SURVEY ARTICLE

TOPOLOGY OF SMOOTH MANIFOLDS

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Differential topology, like differential geometry, is the study of smooth (or " differential ") manifolds. There are several versions of the definition: the basic requirement for M^m to be a smooth manifold of dimension m is the existence of local coordinate systems, *i.e.* imbeddings $\phi_\alpha : U_\alpha \to \mathbf{R}^m$ (where $\mathbf{R}^m$ denotes Euclidean space of dimension m and $\phi_\alpha(U_\alpha)$ is open) of (open) subsets U_α of M which, together, cover M; and in $U_\alpha \cap U_\beta$, where two coordinate systems overlap, each set of coordinates must be smooth (*i.e.* infinitely differentiable) functions of the other set. If M and N are smooth manifolds, a map $f: M \to N$ is called smooth if its expressions by local coordinate systems in M and N are smooth at each point. Hence in particular we have the notion of smooth imbedding. If $f: M \to N$ and $g: N \to M$ are smooth and inverse to each other, they are called diffeomorphisms: M and N are then to be regarded as different copies of the same manifold. If f and g are merely continuous and inverse to each other, we call them homeomorphisms. Thus homeomorphism is a cruder means of classification than diffeomorphism.

The notion of smooth manifold, due essentially to Poincaré, and codified by Veblen and Whitehead [**1**], gained in concreteness from the theorem of Whitney [**1**], that every smooth manifold M^m can be imbedded smoothly in $\mathbf{R}^r$, which is itself of course a smooth manifold, provided that $r \geqslant 2m+1$. Thus M can be regarded as a submanifold of $\mathbf{R}^r$: locally it will be defined by the vanishing of $(r-m)$ smooth functions with linearly independent differentials, and we will be able to choose m of the coordinates of $\mathbf{R}^r$ as local coordinates in M (though of course the same choice will not in general do throughout M). An important example is the unit sphere S^{r-1} in $\mathbf{R}^r$. We shall denote the disc bounded by S^{r-1} as D^r. This is an example of the more general notion of manifold with boundary, defined as above, but replacing $\mathbf{R}^m$ by a closed half-space. We shall use ∂M to denote the boundary of the manifold M, and write

$$\operatorname{Int} M = M - \partial M.$$

Whitney's theorem is the first of the theorems of " general position ": it states that imbeddings are dense in the space of all maps

$$f : M^m \to \mathbf{R}^r \quad (r \geqslant 2m+1),$$

suitably topologised, provided M is compact. Along the same lines, we may note that the same holds if $\mathbf{R}^r$ is replaced by any r-manifold,

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and that if $m > (p+q)$, a map $f: P^p \to M^m$ will "in general" avoid a q-dimensional submanifold Q^q of M^m. These results are all contained in Thom's transversality theorem [**1**] which is, however, too technical to state here.

We leave here the topic of imbeddings (and the related one of immersions—maps which are locally imbeddings) and refer the reader to a recent survey article by Smale [**1**] and to the papers of Haefliger [**1**] [**2**], Haefliger and Hirsch [**1**] [**2**] and Hirsch [**1**] for a fairly complete general theory and for further references.

Suppose M^m a submanifold of V^v. Then a neighbourhood N of M in V is called a tubular neighbourhood if there is a smooth retraction $r: N \to M$ with the property that each $x \in M$ has a neighbourhood U such that $r | r^{-1}(U)$ is equivalent to the projection map of a product, $D^{v-m} \times U \to U$. We need also the condition that in a change of coordinates, $D^{v-m} \times (U_1 \cap U_2) \to r^{-1}(U_1 \cap U_2) \to D^{v-m} \times (U_1 \cap U_2)$, the disc over each $x \in U_1 \cap U_2$ is mapped to itself by an orthogonal map. For example, if V is Euclidean space and M is compact, we can take N as the set of points within a small distance ϵ of M, and r as projection on the nearest point of M: for $x \in M$, $r^{-1}(x)$ is then a disc of radius ϵ with centre x and perpendicular to M at x.

Existence and essential uniqueness of tubular neighbourhoods are proved in all introductions to the subject (*e.g.* Lang [**1**]). We have defined a kind of fibre bundle, using vectors normal to M in V (the "normal bundle" of M in V); analogously (without an imbedding) we can assemble tangent vectors of M into the "tangent bundle" of M. For more general discussion of bundles see Steenrod [**1**]; for associated bundles, such as the tensor bundles which are the key object of study in differential geometry, see Nomizu [**1**].

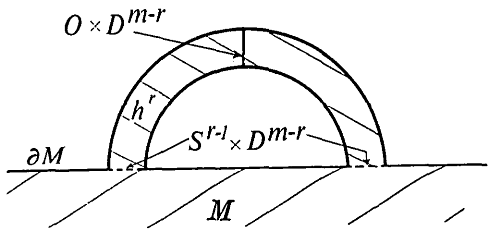
One of the ultimate aims of differential topology is the classification up to diffeomorphism of all smooth manifolds, and while the negative solution of the isomorphism problem for groups shows that this is algorithmically impossible (in dimensions $\geqslant 4$), we can perform it with certain restrictions. The technique is to reduce the problem first to a problem in homotopy theory and then to a problem in algebra. We remind the reader that a continuous map $F: X \times I \to Y$ (where I denotes the unit interval $[0, 1]$) is called a homotopy between the maps $f_0, f_1: X \to Y$, where $f_t(x) = F(x, t)$. Also if $f: X \to Y$ and $g: Y \to X$ are such that the compositions $f \circ g$, $g \circ f$ are homotopic to the identity maps of Y and X respectively, they are called homotopy equivalences between X and Y. The relation of homotopy equivalence is coarser again than that of homeomorphism; correspondingly, there are more algebraic techniques available to solve the equivalence problem (and others)—they form the subject matter of algebraic topology.

Handles

A basic requisite for classification is a reasonably intrinsic and effective way to describe manifolds; this is provided by a handle presentation. Let M^m be a manifold with boundary ∂M, and

$$f: S^{r-1} \times D^{m-r} \to \partial M$$

a smooth imbedding. Take M and $D^r \times D^{m-r}$, glue them together along $S^{r-1} \times D^{m-r}$ (part of the boundary of each). Local coordinate systems in the two can be pieced together to give local coordinate systems which make the result a smooth manifold with boundary (this needs some ingenuity near the " corner " $S^{r-1} \times S^{m-r-1}$). The result is said to be obtained from M by attaching an r-handle: we write it as $M \cup_f h^r$, or as $M \cup h^r$.



