1
Smooth Manifolds

This book is about smooth manifolds. In the simplest terms, these are spaces that locally look like some Euclidean space $\mathbb{R}^n$, and on which one can do calculus. The most familiar examples, aside from Euclidean spaces themselves, are smooth plane curves such as circles and parabolas, and smooth surfaces such as spheres, tori, paraboloids, ellipsoids, and hyperboloids. Higher-dimensional examples include the set of unit vectors in $\mathbb{R}^{n+1}$ (the n-sphere) and graphs of smooth maps between Euclidean spaces.

The simplest examples of manifolds are the topological manifolds, which are topological spaces with certain properties that encode what we mean when we say that they "locally look like" $\mathbb{R}^n$. Such spaces are studied intensively by topologists.

However, many (perhaps most) important applications of manifolds involve calculus. For example, most applications of manifold theory to geometry involve the study of such properties as volume and curvature. Typically, volumes are computed by integration, and curvatures are computed by formulas involving second derivatives, so to extend these ideas to manifolds would require some means of making sense of differentiation and integration on a manifold. The applications of manifold theory to classical mechanics involve solving systems of ordinary differential equations on manifolds, and the applications to general relativity (the theory of gravitation) involve solving a system of partial differential equations.

The first requirement for transferring the ideas of calculus to manifolds is some notion of "smoothness." For the simple examples of manifolds we described above, all of which are subsets of Euclidean spaces, it is fairly easy to describe the meaning of smoothness on an intuitive level. For ex-

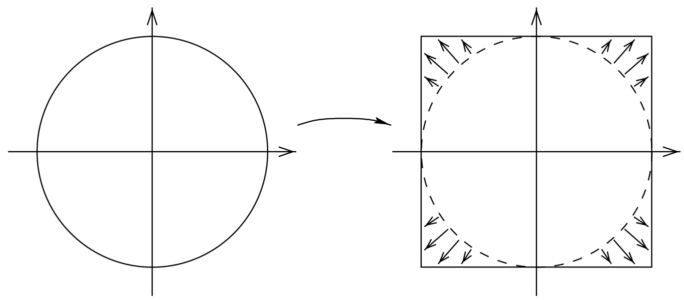

Figure 1.1. A homeomorphism from a circle to a square.

ample, we might want to call a curve "smooth" if it has a tangent line that varies continuously from point to point, and similarly a "smooth surface" should be one that has a tangent plane that varies continuously from point to point. But for more sophisticated applications it is an undue restriction to require smooth manifolds to be subsets of some ambient Euclidean space. The ambient coordinates and the vector space structure of $\mathbb{R}^n$ are superfluous data that often have nothing to do with the problem at hand. It is a tremendous advantage to be able to work with manifolds as abstract topological spaces, without the excess baggage of such an ambient space. For example, in general relativity, spacetime is thought of as a 4-dimensional smooth manifold that carries a certain geometric structure, called a *Lorentz metric*, whose curvature results in gravitational phenomena. In such a model there is no physical meaning that can be assigned to any higher-dimensional ambient space in which the manifold lives, and including such a space in the model would complicate it needlessly. For such reasons, we need to think of smooth manifolds as abstract topological spaces, not necessarily as subsets of larger spaces.

It is not hard to see that there is no way to define a purely topological property that would serve as a criterion for "smoothness," because it cannot be invariant under homeomorphisms. For example, a circle and a square in the plane are homeomorphic topological spaces (Figure 1.1), but we would probably all agree that the circle is "smooth," while the square is not. Thus topological manifolds will not suffice for our purposes. As a consequence, we will think of a smooth manifold as a set with two layers of structure: first a topology, then a smooth structure.

In the first section of this chapter we describe the first of these structures. A topological manifold is a topological space with three special properties that express the notion of being locally like Euclidean space. These properties are shared by Euclidean spaces and by all of the familiar geometric objects that look locally like Euclidean spaces, such as curves and surfaces.

We then prove some important topological properties of manifolds that we will use throughout the book.

In the next section we introduce an additional structure, called a smooth structure, that can be added to a topological manifold to enable us to make sense of derivatives.

Following the basic definitions, we introduce a number of examples of manifolds, so you can have something concrete in mind as you read the general theory. At the end of the chapter we introduce the concept of a smooth manifold with boundary, an important generalization of smooth manifolds that will be important in our study of integration in Chapters 14–16.

Topological Manifolds

In this section we introduce topological manifolds, the most basic type of manifolds. We assume that the reader is familiar with the basic properties of topological spaces, as summarized in the Appendix.

Suppose M is a topological space. We say that M is a *topological manifold of dimension n* or a *topological n-manifold* if it has the following properties:

- M is a *Hausdorff space*: For every pair of points $p, q \in M$, there are disjoint open subsets $U, V \subset M$ such that $p \in U$ and $q \in V$.
- M is *second countable*: There exists a countable basis for the topology of M.
- M is *locally Euclidean of dimension n*: Every point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

The locally Euclidean property means, more specifically, that for each $p \in M$, we can find the following:

- an open set $U \subset M$ containing p;
- an open set $\widetilde{U} \subset \mathbb{R}^n$; and
- a homeomorphism $\varphi\colon U \to \widetilde{U}$.

◇ **Exercise 1.1.** Show that equivalent definitions of locally Euclidean spaces are obtained if instead of requiring U to be homeomorphic to an open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

If M is a topological manifold, we often abbreviate the dimension of M as $\dim M$. In informal writing, one sometimes writes "Let M^n be a manifold" as shorthand for "Let M be a manifold of dimension n." The superscript n is not part of the name of the manifold, and is usually not included in the notation after the first occurrence.

The basic example of a topological n-manifold is, of course, $\mathbb{R}^n$. It is Hausdorff because it is a metric space, and it is second countable because the set of all open balls with rational centers and rational radii is a countable basis.

Requiring that manifolds share these properties helps to ensure that manifolds behave in the ways we expect from our experience with Euclidean spaces. For example, it is easy to verify that in a Hausdorff space, one-point sets are closed and limits of convergent sequences are unique (see Exercise A.5 in the Appendix). The motivation for second countability is a bit less evident, but it will have important consequences throughout the book, mostly based on the existence of partitions of unity (see Chapter 2).

