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TOPOLOGY HOMEWORK PROBLEMS
AUTUMN QUARTER 2010

Please provide plenty of details! Pix are definitely kewl ($\ddot\smile$).

(1) Define $\mathbb{N}\times\mathbb{N}\xrightarrow{d}\mathbb{R}$ by $d(m,n):=\left|\dfrac{1}{m}-\dfrac{1}{n}\right|$.

(a) Verify that d is a distance function on $\mathbb{N}$.

(b) Is the sequence $(n)_{n=1}^{\infty}$ a Cauchy sequence in $(\mathbb{N},d)$? Does it converge in $(\mathbb{N},d)$?

(2) Consider the identity map $(X,d_1)\xrightarrow{\mathsf{id}}(X,d_2)$ where d_1 and d_2 are distance functions on some non-empty set X. Give conditions (both necessary and sufficient, if possible) that describe when: id is a continuous map, id is an open map, id is a homeomorphism.

(3) Let $S\xrightarrow{f}X$ be a function from some set S to some set X. Suppose d is a distance function on X. Define $d_f : S\times S\to\mathbb{R}$ by $d_f(s,t):=d(f(s),f(t))$. Determine conditions on f that guarantee that d_f is a distance function on S. (The metric d_f is called the ***pullback of*** d ***by*** f.)

(4) Let P be *any* set of positive numbers. Prove that there exists a metric space (X,d) with the property that $\{d(x,y)\mid x,y\in X\}=P\cup\{0\}$.

(5) List the distinct topologies on the set $\{a,b\}$.

(6) List the distinct topologies on the set $\{a,b,c\}$.

(7) Let $\mathcal{C}$ be a collection of subsets of some set X. Prove that there is a unique smallest topology on X that contains $\mathcal{C}$; this is called the ***topology generated by*** $\mathcal{C}$.

(8) Let $\{\mathcal{T}_\alpha\}$ be a collection of topologies on some non-empty set X.

(a) Prove that $\bigcap_\alpha \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup_\alpha \mathcal{T}_\alpha$ a topology on X?

(b) Show that there is a unique smallest topology on X that contains each of the $\mathcal{T}_\alpha$, and there is a unique largest topology on X that is contained in each $\mathcal{T}_\alpha$.

(c) Suppose $X:=\{a,b,c\}$ and $\mathcal{T}_1:=\{X,\emptyset,\{a\},\{a,b\}\}$, $\mathcal{T}_2:=\{X,\emptyset,\{a\},\{b,c\}\}$. Find the smallest topology on X that contains both $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology on X contained in each of $\mathcal{T}_1$ and $\mathcal{T}_2$.

(9) Let $\mathcal{B}$ be a basis for some topology $\mathcal{T}$. Prove that $\mathcal{T}$ is the topology generated by $\mathcal{B}$.

(10) Let $\mathcal{B}$ be a basis for some topology $\mathcal{T}$ on some set X. Prove that for each $A\subset X$, the following are equivalent:

(a) $A\in\mathcal{T}$.

(b) $\forall\, a\in A\,,\exists\, U\in\mathcal{T}$ such that $a\in U\subset A$.

(c) A is the union of elements of $\mathcal{T}$.

(d) $\forall\, a\in A\,,\exists\, B\in\mathcal{B}$ such that $a\in B\subset A$.

(e) A is the union of elements of $\mathcal{B}$.

Formulate a Lemma that gives both (b) $\iff$ (c) and (d) $\iff$ (e).

Date: January 4, 2011.

(11) Prove that each of the following collections is a basis for the standard topology on $\mathbb{R}$.
(a) $\{(a,b) \mid a,b \in \mathbb{R} \text{ with } a<b\}$.
(b) $\{(p,q) \mid p,q \in \mathbb{Q} \text{ with } p<q\}$.
(c) $\{(r+1/n, r-1/n) \mid r \in \mathbb{Q}, n \in \mathbb{N}\}$.
Notice that the latter two collections are countable!

(12) Demonstrate that $\{[a,b) \mid a,b \in \mathbb{R} \text{ with } a<b\}$ is a basis for a topology $\mathcal{T}_{\text{low}}$ (called the ***lower limit topology***) on $\mathbb{R}$. Often we write $\mathbb{R}_{\text{low}}$ to indicate the set $\mathbb{R}$ together with its lower limit topology.

Is the collection $\{[p,q) \mid p,q \in \mathbb{Q} \text{ with } p<q\}$ a basis for the lower limit topology on $\mathbb{R}$?

(13) Show that $\{(r,s) \mid 0<r<s<1\} \cup \{[0,r) \cup (s,1) \mid 0<r<s<1\}$ is a basis for a topology on $[0,1)$. Do you "see" what space we get with this topology?

(14) Let $p := (0,1) \in \mathbb{R}^2$ and put $X := \mathbb{R} \cup \{p\}$.

(a) Show that $\{(a,b) \mid a,b \in \mathbb{R} \text{ with } a<b\} \cup \{(-r,0) \cup \{p\} \cup (0,r) \mid r>0\}$ is a basis for a topology on X. (We call X, with this topology, the *line with two origins*.)

(b) Show that $\{(a,b) \mid a,b \in \mathbb{R} \text{ with } a<b\} \cup \{X \setminus [-r,r] \mid r>0\}$ is a basis for a topology on X. Do you "see" what space X is with this topology?

(15) Let $p := (0,0,1) \in \mathbb{R}^3$ and put $X := \mathbb{R}^2 \cup \{p\}$. Show that

$$\{\mathbb{B}^2(z;r) \mid z \in \mathbb{R}^2, r>0\} \bigcup \{X \setminus \mathbb{D}^2(z;r) \mid z \in \mathbb{R}^2, r>0\}$$

is a basis for a topology on X. Do you "see" what space X is with this topology?

(16) Let $\mathcal{S}$ be a collection of subsets of some set X. Let $\mathcal{B}$ be the collection of all sets that can be expressed as a finite intersection of elements of $\mathcal{S}$; thus $B \in \mathcal{B}$ if and only if there are $S_1, \ldots, S_n \in \mathcal{S}$ with $B = S_1 \cap \cdots \cap S_n$.

If $\mathcal{B}$ is a basis for some topology $\mathcal{T}$ on X, then we call $\mathcal{S}$ a ***subbasis for*** $\mathcal{T}$. What conditions on $\mathcal{S}$ and/or $\mathcal{T}$ ensure that $\mathcal{S}$ is a subbasis for $\mathcal{T}$?

Prove that if a collection $\mathcal{S}$ of subsets of some set X is a subbasis for a topology $\mathcal{T}$ on X, then $\mathcal{T}$ is the topology generated by $\mathcal{S}$. (Recall (#7).) When does the converse hold?