The construction may be iterated indefinitely to attach many handles of all dimensions (from 0 to m) to M: this gives a handle decomposition of the result, based on M. Observe what happens to the boundary of M when we attach a handle to M :$f(S^{r-1} \times D^{m-r})$ is removed, and $D^r \times S^{m-r-1}$ takes its place. This kind of change is called a spherical modification (Wallace [**1**]).

There are at least three ways to view the above process: a handle decomposition is a means of describing a given manifold; attaching handles is a means of constructing new manifolds; performing spherical modifications is a means of constructing new manifolds. This construction is the modern form of " scissors and paste " topology: it yields powerful methods. We also note that attaching a handle corresponds to a process (attaching a cell) already very familiar in homotopy theory.

Every manifold has a handle decomposition (Smale [**2**], Wallace [**1**]). In fact suppose the boundary of M expressed as the disjoint union of unbounded (possibly empty) submanifolds $\partial_- M$, $\partial_+ M$. (In this case, M is called a *cobordism* between $\partial_- M$, $\partial_+ M$ which are then cobordant.) Take a real-valued function ϕ, with a minimum along $\partial_- M$ and maximum along $\partial_+ M$, and apply the general position technique. Then $d\phi$ is zero only at isolated points, the critical points of ϕ, at each of which there is a local coordinate system in which ϕ has the form

$$\phi(x_1, \ldots, x_m) = c - x_1{}^2 - \ldots - x_\lambda{}^2 + x_{\lambda+1}^2 + \ldots + x_m{}^2$$

(Morse [**1**]); moreover, the values (c) of ϕ at the critical points are all distinct. The integer λ, the "index" of the singularity, is well-determined. Morse applied this, together with the idea of studying the subsets $M^a = \{P \in M : \phi(P) \leqslant a\}$ as a increases, to obtain information about the homology groups of M; he also applied it to function spaces, with important consequences for the calculus of variations. Next it was noticed (Thom [2]) that information was obtained about the homotopy type of M; an important application of this (Bott [1]) is the so-called Bott periodicity theorem.

It was left to Smale [2] [3] to observe that this is relevant for diffeomorphism problems too. For example if M is compact and unbounded, and the above is the only singularity with $c-\epsilon \leqslant \phi \leqslant c+\epsilon$, then $M^{c+\epsilon} = M^{c-\epsilon} \cup h^\lambda$. For the case above we deduce that if $a < b$ are both non-critical values, then M^a is obtained from M^b by adding a sequence of handles. Thus M always has a handle decomposition based on $\partial_- M$. It follows also that $\partial_+ M$ can be obtained from $\partial_- M$ by a series of spherical modifications, so two manifolds are cobordant if and only if we can proceed from one to the other by a series of spherical modifications.

The little book by Milnor [1] is an excellent reference for this result, and for some of the main applications of Morse theory. In fact the systematic use of functions instead of handles (*cf.* Cerf [3]) avoids the necessity of discussing corners (technically, corners are a major nuisance), but handles are probably more perspicuous.

Smale theory

The first results on classification (up to diffeomorphism, given certain assumptions about homotopy type) were developed by Smale [3], [4], [5]. He used a few basic lemmas, the key one giving a means of cancelling handles. Given $M \cup h^r$, as above, we will call the spheres $S^{r-1} \times O$, $O \times S^{m-r-1}$ the a-sphere and the b-sphere of the handle: the handle is attached along the a-sphere, and the b-sphere appears in the boundary of the result. Then in $(M \cup_f h^r) \cup_g h^s$, the a-sphere of h^s and the b-sphere of h^r both lie in $\partial(M \cup_f h^r)$. If $s \leqslant r$, they are, if in general position, disjoint, and it can then be seen that the order of attaching the two handles can be interchanged. Thus in a handle decomposition, we can always suppose the handles arranged in order of increasing dimension (Smale expresses this with "nice" functions). If $s = r+1$, the spheres will generally intersect transversely in a finite number of points. The key lemma (Smale [5]) states that if they intersect transversly in a single point, then $(M \cup_f h^r) \cup_g h^{r+1}$ is diffeomorphic to M: the handles may be cancelled. Under appropriate hypotheses, this can be strengthened: if everything is simply-connected, and we make certain algebraic assumptions, arguments of Whitney [2] can be used to manoeuvre the spheres to meet transversely in only one point provided $2 \leqslant r \leqslant m-3$ and $m \geqslant 6$.

There are special arguments for the other values of r if $m \geqslant 5$ (but none if $r = 2$, $m = 5$).

This leads to the h-cobordism theorem (Smale [5]). A cobordism W is an h-cobordism if each inclusion $\partial_- W \subset W$, $\partial_+ W \subset W$ is a homotopy equivalence. If, in addition, W is compact, simply-connected, and of dimension $\geqslant 6$, we can apply our techniques to get rid of all the handles, so W is diffeomorphic to $\partial_- W \times I$ (and also $\partial_- W$ to $\partial_+ W$).

From this, Smale deduces several important consequences. For example suppose M^m a submanifold without boundary of the compact manifold W^w, that $M \subset W$ is a homotopy equivalence, that W and ∂W are simply-connected, and that $w \geqslant 6$, $w - m \geqslant 3$. Then if T is a tubular neighbourhood of M in W, $W - \operatorname{Int} T$ turns out to be an h-cobordism; it follows that W is also a tubular neighbourhood of M. In particular, take M a point: then a compact contractible manifold $W^w (w \geqslant 6)$, with simply-connected boundary, is diffeomorphic to the disc D^w. (W is called contractible if it has the homotopy type of a point; equivalently, if the identity map $W \to W$ is homotopic to a map which sends the whole of W to a point). From this follows the Poincaré conjecture: if $\Sigma^w (w \geqslant 6)$ is a smooth manifold, homotopy equivalent to the sphere S^w, we delete the interior of an imbedded disc D^w, then the complement is contractible with boundary S^{w-1}, hence a disc, so Σ^w can be obtained by attaching two copies of D^w along the boundary. In particular, Σ^w is homeomorphic to S^w (*NOT* in general diffeomorphic). The same holds for $w = 5$: we can then show (using surgery: see below) that Σ^5 bounds a contractible 6-manifold, so it is even diffeomorphic to S^5. The Poincaré conjecture was also proved, by the same method, in Wallace [**2**].

