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ALGEBRAIC TOPOLOGY HOMEWORK PROBLEMS
SPRING QUARTER 2011

Please provide plenty of details! Pix are definitely kewl ($\smile$).

There are a few warm-up problems on stuff covered Winter Quarter—you might even recognize some of them.

(1) Find a covering space $(\tilde{X}, p, X)$ and a map $\tilde{X} \overset{\varphi}{\to} X$ such that φ has no lifting through p, but $\varphi \circ \varphi$ does.

(2) (a) Show that every continuous map $\mathsf{P}^2 \to \mathsf{S}^1$ is null-homotopic.
(b) Find a map $\mathsf{T}^2 \to \mathsf{S}^1$ that is not null-homotopic.
(c) Verify that every map $\mathsf{S}^2 \to \mathsf{T}^2$ is null-homotopic.

(3) Let $(\tilde{X}, p, X)$, $(\tilde{Y}, q, X)$ be covering spaces and suppose $(\tilde{X}, p, X) \overset{\varphi}{\to} (\tilde{Y}, q, X)$ is a covering morphism. Confirm that $(\tilde{X}, \varphi, \tilde{Y})$ is also a covering space (i.e., φ is a covering projection).

(4) (a) Construct several different 4-fold connected coverings of the figure eight space.
(b) Construct a covering of the plane onto the Klein bottle.
(c) Construct a 2-fold covering of the torus onto the Klein bottle.

(5) Prove that there does not exist a double covering of the Klein bottle onto the torus.

(6) Prove that every double cover (with a connected locally path connected total space) is normal.

(7) Let $(\tilde{X}, \tilde{x}) \overset{p}{\to} (X, x)$ be a covering space. Prove that $G_{\tilde{x}} := p_* \pi_1(\tilde{X}, \tilde{x})$ is a normal subgroup of $\pi_1(X, x)$ if and only if for each pair of points $x_1, x_2 \in p^{-1}(x)$ there is a covering transformation $\varphi \in \mathsf{CT}(p)$ with $\varphi(x_1) = \varphi(x_2)$. Deduce that $\mathsf{CT}(p)$ acts transitively on fibers if and only if p is normal.

(8) Recall that T^2_n denotes the n-holed torus (i.e., the sphere S^2 with n *handles* attached): $\mathsf{T}^2_n = \#_1^n \mathsf{T}^2$.
(a) Construct a double cover $T^2_3 \to \mathsf{T}^2_2$ and describe the monomorphism induced on the fundamental groups.
(b) Construct an n-fold cover $T^2_{n+1} \to \mathsf{T}^2_2$ and describe the monomorphism induced on the fundamental groups.

(9) Construct a double cover of the Klein bottle to itself and describe the monomorphism induced on the fundamental groups.

(10) Examine the various covering spaces of $\mathsf{S}^1 \vee \mathsf{S}^1$ on the handout given to you in class. Which of these have covering space morphisms between them? Which are isomorphic to which? Which have homeomorphic total spaces, yet are non-isomorphic?

Date: April 11, 2011.

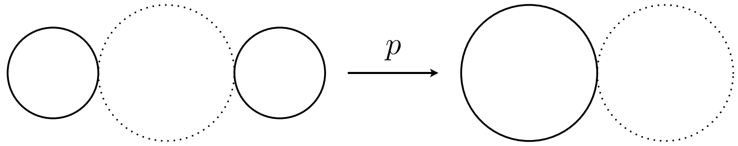

FIGURE 1. A Covering of the Figure Eight Space

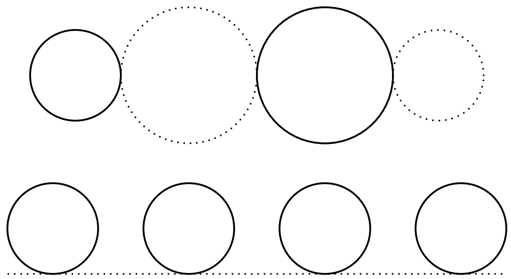

FIGURE 2. Two More Coverings of the Figure Eight

(11) Examine the covering spaces of $\mathsf{FE} := \mathsf{S}^1 \vee \mathsf{S}^1$ on the handout given to you in class.