(17) Prove that $\mathcal{S} := \{(a,\infty) \mid a \in \mathbb{R}\} \cup \{(-\infty,b) \mid b \in \mathbb{R}\}$ is a subbasis for the standard topology on $\mathbb{R}$.

(18) Let $\mathcal{H}$ be the collection of all open half-planes in $\mathbb{R}^2$ each determined by a horizontal or vertical line. Prove that $\mathcal{H}$ is a subbasis for the standard topology on $\mathbb{R}^2$.

(19) Let $\mathcal{C} := \mathcal{C}([0,1],\mathbb{R}) := \{[0,1] \xrightarrow{f} \mathbb{R} \mid f \text{ is continuous}\}$. For each $x \in [0,1]$ and each open $U \subset \mathbb{R}$, put

$$S(x;U) := \{f \in \mathcal{C} \mid f(x) \in U\}.$$

Prove that $\mathcal{S} := \{S(x;U) \mid x \in [0,1], U \subset \mathbb{R} \text{ open}\}$ is a subbasis for a topology $\mathcal{T}_{\text{po}}$ on $\mathcal{C}$; we call $\mathcal{T}_{\text{po}}$ the ***point-open topology*** on $\mathcal{C}([0,1],\mathbb{R})$.

Prove that a sequence $(f_n)_{n=1}^{\infty}$ in $\mathcal{C}$ converges to some $f \in \mathcal{C}$ with respect to the point-open topology $\mathcal{T}_{\text{po}}$ if and only if $(f_n)_{n=1}^{\infty}$ converges pointwise to f on $[0,1]$. (For this reason, $\mathcal{T}_{\text{po}}$ is often called the ***topology of pointwise convergence***.)

What happens if we replace $[0,1]$ by $\mathbb{R}$? What can you say about convergence in $\mathcal{C}(\mathbb{R},\mathbb{R})$ with respect to its point-open topology?

(20) Let $\mathcal{C} := \mathcal{C}([0,1],\mathbb{R}) := \{[0,1] \xrightarrow{f} \mathbb{R} \mid f \text{ is continuous}\}$. For each compact $C \subset [0,1]$ and each open $U \subset \mathbb{R}$, put

$$S(C;U) := \{f \in \mathcal{C} \mid f(C) \in U\}\,.$$

Prove that $\mathcal{S} := \{S(C;U) \mid C \subset [0,1] \text{ compact}\,,\ U \subset \mathbb{R} \text{ open}\}$ is a subbasis for a topology $\mathcal{T}_{\mathrm{co}}$ on $\mathcal{C}$; we call $\mathcal{T}_{\mathrm{co}}$ the ***compact-open topology*** on $\mathcal{C}([0,1],\mathbb{R})$.

Prove that a sequence $(f_n)_{n=1}^{\infty}$ in $\mathcal{C}$ converges to some $f \in \mathcal{C}$ with respect to the compact-open topology $\mathcal{T}_{\mathrm{co}}$ if and only if $(f_n)_{n=1}^{\infty}$ converges uniformly to f on $[0,1]$. (For this reason, the compact-open topology $\mathcal{T}_{\mathrm{co}}$, on $\mathcal{C}([0,1],\mathbb{R})$, could be called the ***topology of uniform convergence*** on $\mathcal{C}([0,1],\mathbb{R})$.)

What happens if we replace $[0,1]$ by $\mathbb{R}$? What can you say about convergence in $\mathcal{C}(\mathbb{R},\mathbb{R})$ with respect to its compact-open topology?

(21) Let (X,d) be a metric space. Prove that $X \times X \xrightarrow{d} \mathbb{R}$ is continuous.

(22) Let $X \xrightarrow{f} Y$ be a map between topological spaces. Suppose $\mathcal{B}$ and $\mathcal{S}$ are a basis and a subbasis (respectively) for the topology on Y. Demonstrate that the following are equivalent:
(a) f is continuous.
(b) $\forall\, B \in \mathcal{B}\,,\ f^{-1}(B)$ is open in X.
(c) $\forall\, S \in \mathcal{S}\,,\ f^{-1}(S)$ is open in X.

(23) Consider the identity map $(X,\mathcal{T}_1) \xrightarrow{\mathsf{id}} (X,\mathcal{T}_2)$ where $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on some non-empty set X. Give conditions (both necessary and sufficient, if possible) that describe when: id is a continuous map, id is an open map, id is a homeomorphism.

Recall that U is a ***neighborhood*** of a point x (in some topological space) provided U is open and $x \in U$. If U happens to belong to an understood basis, we also call it a ***basis neighborhood***.

We say that a map f is continuous at a point x (in some topological space) provided for each neighborhood V of $f(x)$ there is a neighborhood U of x such that $f(U) \subset V$. Here either (or both) of the terms 'neighborhood' can be replace by 'basis neighborhood'. Right?

(24) Prove that a map $X \xrightarrow{f} Y$ between topological spaces is continuous if and only if for each x in X, f is continuous at x.

(25) Let X, Z be topological spaces. Assume $A \subset X$, $B \subset Y \subset Z$ are each given their subspace topologies. Let $X \xrightarrow{f} Y$ be continuous.

(a) Prove that the inclusion map $A \xhookrightarrow{j} X$, defined by $j(x) := x$, is continuous.

(b) Prove that $A \xrightarrow{f|_A} Y$ is continuous.

(c) If $f(X) \subset B$, prove that $X \xrightarrow{f} B$ is continuous.

(c) Prove that $X \xrightarrow{f} Z$ is continuous.

Thus restricting the domain or restricting the target or expanding the target does not destroy continuity!

(26) Let $X \xrightarrow{f} Y$ be a map between two sets. Suppose that Y is a topological space. Find the smallest (i.e., coarsest) topology on X that makes f a continuous map.
(Hint: What sets must be open?)

A map $X \xrightarrow{f} Y$ is called an ***embedding*** if $X \xrightarrow{f} f(X)$ is a homeomorphism.

(27) Define $\mathbb{S}^1 \times \mathbb{R} \xrightarrow{f} \mathbb{R}^2$ by $f(z, r) := e^r z$. Prove that f is an embedding. What is $f(\mathbb{S}^1 \times \mathbb{R})$?
Construct a similar embedding $\mathbb{S}^n \times \mathbb{R} \to \mathbb{R}^{n+1}$, and determine the image of $\mathbb{S}^n \times \mathbb{R}$.