As the Poincaré conjecture is trivial for $w \leqslant 2$, only the cases $w = 3$, $w = 4$ are outstanding. These are equivalent to validity of the corresponding cases of the h-cobordism theorem. The case $w = 5$ of the h-cobordism theorem would be more general; here it has been shown (Barden, unpublished) that if there exists a diffeomorphism $\partial_- W \to \partial_+ W$ in the preferred homotopy class, it extends to a diffeomorphism of $\partial_- W \times I$ onto W. However, it would be more useful to have the diffeomorphism of $\partial_- W$ on $\partial_+ W$ in the conclusion of the theorem. The proof of this case of the theorem breaks down because of the absence of a technique for separating imbedded 2-spheres in 4-manifolds. A counterexample to the most obvious conjecture may be found in Kervaire and Milnor [**1**]; for a discussion of known facts about simply-connected 4-manifolds, see Wall [**1**], [**2**].

Extensions and applications of Smale theory

Another theorem obtained by Smale [5] is the existence, on a simply-connected manifold of dimension $\geqslant 6$ with simply-connected boundary components, of handle decompositions with the smallest possible number of handles necessary to give the correct homology groups. These decompositions provide an extremely effective tool for classifying manifolds, especially when the dimensions of the handles are all near half the dimension of the manifold. Classifications may be found in Smale [**6**], Wall [**3**], [**4**], Tamura [**1**], [**2**], [**3**]. Also, using surgery, the above theorem has been extended to simply-connected closed 5-manifolds, and a complete classification obtained (Barden [**1**]).

Simple-connectivity is not essential in the above. In particular, the h-cobordism theorem continues to hold if we now insist that the inclusion $\partial_- W \subset W$ be a simple homotopy equivalence in the sense of Whitehead [**1**]: this result is due to Mazur [**1**], [**2**]. Mazur's work is full of mistakes, but proofs have also been found by Barden [**2**] and Stallings. The existence of handle decompositions with few handles is replaced by the more complicated "non-stable neighbourhood theorem" (Mazur [**2**]) and its relative version (Mazur [**3**]: beware that the hypotheses given in this paper are inadequate).

From the functional point of view, the h-cobordism theorem appears as follows. Let W be an h-cobordism, $f: W \to [0, 1]$ a general map with $f(\partial_- W) = 0$, $f(\partial_+ W) = 1$. Then we can first arrange the singularities of f in order (make f "nice"), then cancel them, so that after a homotopy f ends without singularities. There are two generalisations of this. Using paths in the function space, and deforming them into singularity-free paths, Cerf [**3**] has shown that if F is a diffeomorphism of $M^m \times I$, leaving each end invariant, then there is a diffeomorphism of $M^m \times I$, agreeing with F on the ends, and preserving the I-coordinate ("concordance implies diffeotopy"); at least, if M is a sphere of dimension $\geqslant 8$. This had been conjectured by Smale, and relates to some problems in Wall [**4**].

Secondly, we can replace I by another 1-manifold: S^1, $\mathbf{R}$, or $\mathbf{R}_+$. Results have been obtained in these cases by using a special kind of surgery to reduce to the h-cobordism theorem. An open manifold W^w is called simply-connected at infinity if every compact subset is contained in a compact subset whose complement is simply-connected. Suppose also that $w \geqslant 6$, and that the homology groups of W are finitely generated. Then there is a proper smooth function on W with only a finite number of critical points, and W is diffeomorphic to the interior of a unique compact manifold. Also, the h-cobordism theorem holds for such W, if we assume simple connectivity. These results are due to Browder, Levine and Livesay [**1**]. We also have the result (Browder [**2**]) that if W^w is the product of $\mathbf{R}$ and a simply-connected topological manifold M, then it is diffeomorphic to the product of $\mathbf{R}$ and a unique smooth manifold N, h-cobordant to M. Browder and Levine have also shown that if W^w is a closed manifold with all $\pi_i(W)$ finitely generated abelian groups, $w \geqslant 6$, and $f: W \to S^1$ induces an isomorphism $\pi_1(W) \to \pi_1(S^1) = \mathbf{Z}$, then f can be deformed to a function with no critical points, *i.e.* the projection map of a fibre bundle. The same holds for any compact W, if we are given that $f|\partial W$ is already the projection of a fibre bundle.

Cobordism theory

We define above the terms " cobordism " and " cobordant ". It is easy to see that being cobordant is an equivalence relation, and it is clearly quite a crude relation. There are many variants on this relation, which the following examples illustrate.

(i) A manifold $M^m \subset \mathbf{R}^{m+r}$ is said to be *framed* if we have an isomorphism of a tubular neighbourhood with the product $M^m \times D^r$. Two such framed manifolds are cobordant (in the " framed " sense) if there is a framed submanifold W of $\mathbf{R}^{m+r} \times [0, 1]$ with $\partial_- W = M_0{}^m \times 0$, $\partial_+ W = M_1{}^m \times 1$, provided with the given framings.

(ii) A manifold and submanifold $M^m \subset V^v$ is called a pair. Two such pairs are cobordant if there is a pair $N^{n+1} \subset W^{v+1}$ with $\partial_- W = V_0$, $\partial_+ W = V_1$, $\partial_- N (= N \cap \partial_- W) = M_0$, $\partial_+ N (= N \cap \partial_+ W) = M_1$.

(iii) A manifold M^m and a map $f: M^m \to X$ are the objects: two such objects are equivalent (" bordant ") if there is a manifold N^{m+1} and map $F: N \to X$, with $\partial_- N = M_0$, $f_0 = F|M_0$ and $\partial_+ N = M_1$, $f_1 = F|M_1$.

It turns out to be essential for the nontriviality of these relations that all the manifolds concerned should be compact: otherwise, for example, M is the boundary of $M \times [0, \infty)$. Also, taking the disjoint union of two manifolds is an addition relation compatible with cobordism, and leads in all cases above (and most others which arise in practice) to the structure of an abelian group on the set of cobordism classes.