In practice, both the Hausdorff and second countability properties are usually easy to check, especially for spaces that are built out of other manifolds, because both properties are inherited by subspaces and products (Lemmas A.5 and A.8). In particular, it follows easily that any open subset of a topological n-manifold is itself a topological n-manifold (with the subspace topology, of course).

The way we have defined topological manifolds, the empty set is a topological n-manifold for every n. For the most part, we will ignore this special case (sometimes without remembering to say so). But because it is useful in certain contexts to allow the empty manifold, we have chosen not to exclude it from the definition.

We should note that some authors choose to omit the Hausdorff property or second countability or both from the definition of manifolds. However, most of the interesting results about manifolds do in fact require these properties, and it is exceedingly rare to encounter a space "in nature" that would be a manifold except for the failure of one or the other of these hypotheses. For a couple of simple examples, see Problems 1-1 and 1-2; for a more involved example (a connected, locally Euclidean, Hausdorff space that is not second countable), see [Lee00, Problem 4-6].

Coordinate Charts

Let M be a topological n-manifold. A *coordinate chart* (or just a *chart*) on M is a pair (U, φ), where U is an open subset of M and $\varphi\colon U \to \widetilde{U}$ is a homeomorphism from U to an open subset $\widetilde{U} = \varphi(U) \subset \mathbb{R}^n$ (Figure 1.2). By definition of a topological manifold, each point $p \in M$ is contained in the domain of some chart (U, φ). If $\varphi(p) = 0$, we say that the chart is *centered at p*. If (U, φ) is any chart whose domain contains p, it is easy to obtain a new chart centered at p by subtracting the constant vector $\varphi(p)$.

Given a chart (U, φ), we call the set U a *coordinate domain*, or a *coordinate neighborhood* of each of its points. If in addition $\varphi(U)$ is an open ball in $\mathbb{R}^n$, then U is called a *coordinate ball*. The map φ is called a (*local*) *coordinate map*, and the component functions $(x^1, \ldots, x^n)$ of φ, defined by

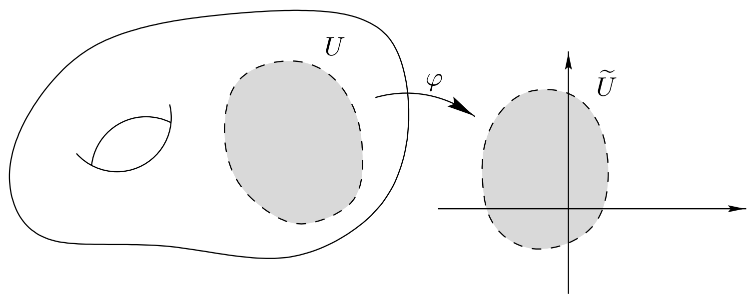

Figure 1.2. A coordinate chart.

$\varphi(p) = (x^1(p), \ldots, x^n(p))$, are called *local coordinates* on U. We will sometimes write things like "(U, φ) is a chart containing p" as shorthand for "(U, φ) is a chart whose domain U contains p." If we wish to emphasize the coordinate functions $(x^1, \ldots, x^n)$ instead of the coordinate map φ, we will sometimes denote the chart by $(U, (x^1, \ldots, x^n))$ or $(U, (x^i))$.

Examples of Topological Manifolds

Here are some simple examples of topological manifolds.

Example 1.1 (Graphs of Continuous Functions). Let $U \subset \mathbb{R}^n$ be an open set, and let $F\colon U \to \mathbb{R}^k$ be a continuous function. The *graph* of F is the subset of $\mathbb{R}^n \times \mathbb{R}^k$ defined by

$$\Gamma(F) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : x \in U \text{ and } y = F(x)\},$$

with the subspace topology. Let $\pi_1\colon \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ denote the projection onto the first factor, and let $\varphi_F\colon \Gamma(F) \to U$ be the restriction of π_1 to $\Gamma(F)$:

$$\varphi_F(x, y) = x, \quad (x, y) \in \Gamma(F).$$

Because φ_F is the restriction of a continuous map, it is continuous; and it is a homeomorphism because it has a continuous inverse given by

$$(\varphi_F)^{-1}(x) = (x, F(x)).$$

Thus $\Gamma(F)$ is a topological manifold of dimension n. In fact, $\Gamma(F)$ is homeomorphic to U itself, and $(\Gamma(F), \varphi_F)$ is a global coordinate chart, called *graph coordinates*. The same observation applies to any subset of $\mathbb{R}^{n+k}$ defined by setting any k of the coordinates (not necessarily the last k) equal to some continuous function of the other n, which are restricted to lie in an open subset of $\mathbb{R}^n$.

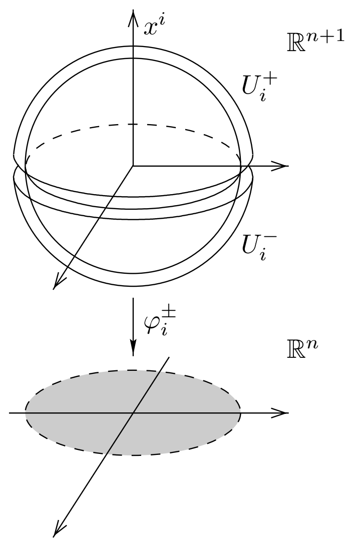

Figure 1.3. Charts for $\mathbb{S}^n$.

Example 1.2 (Spheres). Let $\mathbb{S}^n$ denote the (*unit*) *n-sphere*, which is the set of unit vectors in $\mathbb{R}^{n+1}$:

$$\mathbb{S}^n = \left\{ x \in \mathbb{R}^{n+1} : |x| = 1 \right\},$$

with the subspace topology. It is Hausdorff and second countable because it is a topological subspace of $\mathbb{R}^n$. To show that it is locally Euclidean, for each index $i = 1, \ldots, n+1$ let U_i^+ denote the subset of $\mathbb{S}^n$ where the ith coordinate is positive:

$$U_i^+ = \left\{ \left(x^1, \ldots, x^{n+1}\right) \in \mathbb{S}^n : x^i > 0 \right\}.$$

(See Figure 1.3.) Similarly, U_i^- is the set where $x^i < 0$.