(28) Define $\mathbb{D}^2 \xrightarrow{F} \mathbb{R}^3$ by

$$(x, y, z) := F(r\cos\theta, r\sin\theta) := (k(r)\cos\theta, k(r)\sin\theta, h(r))$$

where, for $0 \le r \le 1$,

$$h(r) := 2r - 1 \quad \text{and} \quad k(r) := \sqrt{1 - h(r)^2}\,.$$

(a) Explain why F maps the circle $\mathbb{S}^1(0; r)$ to a circle in $\mathbb{R}^3$ at 'height' $h(r)$.
(b) Prove that F is continuous and determine $F(\mathbb{D}^2)$.
(c) Is $F : \mathbb{D}^2 \to \mathbb{R}^3$ an embedding?
(d) Is $F : \mathbb{B}^2 \to \mathbb{R}^3$ an embedding?

(29) In each of the following lists, determine which spaces are homeomorphic to which; construct the maps!
(a) $(0, 1), (0, 1], [0, 1], \mathbb{R}$
(b) $\mathbb{B}^2, \mathbb{D}^2, \mathbb{R}^2, \mathbb{S}^2 \setminus \{\text{pt}\}, \mathbb{S}^2_+ := \{(x, y, z) \in \mathbb{S}^2 \mid z \ge 0\}$

(30) Classify, up to homeomorphisms, the non-empty intervals (open, closed, neither, finite, or infinite) in $\mathbb{R}$.

(31) Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. Show that the collection

$$\mathcal{B} := \{U \times V \mid U \in \mathcal{T}\,,\ V \in \mathcal{U}\}$$

is a basis for a topology on $X \times Y$.
This is called the ***product topology*** on $X \times Y$. A word of caution is in order here: this description of the product topology is only valid for finite products. That is, if $(X_1, \mathcal{T}_1), \dots, (X_n, \mathcal{T}_n)$ are topological spaces, then

$$\mathcal{B} := \{U_1 \times \cdots \times U_n \mid U_i \in \mathcal{T}_i\}$$

is a basis for the product topology on $X_1 \times \cdots \times X_n$. But this is **not** the correct definition for the product topology on an *infinite* product space.

(32) Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. Show that the collection

$$\mathcal{S} := \{U \times Y \mid U \in \mathcal{T}\} \bigcup \{X \times V \mid V \in \mathcal{U}\}$$

is a subbasis for the product topology on $X \times Y$.

(33) Prove that the usual (metric) topology on $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ is the same as the product topology. That is, show that the appropriate identity map is a homeomorphism.

(34) Prove that a subspace of a Hausdorff space is Hausdorff, and that a product of two Hausdorff spaces is Hausdorff.

(35) Prove that a topological space X is a Hausdorff space if and only if

$$\Delta := \{(x,x) \mid x \in X\} \quad \text{is a closed subset of } X \times X\,.$$

(36) Show that the Continuity Test (part of the Characteristic Property) for Product Spaces fails if we use the "box topology" instead of the product topology.

(37) Let X be a topological space, Λ a non-empty set, and consider the product space X^Λ. Thus

$$X^\Lambda = \bigtimes_{\lambda\in\Lambda} X_\lambda \quad \text{where each set } X_\lambda := X\,.$$

Consider the map $X \xrightarrow{f} X^\Lambda$ defined by $f(x) := (x_\lambda)_{\lambda\in\Lambda}$ where each $x_\lambda := x$.
Is f an embedding?

(38) Let $\{(X_\lambda, \mathcal{T}_\lambda) \mid \lambda \in \Lambda\}$ be a collection of topological spaces. Let $\bigtimes_{\lambda\in\Lambda} X_\lambda \xrightarrow{\pi_\mu} X_\mu$ denote the usual projection map. Prove that π_μ is a continuous open map.

(39) Let $\{(X_\lambda, \mathcal{T}_\lambda) \mid \lambda \in \Lambda\}$ be a collection of topological spaces. Suppose $\bigtimes_{\lambda\in\Lambda} X_\lambda \xrightarrow{f} Y$ is continuous. Demonstrate that for each $\mu \in \Lambda$ and each fixed $(a_\lambda) \in \bigtimes_{\lambda\in\Lambda} X_\lambda$, the map $X_\mu \xrightarrow{f_\mu} Y$ defined by

$$f_\mu(x) := f((x_\lambda)) \quad \text{where } x_\mu := x \text{ and for } \lambda \neq \mu,\ x_\lambda := a_\lambda$$

is continuous. Does the converse hold? (This is an Advanced Calculus question!)

(40) Let $\{(X_\lambda, \mathcal{T}_\lambda) \mid \lambda \in \Lambda\}$ be a collection of topological spaces. Suppose that for each $\lambda \in \Lambda$ there is a non-empty subset $A_\lambda \subset X_\lambda$. Prove that

$$\mathsf{int}\left[\bigtimes_{\lambda\in\Lambda} A_\lambda\right] \subseteq \bigtimes_{\lambda\in\Lambda} \mathsf{int}[A_\lambda]$$

and equality may not hold, but always

$$\mathsf{cl}\left[\bigtimes_{\lambda\in\Lambda} A_\lambda\right] = \bigtimes_{\lambda\in\Lambda} \mathsf{cl}[A_\lambda]\,.$$

What happens if we use the "box topology" on $\bigtimes_{\lambda\in\Lambda} X_\lambda$?

(41) Recall that we can view the product set $\mathbb{R}^\mathbb{N}$ as the set of all sequences in $\mathbb{R}$. Let $\mathbb{R}^\infty$ denote the subset of $\mathbb{R}^\mathbb{N}$ that consists of all sequences that are 'eventually zero'. Thus $(a_n)_1^\infty$ belongs to $\mathbb{R}^\infty$ precisely when there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n = 0$.

Determine the closure of $\mathbb{R}^\infty$ with respect to each of the product and box topologies on $\mathbb{R}^\mathbb{N}$.

(42) Let $\{(X_\lambda, \mathcal{T}_\lambda) \mid \lambda \in \Lambda\}$ be a collection of topological spaces and $\{Z \xrightarrow{g_\lambda} X_\lambda \mid \lambda \in \Lambda\}$ a collection of continuous maps. Prove that there is a unique continuous map

$$Z \xrightarrow{g} \bigtimes_{\lambda\in\Lambda} X_\lambda$$

with the property that for all $\lambda \in \Lambda$, $g_\lambda = \pi_\lambda \circ g$ (here π_λ are the usual projections).