Consider example (i). It is immaterial whether we consider $\mathbf{R}^{m+r}$ or S^{m+r} (stereographic projection shows $\mathbf{R}^{m+r} \cong S^{m+r}$—point). We have an imbedding $f: M^m \times D^r \to S^{m+r}$, with image N, say. Define a map of N by first using $f^{-1}: N \to M^m \times D^r$, then projecting on D^r, and finally identifying $S^{r-1} = \partial D^r$ to a point (∞). The resulting space is homeomorphic to S^r. Since the frontier of N is mapped to ∞, we can extend to a continuous map of S^{m+r} by mapping the complement of N to ∞. We have thus constructed a continuous map $S^{m+r} \to S^r$. This is called the Pontrjagin-Thom construction, after its use in Pontrjagin [**1**], Thom [**3**].

It is possible to reverse the procedure by putting an arbitrary map $S^{m+r} \to S^r$ in " general position with respect to a point of S^r ", and defining M^m as the inverse image of that point. Closer analysis shows that cobordism of framed manifolds corresponds to homotopy of maps, and hence that the construction defines an isomorphism of the group of framed cobordism classes of manifolds $M^m \subset \mathbf{R}^{m+r}$, and the group of homotopy classes of maps $S^{m+r} \to S^r$, usually denoted $\pi_{m+r}(S^r)$. This result is due to Pontrjagin

[1], and it was originally intended to apply knowledge of smooth manifolds to homotopy theory (though recently, traffic has been mostly—but not all—in the opposite direction).

Thom's great paper [3] provided a comprehensive generalisation, which makes it possible to reduce most of the interesting problems on cobordism to homotopy theory, and settled completely the equivalence problem for unrestricted cobordism (by doing the homotopy theory).

To describe the generalisation, we need some facts from fibre bundle theory. We first give the definitions.

A map $\pi: E \to B$ is a "fibre map with fibre F" if every point in B has a neighbourhood U_α, and a homeomorphism $h_\alpha: U_\alpha \times F \to \pi^{-1}(U_\alpha)$, such that $\pi h_\alpha(b, f) = b$. Suppose $\mathfrak{G}$ a topological group which acts (on the right) as a group of homeomorphisms of F. Then we have a "fibre bundle" if for every pair (α, β) of indices above, there is a map $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathfrak{G}$ such that for $b \in U_\alpha \cap U_\beta$, $f \in F$ we have

$$h_\beta(b, f) = h_\alpha\big(b, f . g_{\alpha\beta}(b)\big).$$

We call E the total space, B the base, π the projection, and $\mathfrak{G}$ the group of the bundle.

The bundles described on p. 2 have fibre the disc D^{v-m}, and group the orthogonal group in $(v-m)$ dimensional space, O_{v-m}; the definition there is modified by requiring all maps in the definition to be smooth. This modification does not affect the bundle theory.

If $\pi: E \to B$ is the projection of a fibre bundle, and $f: B' \to B$ any continuous map, we define a space E' and map $\pi': E' \to B'$ by

$$E' = \{(e, x) : e \in E, x \in B', \pi(e) = f(x)\}$$

and $\pi'(e, x) = x$. It is not difficult to check that this is the projection of another fibre bundle: it is said to be *induced* by f from the former bundle. We also have an induced map $g: E' \to E$ of the total spaces defined by $g(e, x) = e$. Then $\pi g(e, x) = \pi(e) = f(x) = f\pi'(e, x)$.

For any topological group $\mathfrak{G}$, there is a "classifying space" $B_\mathfrak{G}$, and for any space F on which $\mathfrak{G}$ acts, a fibre bundle with projection $\pi_\mathfrak{G}: E_\mathfrak{G} \to B_\mathfrak{G}$, fibre F and group $\mathfrak{G}$. Moreover, if $\pi: E \to B$ is the projection of any other bundle with fibre F and group $\mathfrak{G}$, (and B is paracompact), there exists a classifying map $f: B \to B_\mathfrak{G}$, unique up to homotopy, such that π is (equivalent to) the map induced from $\pi_\mathfrak{G}$ by f.

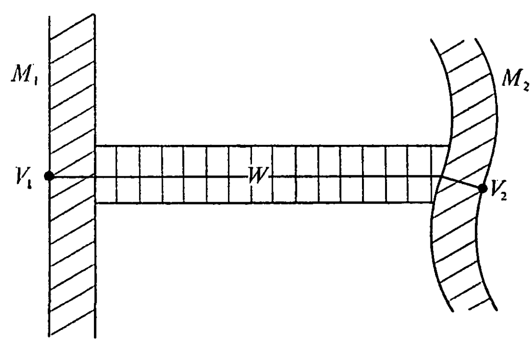
This concludes the necessary facts from bundle theory, for which we refer the reader to Steenrod [1] (though the theory of classifying spaces has been considerably improved since that book was written; see Dold [1] and Milnor [11]).

Let $\mathfrak{G}$ be a topological group, $\mathfrak{G} \to O_r$ a homomorphism, and suppose the normal bundle of an imbedding $M^m \subset \mathbf{R}^{m+r}$ provided with the structure of a bundle with group $\mathfrak{G}$. Then we have a classifying map $f: M \to B_\mathfrak{G}$, and an induced map of total spaces $g: N \to E_{\mathfrak{G}}$, where N is a tubular neighbourhood of M. Now the boundary spheres of fibres form other bundles with total spaces $\dot{N}$, $\dot{E}_{\mathfrak{G}}$ (say), and $g(\dot{N}) \subset \dot{E}_{\mathfrak{G}}$.

Define the Thom space $T_{\mathfrak{G}}$ as $E_{\mathfrak{G}}/\dot{E}_{\mathfrak{G}}$, *i.e.* as $E_{\mathfrak{G}}$ with the subspace $\dot{E}_{\mathfrak{G}}$ identified to a point ∞. Then g induces a map $N \to T_{\mathfrak{G}}$ which sends $\dot{N}$ to ∞; as before, we can extend by mapping the rest of S^{m+r} to ∞, and so obtain an element of $\pi_{m+r}(T_{\mathfrak{G}})$. Thom showed in [**3**] that $\pi_{m+r}(T_{\mathfrak{G}})$ was isomorphic to the group of cobordism classes of closed m-manifolds in $\mathbf{R}^{m+r}$ with a normal bundle with group $\mathfrak{G}$; our first example was the case $\mathfrak{G} = \{1\}$ of this.