Let $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ denote the open unit ball in $\mathbb{R}^n$, and let $f \colon \mathbb{B}^n \to \mathbb{R}$ be the continuous function

$$f(u) = \sqrt{1 - |u|^2}.$$

Then for each $i = 1, \ldots, n+1$, it is easy to check that $U_i^+ \cap \mathbb{S}^n$ is the graph of the function

$$x^i = f\left(x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1}\right),$$

where the hat over x^i indicates that x^i is omitted. Similarly, $U_i^- \cap \mathbb{S}^n$ is the graph of

$$x^i = -f\left(x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1}\right).$$

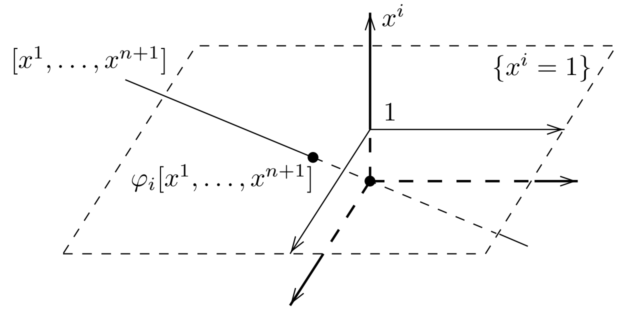

Figure 1.4. A chart for $\mathbb{RP}^n$.

Thus each set $U_i^{\pm} \cap \mathbb{S}^n$ is locally Euclidean of dimension n, and the maps $\varphi_i^{\pm}: U_i^{\pm} \cap \mathbb{S}^n \to \mathbb{B}^n$ given by

$$\varphi_i^{\pm}\left(x^1, \ldots, x^{n+1}\right) = \left(x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1}\right)$$

are graph coordinates for $\mathbb{S}^n$. Since every point in $\mathbb{S}^n$ is in the domain of at least one of these $2n+2$ charts, $\mathbb{S}^n$ is a topological n-manifold.

Example 1.3 (Projective Spaces). The n-dimensional *real projective space*, denoted by $\mathbb{RP}^n$ (or sometimes just $\mathbb{P}^n$), is defined as the set of 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$. We give it the quotient topology determined by the natural map $\pi: \mathbb{R}^{n+1} \smallsetminus \{0\} \to \mathbb{RP}^n$ sending each point $x \in \mathbb{R}^{n+1} \smallsetminus \{0\}$ to the subspace spanned by x. For any point $x \in \mathbb{R}^{n+1} \smallsetminus \{0\}$, let $[x] = \pi(x)$ denote the equivalence class of x in $\mathbb{RP}^n$.

For each $i = 1, \ldots, n+1$, let $\widetilde{U}_i \subset \mathbb{R}^{n+1} \smallsetminus \{0\}$ be the set where $x^i \neq 0$, and let $U_i = \pi(\widetilde{U}_i) \subset \mathbb{RP}^n$. Since $\widetilde{U}_i$ is a saturated open set, U_i is open and $\pi|_{\widetilde{U}_i}: \widetilde{U}_i \to U_i$ is a quotient map (see Lemma A.10). Define a map $\varphi_i: U_i \to \mathbb{R}^n$ by

$$\varphi_i\left[x^1, \ldots, x^{n+1}\right] = \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{n+1}}{x^i}\right).$$

This map is well-defined because its value is unchanged by multiplying x by a nonzero constant. Because $\varphi_i \circ \pi$ is continuous, φ_i is continuous by the characteristic property of quotient maps (Lemma A.10). In fact, φ_i is a homeomorphism, because its inverse is given by

$$\varphi_i^{-1}\left(u^1, \ldots, u^n\right) = \left[u^1, \ldots, u^{i-1}, 1, u^i, \ldots, u^n\right],$$

as you can easily check. Geometrically, if we identify $\mathbb{R}^n$ in the obvious way with the affine subspace where $x^i = 1$, then $\varphi_i[x]$ can be interpreted as the point where the line $[x]$ intersects this subspace (Figure 1.4). Because the sets U_i cover $\mathbb{RP}^n$, this shows that $\mathbb{RP}^n$ is locally Euclidean of dimension n. The Hausdorff and second countability properties are left as exercises.

◇ **Exercise 1.2.** Show that $\mathbb{RP}^n$ is Hausdorff and second countable, and is therefore a topological n-manifold.

◇ **Exercise 1.3.** Show that $\mathbb{RP}^n$ is compact. [Hint: Show that the restriction of π to $\mathbb{S}^n$ is surjective.]

Example 1.4 (Product Manifolds). Suppose $M_1, \dots, M_k$ are topological manifolds of dimensions $n_1, \dots, n_k$, respectively. We will show that the product space $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$. It is Hausdorff and second countable by Lemmas A.5 and A.8, so only the locally Euclidean property needs to be checked. Given any point $(p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$, we can choose a coordinate chart (U_i, φ_i) for each M_i with $p_i \in U_i$. The product map

$$\varphi_1 \times \cdots \times \varphi_k \colon U_1 \times \cdots \times U_k \to \mathbb{R}^{n_1 + \cdots + n_k}$$

is a homeomorphism onto its image, which is an open subset of $\mathbb{R}^{n_1 + \cdots + n_k}$. Thus $M_1 \times \cdots \times M_k$ is a topological manifold of dimension $n_1 + \cdots + n_k$, with charts of the form $(U_1 \times \cdots \times U_k, \varphi_1 \times \cdots \times \varphi_k)$.

Example 1.5 (Tori). For any positive integer n, the *n-torus* is the product space $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. By the discussion above, it is an n-dimensional topological manifold. (The 2-torus is usually called simply "the torus.")

Topological Properties of Manifolds

As topological spaces go, manifolds are quite special, because they share so many important properties with Euclidean spaces. In this section we discuss a few such properties that will be of use to us throughout the book.

The first property we need is that every manifold has a particularly well behaved basis for its topology. If X is a topological space, a subset $K \subset X$ is said to be *precompact* (or *relatively compact*) in X if its closure in X is compact.