(43) Let $\{Z_\lambda \xrightarrow{f_\lambda} X_\lambda \mid \lambda \in \Lambda\}$ a collection of continuous maps. Prove that there is a unique continuous map

$$Z := \bigtimes_{\lambda\in\Lambda} Z_\lambda \xrightarrow{f} \bigtimes_{\lambda\in\Lambda} X_\lambda =: X$$

with the property that for all $\lambda \in \Lambda$, $\rho_\lambda = \pi_\lambda \circ f$; here $\pi_\lambda : X \to X_\lambda$ and $\rho_\lambda : Z \to Z_\lambda$ are the usual projections. We write $f := \bigtimes_{\lambda\in\Lambda} f_\lambda$.

Deduce that when each f_λ is a homeomorphism, so is f.

(44) Let $\{(X_\lambda, \mathcal{T}_\lambda) \mid \lambda \in \Lambda\}$ be a collection of topological spaces. Suppose for each $\lambda \in \Lambda$, Z_λ is a subspace of X_λ. Prove that

$$\bigtimes_{\lambda\in\Lambda} Z_\lambda \text{ is a subspace of } \bigtimes_{\lambda\in\Lambda} X_\lambda\,.$$

(45) Prove that for any topological spaces X, Y, Z

$$X \times Y \approx Y \times X \quad \text{and} \quad (X \times Y) \times Z \approx X \times (Y \times Z)\,.$$

(46) Use pictures to show that $[0, 1] \times [0, 1) \approx (0, 1) \times [0, 1)$.

Thus while $\bigtimes$, is "associative" and "commutative", there is no "cancellation law".

(47) Let X, Y be the closed annuli with *matched* and *mismatched fins* pictured in Figure 1. Draw sketches that suggest a homeomorphism between $X \times [0, 1]$ and $Y \times [0, 1]$. Are X and Y homeomorphic?

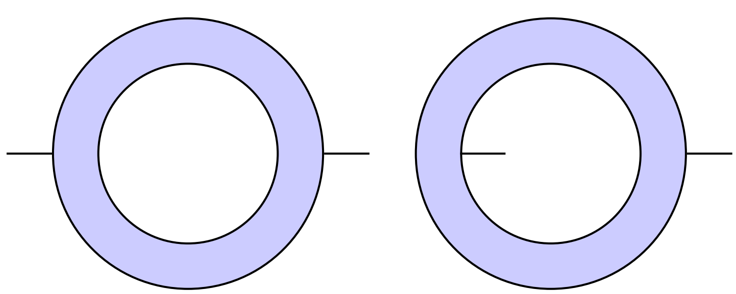
(48) Let W denote the "torus surface" (i.e., the tire tube space in $\mathbb{R}^3$) with an open disk removed (from the surface). Let Z denote the closed unit disk with two smaller open disks removed. Draw sketches that suggest a homeomorphism between $W \times [0, 1]$ and $Z \times [0, 1]$. Are W and X homeomorphic?

(49) Show that for *any* two topological spaces X and Y, there exists a space Z such that $X \times Z \approx Y \times Z$.

(50) Let L be a line in the plane $\mathbb{R}^2$. Describe the subspace topologies on $L \subset \mathbb{R}_{\text{low}} \times \mathbb{R}$ and on $L \subset \mathbb{R}_{\text{low}} \times \mathbb{R}_{\text{low}}$. (These are familiar topologies!)

(51) The *dictionary order* $\prec$ on $\mathbb{R}^2$ is defined by

$$(x, y) \prec (x', y') \iff x < x' \;\text{ or }\; x = x' \text{ and } y < y'\,.$$



X with matched fins $\quad$ Y with mismatched fins

FIGURE 1. Closed annuli with fins

This defines a *total ordering* $\preccurlyeq$ on $\mathbb{R}^2$, and this total ordering induces the so-called *dictionary order topology* $\mathcal{T}_{\text{do}}$ on $\mathbb{R}^2$. (See Lee, especially problem (2-12) on p.37.)

We write $\mathbb{R}^2_{\text{do}} := (\mathbb{R}^2, \mathcal{T}_{\text{do}})$ to denote the set $\mathbb{R}^2$ with its dictionary order topology.

Compare the following topologies on $\mathbb{R}^2$:
(a) $\mathcal{T}_{\text{do}}$
(b) the standard topology
(c) the product topology on $\mathbb{R}_{\text{disc}} \times \mathbb{R}$
(d) the topology described in problem (2-5) on p.37 of Lee

(52) The *dictionary order* on $\mathbb{I}^2$ is just the restriction of the dictionary order on $\mathbb{R}^2$ to $\mathbb{I}^2$, and so $\mathbb{I}^2$ also has a *dictionary order topology.*

Compare the following topologies on $\mathbb{I}^2$:
(a) the standard topology
(b) the dictionary order topology
(c) the subspace topology that $\mathbb{I}^2$ inherits as a subset of $\mathbb{R}^2_{\text{do}}$

(53) Define an equivalence relation on $\mathbb{R}$ by writing $x \sim y$ if $y - x \in \mathbb{Q}$. Prove that $\mathbb{R}/\!\sim$ is an uncountable space with the trivial (i.e., indiscrete) topology.

(54) Define an equivalence relation on $\mathbb{R}^2$ by writing $(x, y) \sim (x', y')$ if $x' - x \in \mathbb{Z}$. Prove that $\mathbb{R}^2/\!\sim$ is a surface (i.e., a 2-manifold). What surface is it?

(55) Define an equivalence relation on $\mathbb{R}^2$ by writing $(x, y) \sim (x', y')$ if $n := x' - x \in \mathbb{Z}$ and $y' = (-1)^n y$. Prove that $\mathbb{R}^2/\!\sim$ is a surface (i.e., a 2-manifold). What is it?

(56) There are four different ways of describing the ***2-dimensional torus*** $\mathbb{T}^2$.
- as the product space $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$, a subspace of $\mathbb{R}^2 \times \mathbb{R}^2$;
- as the ***tire tube surface*** TT in $\mathbb{R}^3$ obtained by rotating the circle $\{(x, y, z) \in \mathbb{R}^3 : (y-2)^2 + z^2 = 1, x = 0\}$ about the z-axis; or, more simply,
$$\mathsf{TT} := \{(x, y, z) : (2 - \sqrt{x^2 + y^2})^2 + z^2 = 1\}\,;$$
- as the quotient space $\mathbb{I}^2/\sim$ where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$, which is called the ***flat torus***;
- as the orbit space $\mathbb{R}^2/\Gamma$ where Γ is the group of all horizontal and vertical translations $(x, y) \mapsto (x + m, y + n)$ with $m, n \in \mathbb{N}$; equivalently, $\mathbb{R}^2/\sim$ where $(x + 1, y) \sim (x, y) \sim (x, y + 1)$.

Demonstrate that these four spaces are homeomorphic.