For unrestricted cobordism we use Whitney's result that M^m has an essentially unique imbedding in $\mathbf{R}^{m+r}$ if $r > m$. The normal bundle automatically has group O_r. Hence the cobordism group $\mathfrak{N}^m$ of closed m-manifolds is isomorphic to $\pi_{m+r}(T_{O_r})$ if $r > m$. If we consider oriented M, the normal bundle is also oriented and has group SO_r, so we must compute $\pi_{m+r}(T_{SO_r})$.

We now indicate how this idea also allows us to reduce problems (ii) and (iii) to homotopy theory. In case (ii), the exposition is again simplified if we suppose V^v framed in M^m and M^m framed in $\mathbf{R}^{m+r}$. Then pairs (M_1, V_1) and (M_2, V_2) are framed cobordant as pairs if and only if M_1 and M_2; also V_1 and V_2 are framed cobordant. For if W is a framed cobordism of V_1 to V_2, glue $W \times D^{m-v}$ to $M_1 \times I$ and $M_2 \times I$; the boundary of the result consists of M_1, M_2, and another piece which (our assumptions imply) bounds a framed manifold.



More generally, if the normal bundle of M^m in $\mathbf{R}^{m+r}$ is to have group $\mathfrak{G}$, and that of V^v in M^m to have group H, then having a cobordism of pairs (M_1, V_1) and (M_2, V_2) is equivalent to having separate cobordisms of M_1 and M_2 (with group $\mathfrak{G}$) and of V_1 and V_2 (in $\mathbf{R}^{m+r}$, with group $\mathfrak{G} \times H$). This is due to Wall [**10**].

Definition (iii) is due to Atiyah [**1**]; and the associated theory may be found there and in the book by Conner and Floyd [**1**]. The reason for the name is that bordism groups of X, denoted $\mathfrak{N}_m(X)$, have the same formal properties as homology groups, and there is a dual set of " cobordism " groups, $\mathfrak{N}^m(X)$.

Suppose that we are considering a closed manifold $M^m \subset \mathbf{R}^{m+r}$, with normal bundle with group $\mathfrak{G}$, and a map $\phi : M \to X$. The classifying map of the bundle $f: M \to B_{\mathfrak{G}}$ is, as before, covered by a map $g: N \to E_{\mathfrak{G}}$ with $g(\dot{N}) \subset \dot{E}_{\mathfrak{G}}$. So we have $g \times \phi\pi : N \to E_{\mathfrak{G}} \times X$ with $(g \times \phi\pi)(\dot{N}) \subset \dot{E}_{\mathfrak{G}} \times X$; now, as before, the Thom construction gives a map $S^{m+r} \to E_{\mathfrak{G}} \times X / \dot{E}_{\mathfrak{G}} \times X$. If we write $T_{\mathfrak{G}}{}^+$ for the disjoint union of $T_{\mathfrak{G}}$ and a point, and $\wedge$ for the ordinary " smash " product of homotopy theory, the last space is no other that $T_{\mathfrak{G}}{}^+ \wedge X$, and Thom's theorem identifies our group with $\pi_{m+r}(T_{\mathfrak{G}}{}^+ \wedge X)$.

In particular, the bordism groups are given by $\mathfrak{N}_m(X) \cong \pi_{m+r}(T_{O_r}{}^+ \wedge X)$ (for $r > m$). One can now define the cobordism group $\mathfrak{N}^m(X)$ as the set $[S^{m+r} X : T_{O_r}]$ of homotopy classes of maps of the $(m+r)$-th suspension of X to T_{O_r} (for $r > m$). These are a typical example of the generalised homology and cohomology theories which have received much attention recently (see *e.g.* G. W. Whitehead [**1**]).

Computations of cobordism groups

For our first example of framed cobordism, we had to compute homotopy groups of spheres, $\pi_{m+r}(S^r)$. There is no simple answer here; we refer the reader to Toda [**1**] for computations with $m < 20$.

More interesting is the case of unrestricted cobordism (Thom [**3**]). To describe the results in this case, we must assume some familiarity with homology theory. The Hurewicz map

$$\pi_{m+r}(T_{O_r}) \to H_{m+r}(T_{O_r};\ \mathbf{Z}_2) \quad (r > m)$$

is a monomorphism. An easy argument shows that this is equivalent to saying that the cobordism class of a manifold M^m is determined by the homomorphism

$$f^* : H^m(B_{O_r};\ \mathbf{Z}_2) \to H^m(M;\ \mathbf{Z}_2) \to \mathbf{Z}_2.$$

Now $H^*(B_{O_r};\ Z_2)$ is a polynomial algebra with generators $w_1 \ldots, w_r$ (w_i in dimension i) called Stiefel-Whitney classes. Monomials in these of dimension m form a base of $H^m\ (B_{O_r};\ \mathbf{Z}_2)$; the values of f^* on these basis elements are called Stiefel-Whitney numbers of M. These determine the cobordism class of M, but are not independent.

The set of cobordism groups in all dimensions, $\mathfrak{N} = \{\mathfrak{N}^m\}$, is a graded ring. Thom also found the structure of $\mathfrak{N}$: it is a polynomial ring modulo 2 with one generator in each dimension not of the form $2^j - 1$. A simplified proof of all this is in Liulevicius [**1**].

If we consider oriented cobordism groups Ω^m, and so replace O_r by SO_r, it is still true that the Hurewicz map

$$\pi_{m+r}(T_{SO_r}) \to H_{m+r}(T_{SO_r};\ \mathbf{Z}) \quad (r > m)$$

is a monomorphism, and hence that oriented cobordism class of M is determined by a characteristic set of numbers; this time, as well as Stiefel-Whitney numbers we need Pontrjagin numbers, which are integers (but analogously defined). Modulo elements of finite order, this is due to Thom [**3**]; it was shown by Milnor [**3**] that the groups Ω^m contain no elements of odd order, and that modulo torsion elements we have a polynomial ring with one generator in each dimension $4k$. The final determination of the Ω^m (including multiplicative structure) was made in Wall [**11**]. In addition to homotopy theory, this last paper used certain exact sequences of which the main one is

$$\Omega^m \xrightarrow{r} \mathfrak{N}^m \xrightarrow{(\partial, d)} \Omega^{m-1} \oplus \mathfrak{N}^{m-2} \xrightarrow{(2,\ 0)} \Omega^{m-1} \xrightarrow{r} \mathfrak{N}^{m-1}.$$

Simplified proofs of this were given by Atiyah [**1**] and Wall [**5**]. Here, r forgets orientation; ∂ and d are defined geometrically, (and come from an earlier paper by Rohlin).