Lemma 1.6. *Every topological manifold has a countable basis of precompact coordinate balls.*

Proof. Let M be a topological n-manifold. First we will prove the lemma in the special case in which M can be covered by a single chart. Suppose $\varphi \colon M \to \widetilde{U} \subset \mathbb{R}^n$ is a global coordinate map, and let $\mathcal{B}$ be the collection of all open balls $B_r(x) \subset \mathbb{R}^n$ such that r is rational, x has rational coordinates, and $\overline{B}_r(x) \subset \widetilde{U}$. Each such ball is precompact in $\widetilde{U}$, and it is easy to check that $\mathcal{B}$ is a countable basis for the topology of $\widetilde{U}$. Because φ is a homeomorphism, it follows that the collection of sets of the form $\varphi^{-1}(B)$ for $B \in \mathcal{B}$ is a countable basis for the topology of M, consisting of precompact coordinate balls, with the restrictions of φ as coordinate maps.

Now let M be an arbitrary n-manifold. By definition, every point of M is in the domain of a chart. Because every open cover of a second countable space has a countable subcover (Lemma A.4), M is covered by countably many charts $\{(U_i, \varphi_i)\}$. By the argument in the preceding paragraph, each coordinate domain U_i has a countable basis of precompact coordinate balls, and the union of all these countable bases is a countable basis for the topology of M. If $V \subset U_i$ is one of these precompact balls, then the closure of V in U_i is compact, hence closed in M. It follows that the closure of V in M the same as its closure in U_i, so V is precompact in M as well. □

A topological space M is said to be *locally compact* if every point has a neighborhood contained in a compact subset of M. If M is Hausdorff, this is equivalent to the requirement that M have a basis of precompact open sets (see [Lee00, Proposition 4.27]). The following corollary is immediate.

Corollary 1.7. *Every topological manifold is locally compact.*

Connectivity

The existence of a basis of coordinate balls has important consequences for the connectivity properties of manifolds. Recall that a topological space X is said to be

- *connected* if there do not exist two disjoint, nonempty, open subsets of X whose union is X;
- *path connected* if every pair of points in X can be joined by a path in X; and
- *locally path connected* if X has a basis of path connected open sets.

(See the Appendix, pages 550–552, for a review of these concepts.) The following proposition shows that connectivity and path connectivity coincide for manifolds.

Proposition 1.8. *Let M be a topological manifold.*

(*a*) *M is locally path connected.*

(*b*) *M is connected if and only if it is path connected.*

(*c*) *The components of M are the same as its path components.*

(*d*) *M has at most countably many components, each of which is an open subset of M and a connected topological manifold.*

Proof. Since every coordinate ball is path connected, part (a) follows from the fact that M has a basis of coordinate balls (Lemma 1.6). Parts (b) and (c) are immediate consequences of (a) (see Lemma A.16). To prove (d), note that each component is open in M by Lemma A.16, so the collection of components is an open cover of M. Because M is second countable, this cover must have a countable subcover. But since the components are all disjoint, the cover must have been countable to begin with, which is to say that M has only countably many components. □

Fundamental Groups of Manifolds

The following result about fundamental groups of manifolds will be important in our study of covering manifolds in Chapters 2 and 9. For a brief review of the fundamental group, see the Appendix, pages 553–555.

Proposition 1.9. *The fundamental group of any topological manifold is countable.*

Proof. Let M be a topological manifold. By Lemma 1.6, there is a countable collection $\mathcal{B}$ of coordinate balls covering M. For any pair of coordinate balls $B, B' \in \mathcal{B}$, the intersection $B \cap B'$ has at most countably many components, each of which is path connected. Let $\mathcal{X}$ be a countable set containing one point from each component of $B \cap B'$ for each $B, B' \in \mathcal{B}$ (including $B = B'$). For each $B \in \mathcal{B}$ and each $x, x' \in \mathcal{X}$ such that $x, x' \in B$, let $p^B_{x,x'}$ be some path from x to x' in B.

Since the fundamental groups based at any two points in the same component of M are isomorphic, and $\mathcal{X}$ contains at least one point in each component of M, we may as well choose a point $q \in \mathcal{X}$ as base point. Define a *special loop* to be a loop based at q that is equal to a finite product of paths of the form $p^B_{x,x'}$. Clearly, the set of special loops is countable, and each special loop determines an element of $\pi_1(M, q)$. To show that $\pi_1(M, q)$ is countable, therefore, it suffices to show that every element of $\pi_1(M, q)$ is represented by a special loop.

Suppose $f \colon [0, 1] \to M$ is any loop based at q. The collection of components of sets of the form $f^{-1}(B)$ as B ranges over $\mathcal{B}$ is an open cover of $[0, 1]$, so by compactness it has a finite subcover. Thus there are finitely many numbers $0 = a_0 < a_1 < \cdots < a_k = 1$ such that $[a_{i-1}, a_i] \subset f^{-1}(B)$ for some $B \subset \mathcal{B}$. For each i, let f_i be the restriction of f to the interval $[a_{i-1}, a_i]$, reparametrized so that its domain is $[0, 1]$, and let $B_i \in \mathcal{B}$ be a coordinate ball containing the image of f_i. For each i, we have $f(a_i) \in B_i \cap B_{i+1}$, and there is some $x_i \in \mathcal{X}$ that lies in the same component of $B_i \cap B_{i+1}$ as $f(a_i)$. Let g_i be a path in $B_i \cap B_{i+1}$ from x_i to $f(a_i)$ (Figure 1.5), with the understanding that $x_0 = x_k = q$, and g_0 and g_k are both equal to the constant path c_q based at q. Then, because $g_i^{-1} \cdot g_i$ is path homotopic to a constant path,

$$
\begin{aligned}
f &\sim f_1 \cdot \cdots \cdot f_k \\
&\sim g_0 \cdot f_1 \cdot g_1^{-1} \cdot g_1 \cdot f_2 \cdot g_2^{-1} \cdot \cdots \cdot g_{k-1}^{-1} \cdot g_{k-1} \cdot f_k \cdot g_k^{-1} \\
&\sim \widetilde{f}_1 \cdot \widetilde{f}_2 \cdot \cdots \cdot \widetilde{f}_n,
\end{aligned}
$$

