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# *DIFFERENTIAL TOPOLOGY HOMEWORK PROBLEMS*
# *SPRING QUARTER 2011*

Please provide plenty of details! Pix are definitely kewl ($\ddot\smile$).

(1) (a) Let $R > \varepsilon > 0$ be given. Prove that there is a smooth function $\mathsf{R}^n \xrightarrow{f} \mathsf{R}$ that satisfies:

- for all $x \in \mathsf{R}^n$, $0 \le f(x) \le 1$ ,
- $\{|x| \le \varepsilon\} = f^{-1}(\{1\})$ , and
- $\{|x| \ge R\} = f^{-1}(\{0\})$ .

(Such a function $f$ is called a smooth *bump* function.)

(b) Let $K$ be a compact subset of $\mathsf{R}^n$ and let $U$ be an open neighborhood of $K$. Construct a smooth function $g : \mathsf{R}^n \to [0, 1]$ that satisfies

$$K \subset g^{-1}(1) \quad \text{and} \quad \mathsf{R}^n \setminus U = g^{-1}(0) .$$

(Start by finding such a $g$ with $g^{-1}(0) \subset \mathsf{R}^n \setminus U$.)

(2) Prove that each smooth atlas on a topological manifold determines a unique maximal smooth atlas. Suggestion: Let $\mathcal{A}$ be a smooth atlas on $M$.

(a) Show that $\mathcal{M} := \{(W, \theta) \mid (W, \theta)$ is a coordinate chart for $M$ that is compatible with all charts in $\mathcal{A}\}$ is a smooth atlas on $M$.

(b) Check that $\mathcal{M} \subset \mathcal{A}$ and that $\mathcal{M}$ is maximal.

(c) Show that if $\mathcal{B}$ is any smooth atlas on $M$ that contains $\mathcal{A}$, then $\mathcal{B} \subset \mathcal{M}$.

(d) Finally, check that if $\mathcal{N}$ is a maximal atlas containing $\mathcal{A}$, then $\mathcal{M} \subset \mathcal{N}$.

(3) Let $M, N$ be smooth manifolds. Prove that $M \xrightarrow{\Phi} N$ is a diffeomorphism if and only if $\Phi$ is a bijection and both $\Phi$, $\Phi^{-1}$ are smooth.

(4) Let $(U, \varphi)$ be a coordinate chart on a smooth manifold $M$. Prove that $U \xrightarrow{\varphi} \varphi(U)$ is a diffeomorphism.

(5) Let $M$ be a smooth manifold. Suppose that $\tilde{M} \xrightarrow{p} M$ is a covering projection. Prove that there is a unique smooth atlas for $\tilde{M}$ relative to which $p$ is a local diffeomorphism. (You need to assume either that $\tilde{M}$ is connected or that it is second countable.)

(6) Let $M \xrightarrow{f} \mathsf{R}$ be a differentiable function (so $M$ is a smooth manifold). Suppose that $(x^1, \dots, x^m)$ and $(y^1, \dots, y^m)$ are coordinates on some open set in $M$. Show that at each point $p$ in this open set,

$$\forall\, 1 \le k \le m\,, \quad \frac{\partial f}{\partial y^k}(p) = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i}(p) \frac{\partial x^i}{\partial y^k}(p)\,.$$

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(7) Define an equivalence relation on $\mathsf{R}^{n+1} \setminus \{0\}$ by saying $x \simeq y$ if $x = ty$ for some $t \in \mathsf{R}$. Let $M$ be the quotient space. For each $i$ $(1 \le i \le n+1)$, let $U_i$ be the set of equivalence classes $[z]$ of points $(z_1, \dots, z_{n+1})$ for which $z_i \neq 0$ and define $\varphi_i : U_i \to \mathsf{R}^n$ by

$$\varphi_i\left([z_1, \dots, z_{n+1}]\right) = \left(\frac{z_1}{z_i}, \dots, \frac{\hat{z_i}}{z_i}, \dots, \frac{z_{n+1}}{z_i}\right) \in \mathsf{R}^n,$$

where $.\hat{.}.$ means that we leave this part out. Prove that the collection $\{(U_i, \varphi_i) : 1 \le i \le n+1\}$ is a smooth atlas for $M$. What (familiar) space is $M$?