In the case when $\mathfrak{G}$ is the unitary group U_r (and $2r > m$) the Hurewicz homomorphism is again (1–1); cobordism class is determined by characteristic numbers (called Chern numbers), and the cobordism ring is a polynomial ring over the integers with a generator in each even dimension. These results are due to Milnor [**3**].

The special unitary group SU_r is at present being studied: its cobordism groups are related to those of U_r by an exact sequence like the one above, but where ∂ and d lower dimensions by 2 and 4 respectively, and " 2 " is replaced by composition with a Hopf map which raises dimension by one and has order 2. (This is due, simultaneously, to Conner and Floyd [**2**] and to Lashof and Rothenberg—unpublished). It follows easily that all the torsion is of order 2. See also Conner and Floyd [**3**].

The symplectic and spinor groups are also of interest, but seem to be a good deal more complicated. The other groups whose cobordism groups are known have the form $O_r \times \mathfrak{G}$ (or SO_r or U_r); but cobordism groups for this are (by a result above) isomorphic to unoriented bordism groups of $T_{\mathfrak{G}}$, hence also (by the so-called Thom isomorphism) to the unoriented bordism groups of $B_{\mathfrak{G}}$. They were computed (with some restrictions on $\mathfrak{G}$ in the SO_r and U_r cases) by Wall [**10**].

Unoriented bordism groups are easy to compute, since there is an isomorphism $\mathfrak{N}_*(X) \cong \mathfrak{N} \otimes H_*(X;\ \mathbf{Z}_2)$ (natural in the technical sense only) for any X. This follows easily from Thom [**3**]. Conner and Floyd [**1**] give some computations in the oriented case, when X is a $K(\mathbf{Z}_p, 1)$; they have since given analogous computations in the unitary case (notes, Seattle, 1963). They also give methods of computation depending on general properties of homology theories. Conner and Floyd apply their results to equivariant cobordism groups and to differentiable periodic maps, which are outside the scope of this survey; however, we refer to

Conner [1] for a useful exact sequence for computing equivariant cobordism groups.

A good expository account of Thom theory is given by Milnor [2].

Homotopy spheres

One of the most germinal papers in differential topology was Milnor [4], in which smooth manifolds are constructed which are homeomorphic but not diffeomorphic to S^7. Further papers (Milnor [5], Shimada [1], Tamura [4]) generalised this to other dimensions. This raises the problem of a complete diffeomorphism classification.

Two groups are of interest. If we wish to classify smooth manifolds obtained by glueing together a pair of discs D^n, we must study the abelian group Γ^n, quotient of the group of diffeomorphisms of S^{n-1} by the subgroup of those which extend to D^n (see Milnor [6] for a discussion). It is easier to compute the group Θ^n, whose elements are h-cobordism classes of homotopy n-spheres (*i.e.* smooth manifolds homotopy equivalent to S^n), and with addition defined by the so-called connected sum of two manifolds—let f_1, f_2 imbed n-discs in the manifolds, delete the interiors of the discs, and glue the boundaries by $f_2 \circ f_1^{-1}$.

Smale theory gives a natural isomorphism $\Gamma^n \cong \Theta^n$ for $n \geqslant 6$; also (using the computation $\Theta^5 = 0$) for $n = 5$. It is elementary that both groups vanish for $n \leqslant 2$; several proofs that $\Gamma^3 = 0$ are known, and Cerf [2] has shown $\Gamma^4 = 0$. As $\Theta^4 = 0$, only Θ^3 is in limbo: the Poincaré conjecture, that every homotopy 3-sphere is homeomorphic to S^3, would imply $\Theta^3 = 0$.

The group Θ^n is computed using the method of surgery, initiated in 1959 in mimeographed notes by Milnor, and published in Milnor [7] and Kervaire and Milnor [2], denoted henceforth by [KM]. The computation is in three stages. Let Σ^n be a homotopy n-sphere: imbed it in Euclidean space—or a sphere—of large dimension, say in S^{n+N}. Can it be framed? The answer turns out to be always yes (using recent results of J. F. Adams). Does it bound a framed manifold in D^{n+N+1}? Cobordism theory shows that there is an obstruction to this, which lies in $\pi_{n+N}(S^N)$. This will vary if we change the framing; we can change it by rotating the frame at each point according to some map $S^n \to SO_N$, and the obstruction changes by some element of the image of a homomorphism

$$J_n : \pi_n(SO_N) \to \pi_{n+N}(S^N).$$

Thus the answer to our second question depends on our computing some element of the cokernel of J_n. (The group $\pi_n(SO_N)$, and the behaviour of J_n are known for all n; $\pi_{n+N}(S^N)$ and hence Coker J_n are always finite, and computations are now available for $n \leqslant 20$. See Toda [1].) Finally we ask (a) which elements of Coker J_n are represented by homotopy n-spheres, and (b) if Σ^n bounds a framed manifold, whether it bounds a contractible manifold (and so determines $0 \in \Theta^n$).

In each case we have a framed manifold, and want a simpler one. The idea of surgery is to take a manifold and by successive spherical modifications, carefully chosen, to make it simpler in some sense. Here we wish to kill the homotopy groups, and to ensure that any map of a sphere S^i into a manifold M^m can be extended to a map of D^{i+1}. If $m > 2i$, any map $f: S^i \to M^m$ can be made an imbedding, by putting it in general position. One also shows that if M^m is framed in S^{m+N}, $f(S^i)$ is framed in M, so we have an imbedding $S^i \times D^{m-i} \to M^m$ and can perform a spherical modification. This has the effect, if $m > 2i+1$, of killing the homotopy class of maps $S^i \to M^m$ that we started from. It is shown in Milnor [7] that by a judiciously chosen sequence of such modifications, any framed M^m, with $m = 2k$ or $2k+1$, can be made $(k-1)$-connected (*i.e.* so that any subcomplex of dimension $\leqslant (k-1)$ can be pulled to a point in M^m). If we could make M k-connected, the Poincaré duality theorem shows that we have attained our objective: for (a) M is a homotopy sphere, for (b) M is contractible.

