $\pi_1(Z,z)$ is the free group on L.
Hint: First, consider the case where I is a finite set (and use induction). Next, show that L is a set of generators. Recall that (images of) loops and path homotopys are compact (and remember HW#(4)).

(79) Consider the ***Hawaiian earring*** space

$$\mathsf{HE} := \bigcup_{1}^{\infty} C_n \qquad \text{where } C_n \text{ is the circle } C_n := \mathsf{S}^1((1/n, 0); 1/n) \subset \mathsf{R}^2.$$

Let $\ell_n := [\lambda_n]$ where λ_n is a loop which generates $\pi_1(C_n, 0)$.

(a) Confirm that $\pi_1(\mathsf{HE}, 0)$ is *not* generated by $\{\ell_n \mid n \in \mathsf{N}\}$.

(b) According to problem #(78) above, HE is not homeomorphic to any wedge of circles. Provide a direct proof of this.

(c) Demonstrate that $\pi_1(\mathsf{HE}, 0)$ is uncountable. (Hint: Recall that any sequence in $\{0,1\}$ can be identified (via a binary expansion) with a real number in $[0,1]$.)

(d) Explain why the fundamental group of a wedge of countably many circles is countable.

(e) What about the spaces $\mathsf{EE}, \mathsf{R}/\mathsf{Z}, \mathsf{I}/\mathsf{M}$ from last quarter's HW#(70)?

(80) Recall the definition of the pseudo-projective plane PP_n of order n; see problem #(22). Determine the fundamental group of PP_n.

(81) Recall the definition of the dunce cap space DC; see problem #(20) and Figure 1. Determine the fundamental group of DC.

(82) (a) Prove that $< a, b \mid abab^{-1} = 1 >$ is a presentation for the fundamental group of the Klein bottle KB.

(b) Show that $< c, d \mid c^2d^2 = 1 >$ is a presentation for the fundamental group of the surface $\mathsf{P}^2_2 := \mathsf{P}^2 \# \mathsf{P}^2$.

(c) Supposedly, $\mathsf{KB} \approx \mathsf{P}^2_2$, so these two presentations should be the same (i.e., they should define isomorphic groups). Verify this. (Hint: Define c by $a = cb$ and conjugate the relation $abab^{-1} = 1$.)

(83) Find a presentation for the fundamental group of $\mathsf{T}^2 \# \mathsf{P}^2$.

(84) Let $(\tilde{X}, p, X)$ be a covering space.

(a) Let $Y \subset X$, $\tilde{Y} := p^{-1}(Y)$, $q = p|_{\tilde{Y}}$. Confirm that $(\tilde{Y}, q, Y)$ is a covering space.

(b) Suppose that X is connected. Demonstrate that each fibre $p^{-1}(x)$ has the same cardinality.

(c) Suppose that X is connected and locally path connected. Prove that if C is a component of $\tilde{X}$, then $p(C) = X$ and $p|_C : C \to X$ is a covering projection. Is $p|_C$ a homeomorphism?

(85) (a) Check that for each $n \in \mathsf{Z}$ the map $z \mapsto z^n$ is a covering $\mathsf{S}^1 \to \mathsf{S}^1$.

(b) Suppose $p(z)$ is a polynomial in $z \in \mathsf{C}$ of degree n. Let F be the set of critical values of p. What theorem in Complex Analysis (or Advanced Calculus) ensures that $p : \mathsf{C} \setminus p^{-1}(F) \to \mathsf{C} \setminus F$ is an n-sheeted covering projection?

(c) Verify that the map $z \mapsto e^z$ is a covering projection from C onto $\mathsf{C}_* := \mathsf{C} \setminus \{0\}$.

(86) (a) Consider the maps $\mathsf{R}^2 \xrightarrow{p} \mathsf{R} \times \mathsf{S}^1 \xrightarrow{q} \mathsf{S}^1$ given by $p(r,t) := (r, e^{2\pi i t})$ and $q(r,z) := z$. Which of p, q are covering projections?

(b) What happens if we replace the cylinder $\mathsf{R} \times \mathsf{S}^1$ with the infinite Möbius plane MP? Recall that the infinite Möbius plane can be defined via

$$\mathsf{MP} := \mathsf{R}^2/\!\sim \qquad \text{where } (x,y) \sim (x+1, -y).$$

(87) Let $\tilde{X} \xrightarrow{p} X$ be a covering projection with $\tilde{X}$ second countable.

(a) Suppose $\tilde{X}$ is an n-manifold and X is Hausdorff. Show that X is an n-manifold.

(a) Suppose X is an n-manifold. Show that $\tilde{X}$ is an n-manifold.

(88) Let $\tilde{X} := \{(z,w) \in \mathsf{C}^2 : w^2 = z \neq 0\}$. (You can think of $\tilde{X}$ as the graph of the 'two-valued' complex square root 'function'.) Prove that the projection $\mathsf{C}^2 \to \mathsf{C}$ onto the first coordinate restricts to a 2-sheeted covering $\tilde{X} \to \mathsf{C}_* := \mathsf{C} \setminus \{0\}$.

(89) Let $\tilde{X} \xrightarrow{p} X$ be an n-sheeted covering projection. Suppose $\tilde{X}, X, M$ are connected m-manifolds. Prove that $X \# M$ has an n-sheeted covering with total space the connected sum of $\tilde{X}$ together with n copies of M. (Suggestion: Do the connected sum operation inside an evenly covered neighborhood.)

(90) Let $\tilde{X} \xrightarrow{p} X$ be a covering projection. Suppose X is compact and for each $x \in X$, $p^{-1}(x)$ is finite. Prove that $\tilde{X}$ is compact.

Prove that $\tilde{X}$ is compact if and only if p is a finite-sheeted covering.