(57) The ***Klein bottle*** KB is the quotient space obtained from the square $\mathbb{I}^2$ via the boundary identifications $(0, y) \sim (1, 1 - y)$ and $(x, 0) \sim (x, 1)$. Prove that KB is a surface.

(58) Let A be a non-degenerate closed annulus in the plane and define an equivalence relation on A by identifying antipodal points on the outer circle and also identifying antipodal points on the inner circle. Show that the resulting quotient space is homeomorphic to the Klein bottle KB.

(59) Let M be the quotient space obtained from the cube $(-1, 1) \times (-1, 1) \times [-1, 1] \subset \mathbb{R}^3$ by identifying, for each $(x, y) \in (-1, 1)^2$, the points $(x, y, 1)$ and $(-x, y, -1)$. Prove that M is a 3-manifold. (You may stipulate that M is Hausdorff and second countable.)

(60) The ***disjoint union*** $X := \coprod_{\lambda \in \Lambda} X_\lambda$ of an indexed collection of sets $\{X_\lambda \mid \lambda \in \Lambda\}$ is characterized by the following two properties:

(i) For each $\lambda \in \Lambda$, there exist an 'injection' $X_\lambda \xrightarrow{j_\lambda} X$.

(ii) For all sets Z and all functions $X_\lambda \xrightarrow{h_\lambda} Z$, there exist a unique function $X \xrightarrow{h} Z$ such that for all $\lambda \in \Lambda$, $h_\lambda = h \circ j_\lambda$.

$$\begin{array}{ccc} X_\lambda & \xrightarrow{j_\lambda} & X \\ & {\scriptstyle h_\lambda} \searrow \quad \swarrow {\scriptstyle h} & \\ & Z & \end{array}$$

You could/should show that for each $x \in X$, there exists a $\lambda \in \Lambda$ and a point $x_\lambda \in X_\lambda$ with $j_\lambda(x_\lambda) = x$; moreover, if $j_\lambda(x_\lambda) = j_\mu(x_\mu)$, then $\lambda = \mu$ and $x_\lambda = x_\mu$.

(a) Check that when the sets X_λ are disjointed (i.e., for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$, $X_\lambda \cap X_\mu = \emptyset$), we can take $X = \bigcup_\lambda X_\lambda$ (the usual union) and for j_λ we can use the natural inclusion $X_\lambda \hookrightarrow \bigcup_\lambda X_\lambda$.

In practice we can always replace each set X_λ with $X_\lambda \times \{\lambda\}$ to obtain disjointed sets. Then $X := \bigcup_\lambda (X_\lambda \times \{\lambda\})$ serves as the disjoint union $\coprod_\lambda X_\lambda$, and the 'injections' $j_\lambda : X_\lambda \to X$ can be defined by $j_\lambda(x) := (x, \lambda)$.

Now let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a collection of topological spaces. The ***disjoint union topology*** $\mathcal{T}_{\mathrm{du}}$ on $X := \coprod_\lambda X_\lambda$ is the largest (i.e., finest) topology on X with the property that each 'injection' $j_\lambda : X_\lambda \to X$ is continuous. Thus (one can prove that) $U \subset X$ is open in X if and only if for all $\lambda \in \Lambda$, $j_\lambda^{-1}(U)$ is open in X_λ. This means that all of the "topological stuff" occurs in each space X_λ separately.

(b) Check that when the sets X_λ are disjointed, a subset of X is open (i.e., belongs to $\mathcal{T}_{\mathrm{du}}$) if and only if its intersection with each X_λ is open in X_λ, and a subset of X is closed if and only if its intersection with each X_λ is closed in X_λ.

(c) Prove that each 'injection' $X_\lambda \xrightarrow{j_\lambda} X$ is an embedding. Because of this fact, typically one identifies X_λ with its image $j_\lambda(X_\lambda) \subset X$.

(d) Prove the following Continuity Test for the disjoint union topology on $X := \coprod_\lambda X_\lambda$.

For every topological space Z,

$$X \xrightarrow{f} Z \text{ is continuous} \iff \forall \lambda \in \Lambda\,, X_\lambda \xrightarrow{f \circ j_\lambda} Z \text{ is continuous}\,.$$

$$\begin{array}{ccc} X_\lambda & \xrightarrow{j_\lambda} & X \\ & {\scriptstyle f_\lambda} \searrow \quad \swarrow {\scriptstyle f} & \\ & Z & \end{array}$$

(e) Prove that the disjoint union topology is the unique topology on X that enjoys the property given in part (c).

(61) For each $n \in \mathbb{N}$, let $X_n := (0, 1)$. Check that $\mathbb{R} \setminus \mathbb{Z} \approx \coprod_{n \in \mathbb{N}} X_n$. (This is <u>not</u> $\mathbb{R}/\mathbb{Z}$!)

(62) For each of the following sets X_n, describe a set $E \subset \mathbb{R}^2$ such that $E \approx \coprod_{n \in \mathbb{N}} X_n$.

$$X_n := \mathbb{I} \text{ or } \mathbb{S}^1 \text{ or } \mathbb{B}^2 \text{ or } \mathbb{I}^2 \text{ or } \mathbb{R}\,.$$

(63) For each $n \in \mathbb{N}$, let H_n be the hyperbola in $\mathbb{R}^2$ described by $xy = 1/n$. Let A be the union of the two coordinate axes in $\mathbb{R}^2$. Determine whether or not $A \bigsqcup \coprod_n H_n$ and $A \cup \bigcup_n H_n \subset \mathbb{R}^2$ are homeomorphic. (Hint: Argue that any homeomorphism would have to map A to A and thus map $\coprod_n H_n$ to $\bigcup_n H_n$, but ....)

Describe a set $E \subset \mathbb{R}^2$ such that $E \approx A \bigsqcup \coprod_{n \in \mathbb{N}} H_n$.

(64) Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a collection of topological spaces. Select points $a_\lambda \in X_\lambda$ and put $A = \{a_\lambda \mid \lambda \in \Lambda\}$. The ***wedge*** of the spaces X_λ (with respect to the points a_λ) is defined by

$$\bigvee_{\lambda \in \Lambda} X_\lambda := X/A = X/\!\sim \qquad \text{where} \quad X := \coprod_{\lambda \in \Lambda} X_\lambda$$

and $a_\lambda \sim a_\mu$ for all $\lambda, \mu \in \Lambda$. Thus a set in $\bigvee_\lambda X_\lambda$ is open (or closed) if and only if its intersection with each X_λ is open (or closed, resp.) in X_λ.