However, the last step is more difficult. For m odd, it can always be accomplished (Wall [6], [KM])—the proofs for $m = 1, 3$ are unrelated to the case $m \geqslant 5$. If $m = 4k$, there is an integer obstruction $\sigma(M)$ called the signature of M, which is divisible by 8 for framed manifolds; if $m = 4k+2$, there is a somewhat mysterious mod 2 obstruction $\Phi(M)$, which arises as the Arf invariant of a certain quadratic form mod 2 (again see [KM]): except when $m = 4$, the vanishing of the obstruction is sufficient to allow us to complete the surgery.

This gives a fairly good answer to (a) and (b) above; to complete it, we note that $\sigma(M)$ vanishes for a closed framed manifold, and that if Σ^{4k-1} bounds a framed manifold M^{4k}, Σ only determines $\sigma(M)$ up to adding some multiple of a certain integer $i_k(M)$, also explicitly known. In particular, all the Θ^n are finite, except possibly Θ^3.

There is a neater way, found by Kervaire, to express these results (intended to be published as a sequel to [KM]). Define two further cobordism groups; A^n by closed manifolds, framed except at a point, and P^n by framed manifolds with homotopy sphere boundaries. There are homomorphisms $\Theta^n \to A^n$ (pick a framing on the contractible complement of a point), $A^n \to P^n$ (remove a neighbourhood of the bad point) and $P^n \to \Theta^{n-1}$ (take the boundary). A simple geometrical argument shows the sequence

$$\ldots P^{n+1} \to \Theta^n \to A^n \to P^n \to \Theta^{n-1} \ldots$$

to be exact. Although our surgery was described for Θ^n, a re-analysis (via handles instead of spherical modifications) shows that they compute P^n as:

$$P^n = 0 \;\; (n \text{ odd}), \quad \cong \mathbf{Z}_2 \;\; (n = 4k+2), \quad \cong \mathbf{Z} \;\; (n = 4k \neq 4)$$

The group A^n lies in another exact sequence

$$\ldots \pi_n(SO_N) \xrightarrow{J} \pi_{n+N}(S^N) \to A^n \to \pi_{n-1}(SO_N) \xrightarrow{J} \pi_{n+N-1}(S^N) \ldots$$

so is known from homotopy theory; the map $A^n \to P^n$ is known when $n = 4k$ (so both are infinite). A key problem is the computation of the map $A^n \to P^n \cong \mathbf{Z}_2$ when $n = 4k+2$; it is nonzero for $n = 2, 6, 14$ [KM] and zero for $n = 10$ (Kervaire [**1**]) and 18 (Kervaire, unpublished). An approach to this has been made by Novikov [**3**] and Brown and Peterson [**1**]; this has now led to a proof (Brown and Peterson, unpublished) that the map is zero when $n = 8k+2 > 2$.

Knots

A more recent application of surgery is to the classification of smooth knots of spheres in spheres. Analogous to Θ^n, consider the set $\Theta_d{}^n$ of homotopy n-spheres in S^{n+d}, modulo h-cobordism (of pairs). This again has a group structure. If $d \geqslant 3$, $n \geqslant 5$, h-cobordant pairs are diffeomorphic, by a further result of Smale [**5**]: this is definitely false for $d = 2$. The condition $n \geqslant 5$, however, is unnecessary if the Poincaré conjecture holds in dimensions 3 and 4. Let $P_d{}^{n+1}$ be the cobordism group defined by framed manifolds in S^{n+d} with boundary a homotopy n-sphere. Let $\mathfrak{G}_d$ be the space of maps of S^{d-1} to itself which are homotopic to the identity; this has as a subspace the rotations, SO_d. Then there is an exact sequence

$$\ldots \Theta_d{}^n \to \pi_n(\mathfrak{G}_d, SO_d) \to P_d{}^n \to \Theta_d{}^{n-1} \ldots$$

proved by similar, but more complicated geometric methods. Now it turns out that to compute $P_d{}^n$ for $d \geqslant 2$ we can proceed as for P^n, but do the surgeries inside the sphere S^{n+d}, and so $P_d{}^n \cong P^n$. In principle this solves the knot problem for $d > 2$ (if $n \geqslant 5$). For $d \geqslant 3$, these results are due to Levine [**1**], (Levine's argument is presented only for $n \geqslant 5$, but is easy to improve), and for $d = 2$ to Kervaire. (Kervaire's arguments work only if n is even). Observe that if $d = 1$, the Σ^n separates S^{n+1} into two contractible manifolds, hence if $n \geqslant 5$ it is unknotted, and in any case $\Theta_1{}^n$ vanishes. Before the paper of Levine [**1**], some information had been obtained by Hsiang, Levine and Szczarba [**1**] for the case $2d > n+1$ and by Haefliger [**3**] for $2d = n+3$. It follows from the exact sequences that $\Theta_d{}^n$ is finite except when $n = 4k-1$, $d \leqslant 2k+1$, which (if $k \neq 1$) is the sum of an infinite cyclic and a finite group; possibly $\Theta_d{}^3$ ($d \geqslant 4$), which is infinite if Θ^3 is; and $\Theta_2{}^{2k+1}$, which is not finitely generated (Fox and Milnor [**1**]—the proof uses Alexander polynomials—if $k = 0$; for $k = 0$, this is due to Kervaire).

If $d = 2$, the failure of the h-cobordism theorem means that $\Theta_2{}^n$ no longer gives a complete classification. For example, we have the following theorem (Wall [**7**]): Let L^r be any cell-complex such that a 2-cell can be attached to make it contractible, $n \geqslant 2r-3$, $n \geqslant 3$. Then there is a smooth knotted n-sphere in S^{n+2} with complement C, which has the same homotopy type as L " up to dimension $n-r+1$ ". From this can be deduced the result (due to Kervaire) that the class of groups $\mathfrak{G} = \pi_1(C)$, where C is the complement of some imbedding of S^n in S^{n+2}, is independent of n if $n \geqslant 3$. Such $\mathfrak{G}$ are characterized by being finitely presented, satisfying $H_1(\mathfrak{G}) = \mathbf{Z}$, $H_2(\mathfrak{G}) = 0$, and having an element whose conjugates generate the group.