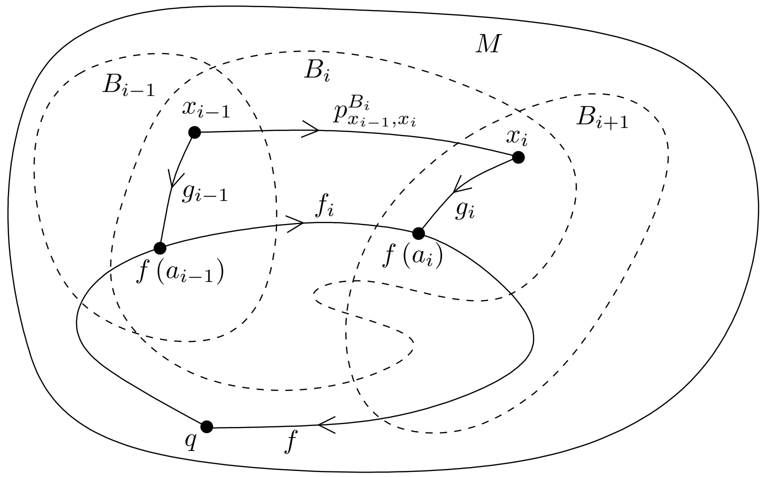

Figure 1.5. The fundamental group of a manifold is countable.

where $\widetilde{f}_i = g_{i-1} \cdot f_i \cdot g_i^{-1}$. For each i, $\widetilde{f}_i$ is a path in B_i from x_{i-1} to x_i. Since B_i is simply connected, $\widetilde{f}_i$ is path homotopic to $p^{B_i}_{x_{i-1},x_i}$. It follows that f is path homotopic to a special loop, as claimed. □

Smooth Structures

The definition of manifolds that we gave in the preceding section is sufficient for studying topological properties of manifolds, such as compactness, connectedness, simple connectedness, and the problem of classifying manifolds up to homeomorphism. However, in the entire theory of topological manifolds there is no mention of calculus. There is a good reason for this: However we might try to make sense of derivatives of functions on a manifold, such derivatives cannot be invariant under homeomorphisms. For example, the map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\varphi(u,v) = \left(u^{1/3}, v^{1/3}\right)$ is a homeomorphism, and it is easy to construct differentiable functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f \circ \varphi$ is not differentiable at the origin. (The function $f(x,y) = x$ is one such.)

To make sense of derivatives of real-valued functions, curves, or maps between manifolds, we will need to introduce a new kind of manifold called a "smooth manifold." It will be a topological manifold with some extra structure in addition to its topology, which will allow us to decide which functions on the manifold are smooth.

The definition will be based on the calculus of maps between Euclidean spaces, so let us begin by reviewing some basic terminology about such maps. If U and V are open subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, a function $F\colon U \to V$ is said to be *smooth* (or C^∞, or *infinitely differentiable*) if each of its component functions has continuous partial derivatives of all orders. If in addition F is bijective and has a smooth inverse map, it is called a *diffeomorphism.* A diffeomorphism is, in particular, a homeomorphism. A review of some of the most important properties of smooth maps is given in the Appendix. (You should be aware that some authors use the word "smooth" in somewhat different senses, for example to mean continuously differentiable or merely differentiable. On the other hand, some use the word "differentiable" to mean what we call "smooth." Throughout this book, "smooth" will for us be synonymous with C^∞.)

To see what additional structure on a topological manifold might be appropriate for discerning which maps are smooth, consider an arbitrary topological n-manifold M. Each point in M is in the domain of a coordinate map $\varphi\colon U \to \widetilde{U} \subset \mathbb{R}^n$. A plausible definition of a smooth function on M would be to say that $f\colon M \to \mathbb{R}$ is smooth if and only if the composite function $f \circ \varphi^{-1}\colon \widetilde{U} \to \mathbb{R}$ is smooth in the sense of ordinary calculus. But this will make sense only if this property is independent of the choice of coordinate chart. To guarantee this independence, we will restrict our attention to "smooth charts." Since smoothness is not a homeomorphism-invariant property, the way to do this is to consider the collection of all smooth charts as a new kind of structure on M.

With this motivation in mind, we now describe the details of the construction.

Let M be a topological n-manifold. If (U,φ), (V,ψ) are two charts such that $U \cap V \neq \varnothing$, the composite map $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$ is called the *transition map* from φ to ψ (Figure 1.6). It is a composition of homeomorphisms, and is therefore itself a homeomorphism. Two charts (U,φ) and (V,ψ) are said to be *smoothly compatible* if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. (Since $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^n$, smoothness of this map is to be interpreted in the ordinary sense of having continuous partial derivatives of all orders.)

We define an *atlas* for M to be a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible with each other.

It often happens in practice that we can prove for *every pair* of coordinate maps φ and ψ in a given atlas that the transition map $\psi \circ \varphi^{-1}$ is smooth. Once we have done this, it is unnecessary to verify directly that $\psi \circ \varphi^{-1}$ is a diffeomorphism, because its inverse $(\psi \circ \varphi^{-1})^{-1} = \varphi \circ \psi^{-1}$ is one of the transition maps we have already shown to be smooth. We will use this observation without further comment when appropriate.

Our plan is to define a "smooth structure" on M by giving a smooth atlas, and to define a function $f\colon M \to \mathbb{R}$ to be smooth if and only if $f \circ \varphi^{-1}$ is smooth in the sense of ordinary calculus for each coordinate chart (U,φ) in the atlas. There is one minor technical problem with this approach: In

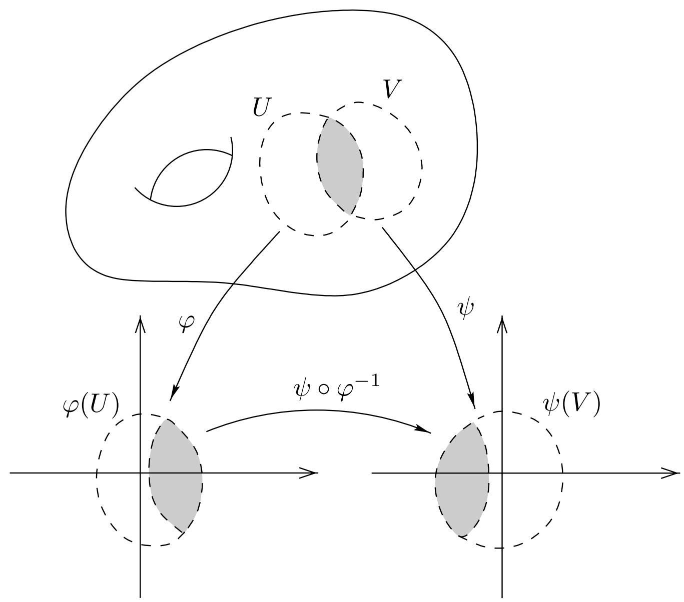

Figure 1.6. A transition map.

general, there will be many possible choices of atlas that give the "same" smooth structure, in that they all determine the same collection of smooth functions on M. For example, consider the following pair of atlases on $\mathbb{R}^n$:

$$\begin{aligned} \mathcal{A}_1 &= \{(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})\} \\ \mathcal{A}_2 &= \left\{\left(B_1(x), \mathrm{Id}_{B_1(x)}\right) : x \in \mathbb{R}^n\right\}. \end{aligned}$$

Although these are different smooth atlases, clearly a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is smooth with respect to either atlas if and only if it is smooth in the sense of ordinary calculus.