(a) Consider topological spaces X and Y with distinguished points $a \in X$ and $b \in Y$. Demonstrate that

$$X \vee Y \approx (X \times \{b\}) \cup (\{a\} \times Y)\ ,$$

where the latter space is regarded as a subspace of $X \times Y$.

(b) Formulate a conjecture regarding $X \vee Y \vee Z$. What about $X_1 \vee \cdots \vee X_n$?

(65) Give a 'geometric' description for the following spaces:

$$\mathbb{R}/[0,1]\,,\ \mathbb{R}/(0,1)\,,\ \mathbb{R}/[0,1)\,,\ \mathbb{R}/\{0,1\}\,.$$

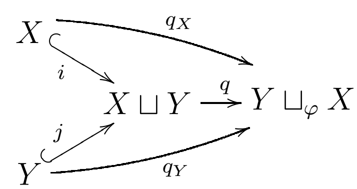
For example, in class we proved that $\mathbb{R}/[0,1] \approx \mathbb{R}$ and we gave a conjecture (but no proof) that $\mathbb{R}/\{0,1\} \approx \mathbb{R} \vee \mathbb{S}^1 \vee \mathbb{R}$.

(66) Let Z be a closed subspace of X and suppose $Z \xrightarrow{\varphi} Y$ is continuous. We construct a space $Y \sqcup_\varphi X$, called the ***adjunction of*** X ***to*** Y ***via*** φ, by attaching X to Y using φ as follows:

$$Y \sqcup_\varphi X \quad \text{is the quotient space} \quad (X \bigsqcup Y)/z \sim \varphi(z)\,;$$

a more precise description of the equivalence relation is that $u \sim v$ if either (i) $u = v$ or (ii) $u, v \in Z$ and $\varphi(u) = \varphi(v)$ or (iii) $u \in Z$ and $v = \varphi(u) \in Y$.

There are natural maps $X \xrightarrow{q_X} Y \sqcup_\varphi X$ and $Y \xrightarrow{q_Y} Y \sqcup_\varphi X$ obtained by precomposing the quotient map $(X \sqcup Y) \xrightarrow{q} (Y \sqcup_\varphi X)$ with the natural inclusions $X \overset{i}{\hookrightarrow} X \sqcup Y$ and $Y \overset{j}{\hookrightarrow} X \sqcup Y$ respectively.



(a) Prove that the natural map $Y \xrightarrow{q_Y} Y \sqcup_\varphi X$ is an embedding onto a closed subspace.

(b) Prove that the natural map $X \setminus Z \xrightarrow{q_X|} Y \sqcup_\varphi X$ is an embedding onto an open subspace. Here $q_X| := q_X|_{X \setminus Z}$.

(c) Suppose $X \xrightarrow{f} W$ and $Y \xrightarrow{g} W$ are continuous maps with $f|_Z = g \circ \varphi$. Demonstrate that there is a unique map $\Psi : Y \sqcup_\varphi X \to W$ with the property that

$$\Psi \circ q_X = f \qquad \text{and} \qquad \Psi \circ q_Y = g\,.$$

(67) One of the main objects of interest to algebraic geometers is so-called ***n-dimensional projective space*** which is defined by $\mathbb{P}^n := \mathbb{S}^n/\sim$ where $x \sim -x$; i.e., we identify so-called antipodal points x and $-x$. (Actually, this is *real* projective space; there is also a complex projective space. Moreover, algebraic geometers use a somewhat different, albeit homeomorphic, description of projective space.)

(a) Prove that $\mathbb{P}^n$ is homeomorphic to the quotient space $\mathbb{D}^n/\approx$ where $x \approx y$ if $x = y$ or $x, y \in \partial\mathbb{D}^n = \mathbb{S}^{n-1}$ and $x = -y$ (i.e., we identify boundary antipodal points).

(b) Demonstrate that $\mathbb{P}^1 \approx \mathbb{S}^1$. Do you think that $\mathbb{P}^2 \approx \mathbb{S}^2$?

(c) Find a space X with the property that $X = U \cup B$ where $U \neq \emptyset$ is open in X and B is homeomorphic to $\mathbb{S}^1 = \partial\mathbb{D}^2$, via some homeomorphism φ, and is such that the adjunction space $\mathbb{D}^2 \sqcup_\varphi X$ is (homeomorphic to) $\mathbb{P}^2$.

(68) Which of the spaces in Figure 2 below are homeomorphic?

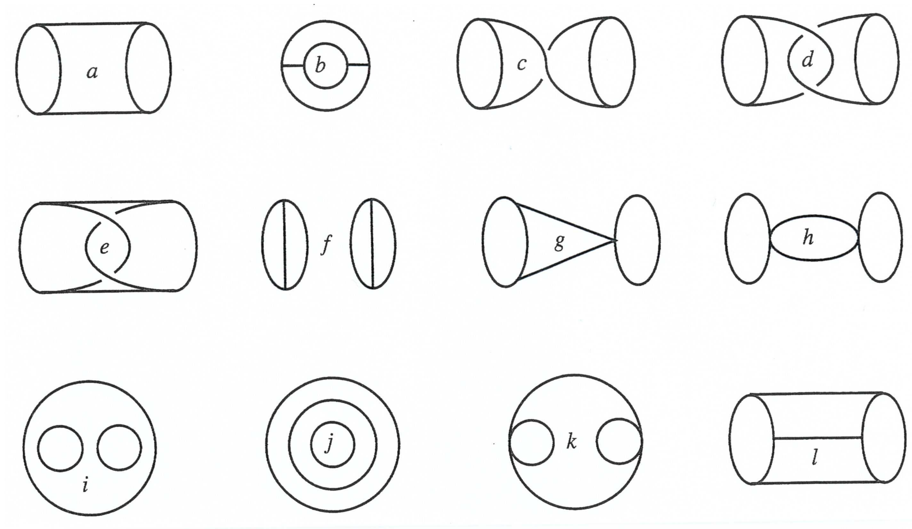


FIGURE 2. Some 'one-dimensional' "wire" topological spaces

(69) The ***natural comb*** NC, ***harmonic comb*** HC, and ***doubled harmonic comb*** DHC are the subspaces of $\mathbb{R}^2$ defined by

$$\mathsf{NC} := ([0,\infty)\times\{0\}) \cup \bigcup_{n=0}^{\infty} (\{n\}\times[0,1])\ ,$$

$$\mathsf{HC} := ([0,1]\times\{0\}) \cup (\{0\}\times[0,1]) \cup \bigcup_{n=1}^{\infty} \{1/n\}\times[0,1]\,,$$

$$\mathsf{DHC} := (\{0\}\times[-1,1]) \cup ([0,1]\times\{1\}) \cup \bigcup_{n=1}^{\infty} (\{1/n\}\times[0,1])$$

$$\cup \bigcup_{n=1}^{\infty} (\{-1/n\}\times[-1,0]) \cup ([-1,0]\times\{-1\})\ .$$

(a) Find two of $\mathsf{NC}, \mathsf{HC}, \mathsf{HC}_0 := \mathsf{HC}\setminus(\{0\}\times\mathbb{I}), \mathsf{DHC}$ that are homeomorphic.