(8) Recall that the ***general linear group*** $\mathcal{GL}(n, \mathsf{R})$ is the subset of $\mathcal{M}(n, \mathsf{R})$ (the $n \times n$ matrices with real coefficients) of all non-singular $n \times n$ matrices.

(a) Verify that $\mathcal{GL}(n, \mathsf{R})$ is a submanifold of $\mathcal{M}(n, \mathsf{R})$. What is its dimension?

(b) Is $\mathcal{GL}(n, \mathsf{R})$ compact? Is it connected?

(9) Prove that the smooth sphere $\mathsf{S}^n$ (e.g., with a differential structure given by using stereographic projection) is diffeomorphic to $\mathsf{S}^n$ with the differential structure it inherits as a submanifold of $\mathsf{R}^{n+1}$.

(10) (a) Let $(-\varepsilon, \varepsilon) \xrightarrow{\gamma} M$ be a smooth path in some $m$-manifold $M$ with $\gamma(0) = p$. Suppose $(U, \varphi)$ and $(V, \psi)$ are coordinate charts about $p$. Prove that

$$(\psi \circ \gamma)'(0) = D(\psi \circ \varphi^{-1})(\varphi(p))[(\varphi \circ \gamma)'(0)]\,.$$

(b) Let $(-\varepsilon, \varepsilon) \xrightarrow{\alpha,\beta} M$ be smooth paths in some $m$-manifold $M$ with $\alpha(0) = p = \beta(0)$. Suppose $(U, \varphi)$ is a coordinate chart about $p$ and $(\varphi \circ \alpha)'(0) = (\varphi \circ \beta)'(0)$. Prove that for all coordinate charts $(V, \psi)$ about $p$, $(\psi \circ \alpha)'(0) = (\psi \circ \beta)'(0)$.

(11) Let $(U, \phi)$ be a coordinate chart (for some smooth $m$-manifold $M$) centered at the point $p$; so, $\varphi(p) = 0$ and $\{\frac{\partial}{\partial x^i}|_p \mid 1 \le i \le m\}$ is a basis for the tangent space $T_p(M)$, where $(x^1, \dots, x^m) = \varphi$. Let $\{\partial_i := D_{e_i} \mid 1 \le i \le m\}$ be the standard/usual/ordinary basis for $T_0(\mathsf{R}^m)$.

Corroborate that $T_p(M) \xrightarrow{\varphi_*} T_0(\mathsf{R}^m)$ is the map that sends $\left.\dfrac{\partial}{\partial x^i}\right|_p \mapsto \partial_i$ for $1 \le i \le m$.

(12) Let $M$, $N$ be smooth manifolds.

(a) Define a smooth structure for $M \times N$ and verify that

$$T_{(p,q)}(M \times N) \cong T_p(M) \times T_q(N)\,.$$

(b) Let $M \xrightarrow{F} M \times N$ be the map $F(p) := (p, q)$ where $q \in N$ is some fixed point. Check that $F$ is smooth and that $F_* : T_p(M) \to T_{(p,q)}(M \times N)$ is given by $F_*(X) = (X, 0)$.

(c) Let $M \xrightarrow{G} M \times M$ be the map $G(p) := (p, p)$. Show that $G$ is smooth and that $G_*(X) = (X, X)$.

(d) Let $M \xrightarrow{\vartheta} N$ be a smooth map and define $M \xrightarrow{\Theta} M \times N$ by $\Theta(p) := (p, \vartheta(p))$. Prove that $\Theta$ is smooth and that $\Theta_*(X) = (X, \vartheta_*(X))$.

(13) Prove that the product $F_1 \times F_2 : M_1 \times M_1 \to N_1 \times N_2$ is a diffeomorphism if both $F_1 : M_1 \to N_1$ and $F_2 : M_2 \to N_2$ are diffeomorphisms.