We could choose to define a smooth structure as an equivalence class of smooth atlases under an appropriate equivalence relation. However, it is more straightforward to make the following definition: A smooth atlas $\mathcal{A}$ on M is *maximal* if it is not contained in any strictly larger smooth atlas. This just means that any chart that is smoothly compatible with every chart in $\mathcal{A}$ is already in $\mathcal{A}$. (Such a smooth atlas is also said to be *complete*.)

Now we can define the main concept of this chapter. A *smooth structure* on a topological n-manifold M is a maximal smooth atlas. A *smooth manifold* is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M. When the smooth structure is understood, we usually omit mention of it and just say "M is a smooth manifold." Smooth structures are also called *differentiable structures* or C^∞ *structures* by some authors. We will use the term *smooth manifold structure* to mean a manifold topology together with a smooth structure.

We emphasize that a smooth structure is an additional piece of data that must be added to a topological manifold before we are entitled to talk about a "smooth manifold." In fact, a given topological manifold may have many different smooth structures (see Example 1.14 and Problem 1-3). And it should be noted that it is not always possible to find a smooth structure on a given topological manifold: There exist topological manifolds that admit no smooth structures at all. (The first example was a compact 10-dimensional manifold found in 1960 by Michel Kervaire [Ker60].)

It is generally not very convenient to define a smooth structure by explicitly describing a maximal smooth atlas, because such an atlas contains very many charts. Fortunately, we need only specify *some* smooth atlas, as the next lemma shows.

Lemma 1.10. *Let M be a topological manifold.*

(*a*) *Every smooth atlas for M is contained in a unique maximal smooth atlas.*

(*b*) *Two smooth atlases for M determine the same maximal smooth atlas if and only if their union is a smooth atlas.*

Proof. Let $\mathcal{A}$ be a smooth atlas for M, and let $\overline{\mathcal{A}}$ denote the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$. To show that $\overline{\mathcal{A}}$ is a smooth atlas, we need to show that any two charts of $\overline{\mathcal{A}}$ are smoothly compatible with each other, which is to say that for any (U,φ), $(V,\psi) \in \overline{\mathcal{A}}$, $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ is smooth.

Let $x = \varphi(p) \in \varphi(U \cap V)$ be arbitrary. Because the domains of the charts in $\mathcal{A}$ cover M, there is some chart $(W,\theta) \in \mathcal{A}$ such that $p \in W$ (Figure 1.7). Since every chart in $\overline{\mathcal{A}}$ is smoothly compatible with (W,θ), both of the maps $\theta \circ \varphi^{-1}$ and $\psi \circ \theta^{-1}$ are smooth where they are defined. Since $p \in U \cap V \cap W$, it follows that $\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1})$ is smooth on a neighborhood of x. Thus $\psi \circ \varphi^{-1}$ is smooth in a neighborhood of each point in $\varphi(U \cap V)$. Therefore, $\overline{\mathcal{A}}$ is a smooth atlas. To check that it is maximal, just note that any chart that is smoothly compatible with every chart in $\overline{\mathcal{A}}$ must in particular be smoothly compatible with every chart in $\mathcal{A}$, so it is already in $\overline{\mathcal{A}}$. This proves the existence of a maximal smooth atlas containing $\mathcal{A}$. If $\mathcal{B}$ is any other maximal smooth atlas containing $\mathcal{A}$, each of its charts is smoothly compatible with each chart in $\mathcal{A}$, so $\mathcal{B} \subset \overline{\mathcal{A}}$. By maximality of $\mathcal{B}$, $\mathcal{B} = \overline{\mathcal{A}}$.

The proof of (b) is left as an exercise. □

◇ **Exercise 1.4.** Prove Lemma 1.10(b).

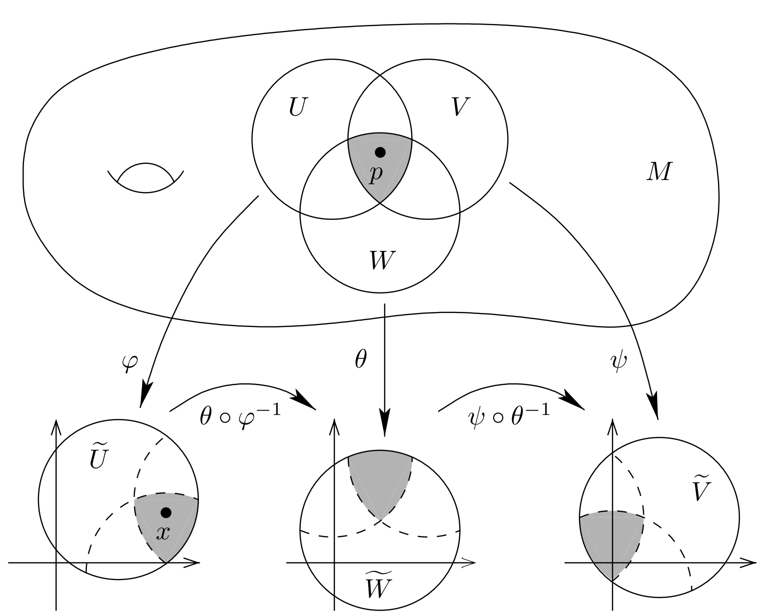

Figure 1.7. Proof of Lemma 1.10(a).

For example, if a topological manifold M can be covered by a single chart, the smooth compatibility condition is trivially satisfied, so any such chart automatically determines a smooth structure on M.