(b) Find two of the spaces $\mathsf{NC}, \mathsf{HC}, \mathsf{HC}_0, \mathsf{DHC}$ that are not homeomorphic.

(70) The ***Hawaiian earring*** and ***expanding earring*** are the subspaces of $\mathbb{R}^2$ defined by

$$\mathsf{HE} := \bigcup_{1}^{\infty} \mathbb{S}^1((1/n,0);1/n) \quad \text{and} \quad \mathsf{EE} := \bigcup_{1}^{\infty} \mathbb{S}^1((n,0);n)$$

where $\mathbb{S}^1(z;r)$ is the circle in $\mathbb{R}^2$ with center z and radius r.

Which of the following spaces are homeomorphic?

$$\mathsf{HE}\ ,\ \mathsf{EE}\ ,\ \mathbb{R}/\mathbb{Z}\ ,\ \mathbb{I}/\mathbb{M}\ ,\ \mathbb{R}/\{\pm 2^n : n\in\mathbb{N}\}$$

Here $\mathbb{M} := \{0,1,1/2,1/3,1/4,\dots\}$.

For each $n\in\mathbb{N}$, put $X_n := \mathbb{S}^1$ and let $a_n := (1,0)\in\mathbb{S}^1$. Set $X := \bigvee_{n\in\mathbb{N}} X_n$. Which of the spaces $\mathsf{HE}, \mathsf{EE}, \mathbb{R}/\mathbb{Z}, \mathbb{I}/\mathbb{M}$ are homeomorphic to X?

(71) Let (A_n) be a sequence (finite or infinite) of connected subspaces of some topological space X. Suppose that for all n, $A_n \cap A_{n+1} \neq \emptyset$. Prove that $\bigcup_n A_n$ is connected.

(72) Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be a family of connected subspaces of some topological space X. Suppose A is another connected subspace of X and for all $\lambda \in \Lambda$, $A \cap A_\lambda \neq \emptyset$. Prove that $A \cup \bigcup_\lambda A_\lambda$ is connected.

(73) Suppose a connected subspace C of X intersects both $A \subset X$ and $X \setminus A$. Prove that C meets ∂A.

(74) Prove that any uncountable set, with its countable complement topology (see Lee, p.36, #(2-3), $\mathcal{T}_3$), is a connected space. Characterize its connected and disconnected subspaces.

(75) Determine the homeomorphism types of connected spaces with exactly three points.

(76) Describe, up to homeomorphisms, the connected spaces that can be constructed from four compact intervals via identifications among their endpoints.

(77) A topological space is ***totally disconnected*** if its only connected subspaces are one-point sets. Show that every discrete space is totally disconnected. Does the converse hold?

(78) Prove that any product of totally disconnected spaces is totally disconnected.

(79) Prove that if U is a dense open subset of $\mathbb{I} = [0,1]$, then $\mathbb{I} \setminus U$ is totally disconnected.

(80) Let X, Y be connected topological spaces. Suppose $A \subsetneq X$ and $B \subsetneq Y$. Prove that $(X \times Y) \setminus (A \times B)$ is connected.

(81) Let $X \xrightarrow{p} Y$ be an identification map. Suppose that Y is connected and that for all $y \in Y$, $p^{-1}(y)$ is also connected. Prove that X is connected.

(82) Let S be a connected subspace of a connected space X. Suppose that $\{A, B\}$ is a separation of $X \setminus S$. Prove that both $A \cup S$ and $B \cup S$ are connected.

(83) If S is a connected subspace of a topological space X, are either its interior or boundary connected? If both the interior and boundary of S are connected, must S be connected?

(84) Let $\mathbb{S}^1 \xrightarrow{f} \mathbb{R}$ be continuous. Prove that there exists an $s \in \mathbb{S}^1$ with $f(s) = f(-s)$.

(85) (a) If A is a path connected subspace of X, is $\bar{A}$ path connected?

(b) If $f : X \to Y$ is continuous and X is path connected, is $f(X)$ path connected?

(c) Is a product of path connected spaces path connected?

(d) If $\{A_\lambda \mid \lambda \in \Lambda\}$ is a collection of path connected subspaces of X, and $\bigcap_\lambda A_\lambda \neq \emptyset$, is $\bigcup_\lambda A_\lambda$ path connected?

(86) Show that for *any* countable set $S \subset \mathbb{R}^2$, $\mathbb{R}^2 \setminus S$ is path connected.

(87) Determine the components of the space $(\mathbb{I} \setminus \mathbb{M})^2 = (\mathbb{I} \setminus \mathbb{M}) \times (\mathbb{I} \setminus \mathbb{M})$ where $\mathbb{I} := [0,1]$ and $\mathbb{M} := \{1/n \mid n \in \mathbb{N}\}$.

(88) What are the components and path components of $\mathbb{R}_{\text{low}}$? Describe all continuous maps $f : \mathbb{R} \to \mathbb{R}_{\text{low}}$.

A topological space X is said to be ***locally connected at*** x (a point of X) provided for each neighborhood U of x there exists a *connected* neighborhood V of x with $V \subset U$. A space X is ***locally connected*** if it is locally connected at each of its points.

A topological space X is said to be ***locally path connected at*** x (a point of X) provided for each neighborhood U of x there exists a *path connected* neighborhood V of x with $V \subset U$. A space X is ***locally path connected*** if it is locally path connected at each of its points.

(89) The ***harmonic rake*** HR and ***natural rake*** NR subspaces of $\mathbb{R}^2$ are pictured below and defined by

$$\mathsf{HR} := (\{0\} \times [0,1]) \cup \bigcup_{n=1}^{\infty} ([c, a_n]) \quad \text{and} \quad \mathsf{NR} := (\{0\} \times [0,1]) \cup \bigcup_{n=1}^{\infty} ([c, b_n])$$

where $c := (0,1)$ and $a_n := (1/n, 0)$ and $b_n := (n, 0)$.

Determine the sets of all points at which HR or NR is locally connected or locally path connected.