(a) What are the covering transformation groups?

(b) Which of these are normal covering spaces? What are the corresponding subgroups of $\pi_1(\mathsf{FE})$?

(c) For the non-normal coverings: What are the corresponding conjugacy classes of subgroups of $\pi_1(\mathsf{FE})$?

(12) Figures 1 and 2 show more coverings of $FE := \mathsf{S}^1 \vee \mathsf{S}^1$.

(a) What are the associated covering transformation groups?

(b) Which of these are normal covering spaces? What are the corresponding subgroups of $\pi_1(\mathsf{FE})$?

(c) For the non-normal coverings: What are the corresponding conjugacy classes of subgroups of $\pi_1(\mathsf{FE})$?

(13) (a) Confirm that every homomorphism of $\pi_1(\mathsf{S}^1)$ is induced by some continuous self-map of S^1.

(b) Fill in the details for the example given in class where we listed all possible covering spaces of the circle S^1.

(14) Here we investigate the correspondence between covers of the torus $\mathsf{T}^2 = \mathsf{S}^1 \times \mathsf{S}^1$ and subgroups of $\pi_1(\mathsf{T}^2, (1,1)) \cong \mathsf{Z} \times \mathsf{Z}$.

(a) Find the cover of T^2 corresponding to the subgroup generated by $m \times 0$ where $m \in \mathsf{N}$.

(b) Find the cover of T^2 corresponding to the subgroup generated by $\{m \times 0, 0 \times n\}$ where $m, n \in \mathsf{N}$.

(15) Let $z = (1,1) \in \mathsf{T}^2$.

(a) Prove that every automorphism of $\pi_1(\mathsf{T}^2, z)$ is induced by some self-homeomorphism of T^2 that maps z to itself.

(b) Prove that every covering of T^2 is homeomorphic to either R^2 or $\mathsf{S}^1 \times \mathsf{R}^1$ or T^2. (Hint from algebra: If F is a free abelian group of rank 2 and $G < F$ is a non-trivial subgroup, then there is a basis a, b for F such that either (i) for some $m \in \mathsf{N}$, $\{ma\}$ is a basis for G or (ii) for some $m, n \in \mathsf{N}$, $\{ma, nb\}$ is a basis for G.)

(16) Determine all covering spaces of the torus T^2. Note that this problem is *different* from the similar one right above; here you're being asked to determine all covering spaces $(\tilde{\mathsf{T}}, p, \mathsf{T}^2)$, so not only do you want to find $\tilde{\mathsf{T}}$ but also p. For each such $(\tilde{\mathsf{T}}, p, \mathsf{T}^2)$ you should also identify the corresponding conjugacy class of subgroups of $\pi(\mathsf{T}^2)$.

(17) Now do the same thing as above for the Klein bottle KB.

(18) (a) Find all possible double covers of $\mathsf{S}^1 \vee \mathsf{S}^1$ and their associated conjugacy classes. What is the algebraic significance of your list?

(b) Find all possible double covers of $\mathsf{S}^1 \vee \mathsf{S}^1 \vee \mathsf{S}^1$ and their associated conjugacy classes. What is the algebraic significance of your list?

(19) Find all 3-sheeted covering spaces of the figure-eight space $\mathsf{S}^1 \vee \mathsf{S}^1$. For each covering space identify the corresponding conjugacy class of subgroups of $\pi(\mathsf{S}^1 \vee \mathsf{S}^1)$. What is the algebraic significance of your results?

(20) Recall that the fundamental group G of the Klein bottle KB has a presentation $G =< a, b \mid aba = b >$

(a) Determine the covering spaces $A \xrightarrow{p} \mathsf{KB}$ and $B \xrightarrow{q} \mathsf{KB}$ that correspond to the infinite cyclic subgroups $< a >$ and $< b >$ of G.

(b) Are there any other (i.e., non-equivalent) coverings $A \to \mathsf{KB}$ or $B \to \mathsf{KB}$ with the same total spaces A, B as in (b)?

(c) Find the covering space $T \xrightarrow{r} \mathsf{KB}$ that corresponds to the subgroup of G generated by a and b^2. (Notice that $ab^2 = b^2a$, right?)