(91) Let $X \xrightarrow{q} Y \xrightarrow{r} Z$ be covering projections. Suppose that for each $z \in Z$, $r^{-1}(z)$ is finite. Prove that $p := r \circ q$ is a covering map. What if some $r^{-1}(z)$ were not finite?

(92) Suppose $X \xrightarrow{p} Z$, $X \xrightarrow{q} Y$, $Y \xrightarrow{r} Z$ are all continuous maps. Assume that $p = r \circ q$. Consider the assertion: If two of these maps are covering projections, then so is the third. Demonstrate the validity of this claim when the 'two' maps are p, r or p, q. Determine whether or not it holds for q, r. What if you knew, e.g., that r were an n-fold covering?

(93) Let $m, n \in \mathsf{N}$.

(a) Prove that the composition of an m-fold covering and an n-fold covering is an mn-fold covering.

(b) Find an example of two covering projections whose composition is not a covering.

(94) Let $\mathsf{R}_+ := \{r \in \mathsf{R} : r > 0\}$, $\mathsf{R}^2_* := \mathsf{R}^2 \setminus \{0\}$, and define $\mathsf{R} \times \mathsf{R}_+ \xrightarrow{p} \mathsf{R}^2_*$ by

$$p(t,r) := (r\cos(2\pi i t), r\sin(2\pi i t)) = re^{2\pi i t}\,.$$

(a) Prove that p is a covering projection.

(b) Find lifts of the paths $\alpha, \beta, \gamma := \alpha \star \beta$ where $\alpha(t) := (2-t, 0)$ and $\beta(t) := ((1+t)\cos(2\pi i t), (1+t)\sin(2\pi i t)) = (1+t)e^{2\pi i t}$.

(95) Let $(\tilde{X}, p, X)$ and $(\tilde{Y}, q, Y)$ be covering spaces. Show that $(\tilde{X} \times \tilde{Y}, p \times q, X \times Y)$ is a covering space. Use this to find a covering space for the torus.

(96) Consider the covering projection $\mathsf{R}^2 \xrightarrow{\exp \times \exp} \mathsf{T}^2$ where $\exp(t) := e^{2\pi i t}$; cf. #(95). Let λ be the path in $\mathsf{T}^2 = \mathsf{S}^1 \times \mathsf{S}^1$ defined by

$$\lambda(t) := (e^{2\pi i t}, e^{4\pi i t}) = (\exp(t), \exp(2t))\,.$$

Sketch a picture for the trajectory of λ when T^2 is identified with the tire tube space TT. Find a lift $\tilde{\lambda}$ of λ to R^2 and sketch its trajectory.

(97) Recall that there are four different ways of 'defining' the *2-dimensional torus* T^2; see #(10). Each of these 'definitions' provides a different way to determine the fundamental group $\pi_1(\mathsf{T}^2)$.

(a) Use the definition $\mathsf{T}^2 := \mathsf{S}^1 \times \mathsf{S}^1$, in conjunction with HW #(63), to determine $\pi_1(\mathsf{T}^2, (1,1))$.

(b) Consider the covering map $\mathsf{R}^2 \xrightarrow{\exp\times\exp} \mathsf{T}^2$; cf. #(95). Mimic our proof that $\pi_1(\mathsf{S}^1, 1) \cong \mathsf{Z}$ to confirm your answer to (a). Provide a detailed explanation why the 2-torus has an *abelian* fundamental group.

(c) What goes wrong with these arguments if we attempt to use them to find the fundamental group of the Klein bottle? Recall that the Klein bottle can be defined via $\mathsf{KB} = \mathsf{R}^2/\sim$ where $(x+1, -y) \sim (x, y) \sim (x, y+1)$.

(98) Let $\tilde{X} \xrightarrow{p} X$ be a covering projection. Suppose $\tilde{X}$ is path connected and X is simply connected. Demonstrate that p is a homeomorphism.

(99) Let $\tilde{X} \xrightarrow{p} X$ be a covering projection. Suppose $F : \mathsf{I}^2 \to X$ is continuous. Prove that there exists a $\delta > 0$ such that if $S \subset \mathsf{I}^2$ is any $\delta \times \delta$ subsquare of I^2, then $F(S)$ lies inside an evenly covered neighborhood. (A $\delta \times \delta$ square in I^2 is a set of the form $[a, a+\delta] \times [b, b+\delta]$ for some $a, a+\delta, b, b+\delta \in \mathsf{I}$.)

(100) (a) Construct several different 4-fold connected coverings of the figure eight space.

(b) Construct a covering of the plane onto the Klein bottle.

(c) Construct a 2-fold covering of the torus onto the Klein bottle.

(101) Prove continuity for the lift $\tilde{\varphi}$ of φ constructed in our proof of the *Lifting Criterion Theorem*.

(102) Find a covering space $(\tilde{X}, p, X)$ and a map $\tilde{X} \xrightarrow{\varphi} X$ such that φ has no lifting through p, but $\varphi \circ \varphi$ does.

(103) (a) Show that every continuous map $\mathsf{P}^2 \to \mathsf{S}^1$ is null-homotopic.

(b) Find a map $\mathsf{T}^2 \to \mathsf{S}^1$ that is not null-homotopic.

(104) Verify that every map $\mathsf{S}^2 \to \mathsf{T}^2$ is null-homotopic.

(105) Prove that there does not exist a double covering of the Klein bottle onto the torus.

(106) Let $(\tilde{X}, p, X)$, $(\tilde{Y}, q, X)$ be covering spaces and suppose $(\tilde{X}, p, X) \xrightarrow{\varphi} (\tilde{Y}, q, X)$ is a covering morphism. Confirm that $(\tilde{X}, \varphi, \tilde{Y})$ is also a covering space (i.e., φ is a covering projection).

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