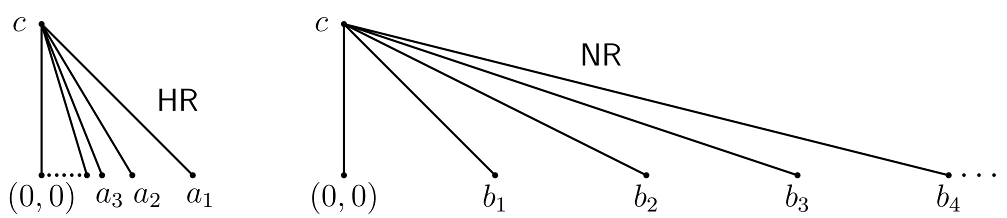


FIGURE 3. The Harmonic and Natural Rakes

(90) Is local connectivity preserved by continuous maps? By homeomorphisms?

(91) If S is a locally connected subspace of some space X, is $\bar{S}$ locally connected?

(92) Find a subset of $\mathbb{R}^2$ that is path connected but is not locally connected anywhere.

(93) Prove that for any topological space $(X, \mathcal{T})$, the following are equivalent:
  (a) X is locally connected.
  (b) Every component of every open subspace of X is open.
  (c) There is a basis for $\mathcal{T}$ consisting of connected sets.

Formulate (and prove) a similar result for locally path connected spaces.

(94) Suppose that X is a locally path connected space. Prove that:
  (a) Every open connected subspace of X is path connected.
  (b) The components and path components of X are exactly the same.
  (c) All components of X are both open and closed.

(95) Let $S := \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \mid r \geq 0, \theta \in [0, 2\pi] \cap \mathbb{Q}\}$ (the set of "polar rays with rational radian angle").
  (a) Prove that S is a connected subspace of $\mathbb{R}^2$. Is it path connected?
  (b) Determine the components of the subspace $S \setminus \{(0,0)\}$.
  (c) Is S locally connected?

(96) Prove that a connected metric space having more than one point is uncountable.

(97) Let $X \xrightarrow{f} Y$ be a map of topological spaces with Y compact and Hausdorff. Prove that f is continuous if and only if the *graph of* f,

$$G_f := \{(x, f(x)) \mid x \in X\}$$

is a closed subspace of $X \times Y$.

(98) Let A, B be compact subspaces of X, Y respectively. Suppose W is an open set in $X \times Y$ that contains $A \times B$. Prove that there are open sets $U \subset X$ and $V \subset Y$ such that $A \times B \subset U \times V \subset W$.

(99) Let $X \xrightarrow{p} Y$ be a continuous closed surjection. Suppose that Y is compact and that for each $y \in Y$, $p^{-1}(y) \subset X$ is also compact. Prove that X is compact.
(Hint: Show that for any open set $U \supset p^{-1}(y)$ there exists an open neighborhood V of y such that $p^{-1}(V) \subset U$.)

(100) Let X be a metric space and $\emptyset \neq A \subset X$. Recall that the *distance from* x *to* A is

$$\operatorname{dist}(x, A) := \inf_{a \in A} |x - a| .$$

(a) Prove that $\operatorname{dist}(x, A) = 0$ if and only if $x \in \bar{A}$. Deduce that A is closed if and only if for all $x \in X \setminus A$, $\operatorname{dist}(x, A) > 0$.
(b) Show that when A is compact, for each $x \in X$ there exists an $a \in A$ such that $\operatorname{dist}(x, A) = |x - a|$.
(c) The ε-*neighborhood of* A is $\mathsf{N}(A; \varepsilon) := \{x \in X \mid \operatorname{dist}(x, A) < \varepsilon\}$.
Show that $\mathsf{N}(A; \varepsilon) = \bigcup_{a \in A} \mathsf{B}(a; \varepsilon)$.
(d) Prove that when A is compact, for each open $U \supset A$ there exists an $\varepsilon > 0$ such that $U \supset \mathsf{N}(A; \varepsilon)$.
(e) Do either (b) or (d) hold for closed sets A?

(101) Let X be a metric space and $\emptyset \neq A, B \subset X$. Recall that the *distance from* A *to* B is

$$\operatorname{dist}(A, B) := \inf_{a \in A, b \in B} |a - b| .$$

(a) Prove that when A is compact, B is closed, and $A \cap B = \emptyset$, $\operatorname{dist}(A, B) > 0$.
(b) If A and B are closed with $A \cap B = \emptyset$, is $\operatorname{dist}(A, B) > 0$?

(102) Let X be a metric space and let $\mathcal{H}$ denote the collection of all n on-empty closed bounded subsets of X. For $A, B \in \mathcal{H}$, define

$$d_{\mathcal{H}}(A, B) := \inf \{\varepsilon > 0 \mid A \subset \mathsf{N}(B; \varepsilon) \text{ and } B \subset \mathsf{N}(A; \varepsilon)\} .$$

(a) Prove that $d_{\mathcal{H}}$ is a distance function on $\mathcal{H}$, so $(\mathcal{H}, d_{\mathcal{H}})$ is a metric space.
(b) Show that when X is complete, so is $\mathcal{H}$.
(c) Show that when X is totally bounded, so is $\mathcal{H}$.
(d) Show that when X is compact, so is $\mathcal{H}$.

(103) Investigate the the following claims.
(a) Every compact subspace of a topological space has compact closure.
(b) No compact subspace of a topological space has compact interior.

(104) (a) Is the quotient space $\mathbb{R}/\mathbb{Z}$ compact?

(b) Is the quotient space $\mathbb{R}^2/\mathsf{L}$ compact? Here the 'integer' lattice
$$\mathsf{L} := (\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z})$$
is identified to a point.

(105) (a) Is the quotient space $\mathbb{R}/(\mathbb{R} \setminus [0,1])$ compact or Hausdorff?
(b) Is the quotient space $\mathbb{R}/(\mathbb{R} \setminus (0,1))$ compact or Hausdorff?

(106) Prove that $S \subset \mathbb{R}$ is compact if and only if every continuous map $f : S \to \mathbb{R}$ is bounded and attains a maximum value on S.

(107) Find a Lebesgue number for each of the following coverings.
(a) The cover $\mathcal{U}_r := \{(n-r, n+r) \mid n \in \mathbb{Z}\}$ of $\mathbb{R}$ (where $r > 0$).
(b) The cover $\mathcal{U}_r := \{\mathbb{B}^2(x;r) \mid x \in \mathbb{Z}^2\}$ of $\mathbb{R}^2$ (where $r > 1$).
(c) The cover $\{X \setminus \{x\} \mid x \in X\}$ of a compact metric space X.
(d) The cover $\mathcal{U}_r := \{\mathsf{B}(x;r) \mid x \in X\}$ of a compact metric space X (where $r > 0$).

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