It is worth mentioning that the notion of smooth structure can be generalized in several different ways by changing the compatibility requirement for charts. For example, if we replace the requirement that charts be smoothly compatible by the weaker requirement that each transition map $\psi \circ \varphi^{-1}$ (and its inverse) be of class C^k, we obtain the definition of a *C^k structure*. Similarly, if we require that each transition map be real-analytic (i.e., expressible as a convergent power series in a neighborhood of each point), we obtain the definition of a *real-analytic structure*, also called a *C^ω structure*. If M has even dimension $n = 2m$, we can identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$ and require that the transition maps be complex-analytic; this determines a *complex-analytic structure*. A manifold endowed with one of these structures is called a *C^k manifold*, *real-analytic manifold*, or *complex manifold*, respectively. (Note that a C^0 manifold is just a topological manifold.) We will not treat any of these other kinds of manifolds in this book, but they play important roles in analysis, so it is useful to know the definitions.

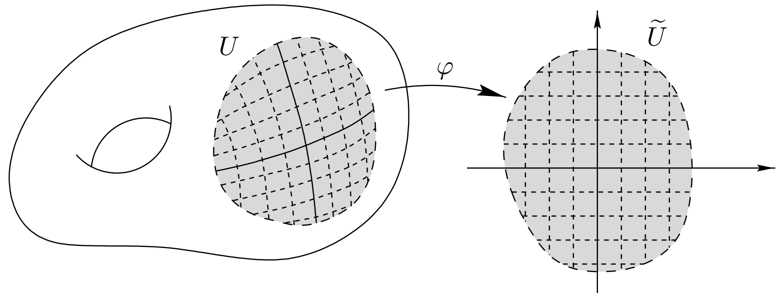

Figure 1.8. A coordinate grid.

Local Coordinate Representations

If M is a smooth manifold, any chart (U, φ) contained in the given maximal smooth atlas will be called a *smooth chart*, and the corresponding coordinate map φ will be called a *smooth coordinate map*. It is useful also to introduce the terms *smooth coordinate domain* or *smooth coordinate neighborhood* for the domain of a smooth coordinate chart. A *smooth coordinate ball* will mean a smooth coordinate domain whose image under a smooth coordinate map is a ball in Euclidean space.

The next lemma gives a slight improvement on Lemma 1.6 for smooth manifolds. Its proof is a straightforward adaptation of the proof of that lemma.

Lemma 1.11. *Every smooth manifold has a countable basis of precompact smooth coordinate balls.*

◇ **Exercise 1.5.** Prove Lemma 1.11.

Here is how one usually thinks about coordinate charts on a smooth manifold. Once we choose a smooth chart (U, φ) on M, the coordinate map $\varphi\colon U \to \widetilde{U} \subset \mathbb{R}^n$ can be thought of as giving an *identification* between U and $\widetilde{U}$. Using this identification, we can think of U simultaneously as an open subset of M and (at least temporarily while we work with this chart) as an open subset of $\mathbb{R}^n$. You can visualize this identification by thinking of a "grid" drawn on U representing the inverse images of the coordinate lines under φ (Figure 1.8). Under this identification, we can represent a point $p \in U$ by its coordinates $(x^1, \dots, x^n) = \varphi(p)$, and think of this n-tuple as *being* the point p. We will typically express this by saying "$(x^1, \dots, x^n)$ is the (local) coordinate representation for p" or "$p = (x^1, \dots, x^n)$ in local coordinates."

Another way to look at it is that by means of our identification $U \leftrightarrow \widetilde{U}$, we can think of φ as the identity map and suppress it from the notation. This takes a bit of getting used to, but the payoff is a huge simplification of the notation in many situations. You just need to remember that the identification is in general only local, and depends heavily on the choice of coordinate chart.

For example, if $M = \mathbb{R}^2$, let $U = \{(x,y) : x > 0\} \subset M$ be the open right half-plane, and let $\varphi\colon U \to \mathbb{R}^2$ be the *polar coordinate map* $\varphi(x,y) = (r,\theta) = \left(\sqrt{x^2+y^2}, \tan^{-1} y/x\right)$. We can write a given point $p \in U$ either as $p = (x,y)$ in standard coordinates or as $p = (r,\theta)$ in polar coordinates, where the two coordinate representations are related by $(r,\theta) = \left(\sqrt{x^2+y^2}, \tan^{-1} y/x\right)$ and $(x,y) = (r\cos\theta, r\sin\theta)$.

Examples of Smooth Manifolds

Before proceeding further with the general theory, let us survey some examples of smooth manifolds.

Example 1.12 (Zero-Dimensional Manifolds). A zero-dimensional topological manifold M is just a countable discrete space. For each point $p \in M$, the only neighborhood of p that is homeomorphic to an open subset of $\mathbb{R}^0$ is $\{p\}$ itself, and there is exactly one coordinate map $\varphi\colon \{p\} \to \mathbb{R}^0$. Thus the set of all charts on M trivially satisfies the smooth compatibility condition, and every zero-dimensional manifold has a unique smooth structure.

Example 1.13 (Euclidean Spaces). $\mathbb{R}^n$ is a smooth n-manifold with the smooth structure determined by the atlas consisting of the single chart $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$. We call this the *standard smooth structure*, and the resulting coordinate map *standard coordinates*. Unless we explicitly specify otherwise, we will always use this smooth structure on $\mathbb{R}^n$.

Example 1.14 (Another Smooth Structure on the Real Line). Consider the homeomorphism $\psi\colon \mathbb{R} \to \mathbb{R}$ given by

$$\psi(x) = x^3. \tag{1.1}$$

The atlas consisting of the single chart $(\mathbb{R}, \psi)$ defines a smooth structure on $\mathbb{R}$. This chart is not smoothly compatible with the standard smooth structure, because the transition map $\mathrm{Id}_{\mathbb{R}^n} \circ \psi^{-1}(y) = y^{1/3}$ is not smooth at the origin. Therefore, the smooth structure defined on $\mathbb{R}$ by ψ is not the same as the standard one. Using similar ideas, it is not hard to construct many distinct smooth structures on any given positive-dimensional topological manifold, as long as it has one smooth structure to begin with (see Problem 1-3).