(14) Prove that the following maps are smooth, and determine their derivatives:
  (a) the product $\mathcal{GL}(n, \mathsf{R}) \times \mathcal{GL}(n, \mathsf{R}) \to \mathcal{GL}(n, \mathsf{R})$, $(A, B) \mapsto AB$.
  (b) the 'left-translation' $L_A : \mathcal{GL}(n, \mathsf{R}) \to \mathcal{GL}(n, \mathsf{R})$, $M \mapsto L_A(M) := AM$ (for a given $A \in \mathcal{GL}(n, \mathsf{R})$).
  (c) the exponential map $\mathsf{R} \to \mathsf{S}^1$, $t \mapsto \exp(it) = (\cos(t), \sin(t)) \in \mathsf{R}^2$.
  (d) the product $\mathsf{S}^1 \times \mathsf{S}^1 \to \mathsf{S}^1$, $(z, w) \mapsto zw$.
  (e) the 'power' $\mathsf{S}^1 \to \mathsf{S}^1$, $z \mapsto z^n$ (where $n \in \mathsf{Z}$).

(15) (a) Prove that the 'left-translation' $L_A$ is a diffeomorphism.
  (b) Prove that the exponential map $\mathsf{R} \to \mathsf{S}^1$ is a local diffeomorphism.

(16) Let $M \xrightarrow{\vartheta} N$ be a smooth map. Recall that the ***graph*** of $\vartheta$ is

$$\mathrm{Gr}(\vartheta) := \{(p, \vartheta(p)) \in M \times N \mid p \in M\}.$$

Demonstrate that for each $(p, q) \in \mathrm{Gr}(\vartheta)$ we have

$$T_{(p,q)}(\mathrm{Gr}(\vartheta)) = \mathrm{Gr}(\vartheta_*) \subset T_{(p,q)}(M \times N),$$

where, of course, $T_p(M) \xrightarrow{\vartheta_*} T_q(N)$.

(17) Suppose $M \xrightarrow{F} N$ is $\mathcal{C}^\infty$ and $M$ is connected. Prove that $F_* = 0$ if and only if $F$ is a constant map.

(18) Find the derivative $F_*$ of each of the following maps.
  (a) $\mathsf{S}^1 \xrightarrow{F} \mathsf{S}^1$ is $F(z) := \lambda z$ for some fixed $\lambda \in \mathsf{S}^1$.
  (b) $\mathsf{S}^1 \xrightarrow{F} \mathsf{S}^1$ is $F(z) := z^n$ for some fixed $n \in \mathsf{Z}$.
  (c) $\mathsf{T}^2 \xrightarrow{F} \mathsf{S}^1$ is $F(z, w) := zw$.
  (d) $\mathsf{T}^2 \xrightarrow{F} \mathsf{S}^2$ is $F((x, y), (u, v)) := (ux, uy, v)$.

(19) Find non-trivial smooth maps (and calculate their derivatives),

$$\mathsf{S}^2 \to \mathsf{P}^2 \quad \mathsf{R}^2 \to \mathsf{KB}\,, \quad \mathsf{T}^2 \to \mathsf{KB}\,.$$

(20) Recall that the ***special linear group*** $\mathcal{SL}(n, \mathsf{R})$ is the subset of $\mathcal{M}(n, \mathsf{R})$ of matrices having determinant 1.
  (a) Verify that $\mathcal{SL}(n, \mathsf{R})$ is a submanifold of $\mathcal{M}(n, \mathsf{R})$. What is its dimension?
  (b) Is $\mathcal{SL}(n, \mathsf{R})$ compact? Is it connected?
  (c) Determine the tangent space for $\mathcal{SL}(n, \mathsf{R})$ at the identity.

(21) Recall that the ***orthogonal group*** $\mathcal{O}(n)$ is the subset of $\mathcal{M}(n, \mathsf{R})$ of all matrices $A$ satisfying $AA^{\mathrm{tr}} = I$.
  (a) Verify that $\mathcal{O}(n)$ is an $n(n-1)/2$-dimensional submanifold of $\mathcal{M}(n, \mathsf{R})$.
  (b) Is $\mathcal{O}(n)$ compact? Is it connected?
  (c) Determine the tangent space for $\mathcal{O}(n)$ at the identity.

(22) Prove that $\mathcal{GL}(n, \mathsf{R})$ is homeomorphic to $\mathcal{O}(n) \times \mathsf{R}^{n(n-1)/2}$.
(Hint: See Spivak, p.144, #31(h).)

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