(Hint: What are the covering transformations for the universal cover of KB?)

(21) (a) Find spaces whose fundamental groups are isomorphic to: $n\mathsf{Z} \times m\mathsf{Z}$, $n\mathsf{Z} \star m\mathsf{Z}$

(b) Prove that for each finitely presented group G, there is a compact Hausdorff space whose fundamental group is isomorphic to G.

(22) Give a list of all possible covering spaces with base space the n-dimensional torus $\mathsf{T}^n = \mathsf{S}^1 \times \cdots \times \mathsf{S}^1$ (with n factors). How might you go about *proving* that your list is correct?

(23) For each pair X, Y of spaces chosen from R^2, S^2, T^2, P^2, KB, determine whether or not there is a covering projection $X \to Y$.

(24) (a) Find a group H of self-homeomorphisms of the torus T^2, with order 2, such that $\mathsf{T}^2/H \cong \mathsf{T}^2$.

(b) Find a group H of self-homeomorphisms of the torus T^2, with order 2, such that $\mathsf{T}^2/H \cong \mathsf{KB}$.

(25) Let $(\tilde{X}, p, X)$ be a covering space.

(a) Confirm that p is a homeomorphism if and only if p_* is an isomorphism.

(b) Show that if p is finitely-sheeted, then every covering space morphism from $(\tilde{X}, p, X)$ to itself is actually an isomorphism (i.e., a covering transformation).

(26) Determine universal covering spaces for the following spaces:

$$\mathsf{S}^2 \cup I, \quad \mathsf{S}^2 \vee \mathsf{S}^1, \quad \mathsf{TT} \cup D_1, \quad \mathsf{TT} \cup D_2, \quad \mathsf{TT} \cup D_1 \cup D_2,$$

where S^2 is the unit sphere in R^3, $I := \{(x,y,z) \in \mathsf{R}^3 : x = 0 = y, -1 \le z \le 1\}$, TT is the tire tube space in R^3 obtained by rotating the circle $\{(x,y,z) \in \mathsf{R}^3 : (y-2)^2 + z^2 = 1, x = 0\}$ about the z-axis,

$$D_1 := \{(x,y,z) \in \mathsf{R}^3 : (x-2)^2 + z^2 \le 1, y = 0\},$$
$$D_2 := \{(x,y,z) \in \mathsf{R}^3 : x^2 + y^2 \le 1, z = 0\}.$$

Each space $\mathsf{TT} \cup D_i$ $(i = 1, 2)$ given above is called a ***torus with a membrane***. What can you say about $\mathsf{TT} \cup D_1$ versus $\mathsf{TT} \cup D_2$? Does it matter whether we add a 'vertical' or a 'horizontal' membrane? (Perhaps the adjectives 'meridianal' and 'latitudinal' are more accurate.) What if we replace the torus with the Klein bottle? What is(are) the universal covering space(s) of a Klein bottle with a membrane?

(27) Consider the ***Hawaiian earring*** space

$$\mathsf{HE} := \bigcup_1^\infty C_n \qquad \text{where } C_n \text{ is the circle } C_n := \mathsf{S}^1((1/n, 0); 1/n) \subset \mathsf{R}^2.$$

(a) Prove that HE is not semi-locally simply connected.

(b) Prove that $\mathsf{Cone}(\mathsf{HE})$ is simply connected but not locally simply connected.

(c) Consider the *doubled* Hawaiian earring $\mathsf{DHE} := \mathsf{HE} \cup \mathsf{HE}^*$ where HE^* is the reflection of HE across the y-axis. Let $X \subset \mathsf{R}^3$ be the space obtained by forming the cone over HE with apex $(0,0,1)$ together with the cone over (or should I say *under*) HE^* with apex $(0,0,-1)$. Thus X is a union of two simply connected spaces whose intersection consists of the origin $0 \in \mathsf{R}^3$. Prove that X is not simply connected. (Suggestion: any loop in X, based at 0, that 'goes around' an infinite number of the circles in HE and also 'goes around' an infinite number of the circles in HE^* is not homotopic to a constant.)

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