These show the variety of knots when $d = 2$. In the opposite direction we have the following result of Levine. Let $n \geqslant 4$, K be a knotted n-sphere in S^{n+2}, with complement C such that all $\pi_i(C)$ are finitely generated abelian groups (in particular $\pi_1(C) \cong \mathbf{Z}$). Then C is a smooth fibre bundle over a circle, such that the closure in S^{n+2} of each fibre is a submanifold with boundary K. In particular if C is homotopy equivalent to a circle, K is unknotted. (Levine [**2**]). This is an extension of an analogous result when $n = 1$, due to Neuwirth [**1**] and Stallings [**1**].

Finally, there is a general construction of spinning, due in origin to Artin, and generalised recently by Zeeman [**1**] to twist-spinning (in a somewhat different setting). A comprehensive recent generalisation of this is due to Hsiang and Sanderson [**1**]; they have a formulation which, on h-cobordism classes, yields a map

$$\phi : \Theta_d^n \times \pi_e(SO_n) \times \pi_e(SO_d) \to \Theta_d^{n+e}:$$

when $d = 2$, the construction applies to actual knots, not merely equivalence classes. The map ϕ seems to generalise pairings found by Milnor [**5**], [**8**], Munkres [**1**] and Novikov [**1**] and used (*inter alia*) to study homotopy groups of diffeomorphisms of spheres.

Diffeomorphism classifications

We have already mentioned, as applications of Smale theory, numerous classifications which have been performed under strict assumptions on homotopy type. A more general approach is due to Novikov [**2**]. Let M^m be a smooth, compact, simply-connected manifold without boundary. We seek to classify up to diffeomorphism other such manifolds, of the same homotopy type as M. First note that if $K \geqslant m+1$, there is an essentially unique imbedding of M^m in $\mathbf{R}^{m+K}$: the normal bundle of this is our first diffeomorphism invariant of M. Let N^{m+K} be a corresponding tubular neighbourhood, $\dot{N}$ its boundary, T obtained from N by identifying $\dot{N}$ to a point. Then T is also obtained from $\mathbf{R}^{m+K}$—or let us say, S^{m+K}—by identifying also the complement of N to a point. The corresponding map $S^{n+K} \to T$ defines a homotopy class of degree 1 in $\pi_{m+K}(T)$; this (or rather, its equivalence class under bundle automorphisms of N) is the second diffeomorphism invariant.

There remain two problems, first to see which invariants correspond to smooth manifolds and second to see when two manifolds can have the same invariants: these are treated similarly, so we discuss the first.

Given a map $S^{m+K} \to T$ of degree 1, we put it in " general position "; then the inverse image of M is a smooth submanifold V^m of $\mathbf{R}^{m+K}$. Now proceed (as with homotopy spheres) to do surgery, only instead of killing the homotopy groups of V, we kill the relative homotopy groups of the map $V \to M$, so as to make the map a homotopy equivalence. It turns out that the same arguments apply, with the result, first that if $m \geqslant 6$, we can obtain a homotopy equivalence, provided m is odd, or $m = 4k$ and the signature satisfies a certain condition, or $m = 4k+2$ and a mod 2 condition (related to the Arf invariant) is satisfied. Next, if $m \geqslant 5$, the resulting manifold is unique up to connected sum with a homotopy sphere Σ^m which bounds a framed manifold (the proof of this uses the h-cobordism theorem).

The idea was extended by Browder [1] to the case where M is not even a smooth manifold, but just a C.W. complex satisfying Poincaré duality—given a bundle N, defining a " Thom space " T, and a homotopy class of degree 1 in $\pi_{m+K}(T)$, the above results remain valid. In particular, if $m \not\equiv 2 \pmod 4$, we obtain a necessary and sufficient condition for a C.W. complex which is simply connected to have the homotopy type of a smooth manifold. Combining this with an earlier result, Browder deduces that if M is a finite simply-connected C.W. complex which has a continuous multiplication with 2-sided unit, then (subject to reservations when $m \equiv 2 \pmod 4$) M has the homotopy type of a closed framed smooth manifold.

Extensions have been obtained by Wall [**8**] and Golo [**1**], by showing that the results can be extended to manifolds with boundary; indeed, if both manifold and boundary are connected and simply-connected, a stronger result is obtained: surgery is always possible, and the resulting manifold is always unique, under suitable dimensional restrictions (slightly different for the two results)—which need only be of the form $m \geqslant 6$, $2K \geqslant m+1$. A further paper of Wall [**7**] investigates the non simply-connected case: here surgery in the middle dimension is held up by formidable algebraic problems, which are only (as yet) tackled when the fundamental group is cyclic of prime order. This can be applied to the problem of diffeomorphism classification of smooth manifolds homotopy equivalent to real projective space P^n: here some nontrivial examples are given by Hirsch and Milnor [**1**]; the results of Wall [**7**], with those of another unpublished paper of W. Browder and G. R. Livesay, prove the number of classes finite if $n \geqslant 5$.

Other topics

Our selection of subjects for discussion was made for homogeneity of method. This leads to certain omissions, of which we have already mentioned immersions and imbeddings, and differentiable periodic maps

We have also omitted most results on 2- and 3-manifolds, which are usually obtained by rather special techniques: I do not know any good survey articles on the subject since Papakyriakopoulos [**1**]: see also the book containing Stallings [**1**]. The study of singularities of differentiable mappings is in a state of flux at present, being dominated by unpublished results of Thom and a recent theorem of Malgrange (*Cartan seminar notes*, 1962/63), so is not ripe for survey. However, the theory of smoothings of combinatorial manifolds, starting in 1958, has reached fairly definitive results (not quite yet published): however, most of the difficulties here arise from the combinatorial manifolds, and the theory is not directly relevant to the study of smooth manifolds *per se* (though one good application is given by Hirsch and Milnor [**1**], and no doubt there will be others). Good preliminary reading for this is a little book by Munkres [**2**].

The following are introductory accounts of differential topology. A book by Lang [**1**] gives only the most elementary geometry, but that in a highly abstract setting. A long paper by Cerf [**1**] contains many foundational results about spaces of smooth maps between smooth manifolds. We have already mentioned Milnor [**1**] and Munkres [**2**]. There are also duplicated lecture notes by Milnor [**9**], [**10**] and Wall [**9**]: it was the former of these which first introduced the term " differential topology ".

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