Example 1.15 (Finite-Dimensional Vector Spaces). Let V be a finite-dimensional vector space. Any norm on V determines a topology, which is independent of the choice of norm (Exercise A.53). With this topology, V has a natural smooth manifold structure defined as follows. Any (ordered) basis $(E_1, \dots, E_n)$ for V defines a basis isomorphism $E\colon \mathbb{R}^n \to V$ by

$$E(x) = \sum_{i=1}^{n} x^i E_i.$$

This map is a homeomorphism, so the atlas consisting of the single chart (V, E^{-1}) defines a smooth structure. To see that this smooth structure is independent of the choice of basis, let $\bigl(\widetilde{E}_1, \dots, \widetilde{E}_n\bigr)$ be any other basis and let $\widetilde{E}(x) = \sum_j x^j \widetilde{E}_j$ be the corresponding isomorphism. There is some invertible matrix $\bigl(A_i^j\bigr)$ such that $E_i = \sum_j A_i^j \widetilde{E}_j$ for each i. The transition map between the two charts is then given by $\widetilde{E}^{-1} \circ E(x) = \widetilde{x}$, where $\widetilde{x} = \bigl(\widetilde{x}^1, \dots, \widetilde{x}^n\bigr)$ is determined by

$$\sum_{j=1}^{n} \widetilde{x}^j \widetilde{E}_j = \sum_{i=1}^{n} x^i E_i = \sum_{i,j=1}^{n} x^i A_i^j \widetilde{E}_j.$$

It follows that $\widetilde{x}^j = \sum_i A_i^j x^i$. Thus the map from x to $\widetilde{x}$ is an invertible linear map and hence a diffeomorphism, so the two charts are smoothly compatible. This shows that the union of the two charts determined by any two bases is still a smooth atlas, and thus all bases determine the same smooth structure. We will call this the *standard smooth structure* on V.

The Einstein Summation Convention

This is a good place to pause and introduce an important notational convention that we will use throughout the book. Because of the proliferation of summations such as $\sum_i x^i E_i$ in this subject, we will often abbreviate such a sum by omitting the summation sign, as in

$$E(x) = x^i E_i.$$

We interpret any such expression according to the following rule, called the *Einstein summation convention*: If the same index name (such as i in the expression above) appears exactly twice in any monomial term, once as an upper index and once as a lower index, that term is understood to be summed over all possible values of that index, generally from 1 to the dimension of the space in question. This simple idea was introduced by Einstein to reduce the complexity of the expressions arising in the study of smooth manifolds by eliminating the necessity of explicitly writing summation signs.

Another important aspect of the summation convention is the positions of the indices. We will always write basis vectors (such as E_i) with lower indices, and components of a vector with respect to a basis (such as x^i) with upper indices. These index conventions help to ensure that, in summations that make mathematical sense, any index to be summed over will typically appear twice in any given term, once as a lower index and once as an upper index. Any index that is implicitly summed over is a "dummy index," meaning that the value of such an expression is unchanged if a different name is substituted for each dummy index. For example, $x^i E_i$ and $x^j E_j$ mean exactly the same thing.

Since the coordinates of a point $\left(x^1, \dots, x^n\right) \in \mathbb{R}^n$ are also its components with respect to the standard basis, in order to be consistent with our convention of writing components of vectors with upper indices, we need to use upper indices for these coordinates, and we will do so throughout this book. Although this may seem awkward at first, in combination with the summation convention it offers enormous advantages when we work with complicated indexed sums, not the least of which is that expressions that are not mathematically meaningful often betray themselves quickly by violating the index convention. (The main exceptions are expressions involving the Euclidean dot product $x \cdot y = \sum_i x^i y^i$, in which the same index appears twice in the upper position, and the standard symplectic form on $\mathbb{R}^{2n}$, which we will define in Chapter 12. We will always explicitly write summation signs in such expressions.)

More Examples

Now we continue with our examples of smooth manifolds.

Example 1.16 (Matrices). Let $\mathrm{M}(m \times n, \mathbb{R})$ denote the space of $m \times n$ matrices with real entries. It is a vector space of dimension mn under matrix addition and scalar multiplication. Thus $\mathrm{M}(m \times n, \mathbb{R})$ is a smooth mn-dimensional manifold. Similarly, the space $\mathrm{M}(m \times n, \mathbb{C})$ of $m \times n$ complex matrices is a vector space of dimension $2mn$ over $\mathbb{R}$, and thus a smooth manifold of dimension $2mn$. In the special case $m = n$ (square matrices), we will abbreviate $\mathrm{M}(n \times n, \mathbb{R})$ and $\mathrm{M}(n \times n, \mathbb{C})$ by $\mathrm{M}(n, \mathbb{R})$ and $\mathrm{M}(n, \mathbb{C})$, respectively.

Example 1.17 (Open Submanifolds). Let U be any open subset of $\mathbb{R}^n$. Then U is a topological n-manifold, and the single chart (U, Id_U) defines a smooth structure on U.

More generally, let M be a smooth n-manifold and let $U \subset M$ be any open subset. Define an atlas on U by

$$\mathcal{A}_U = \{\text{smooth charts } (V, \varphi) \text{ for } M \text{ such that } V \subset U\}.$$

Any point $p \in U$ is contained in the domain of some chart (W, φ) for M; if we set $V = W \cap U$, then $(V, \varphi|_V)$ is a chart in A_U whose domain contains p. Therefore, U is covered by the domains of charts in $\mathcal{A}_U$, and it is easy to verify that this is a smooth atlas for U. Thus any open subset of M is itself a smooth n-manifold in a natural way. Endowed with this smooth structure, we call any open subset an *open submanifold* of M. (We will define a more general class of submanifolds in Chapter 8.)

Example 1.18 (The General Linear Group). The *general linear group* $\mathrm{GL}(n,\mathbb{R})$ is the set of invertible $n\times n$ matrices with real entries. It is a smooth n^2-dimensional manifold because it is an open subset of the n^2-dimensional vector space $\mathrm{M}(n,\mathbb{R})$, namely the set where the (continuous) determinant function is nonzero.

Example 1.19 (Matrices of Maximal Rank). The previous example has a natural generalization to rectangular matrices of maximal rank. Suppose $m<n$, and let $\mathrm{M}_m(m\times n,\mathbb{R})$ denote the subset of $\mathrm{M}(m\times n,\mathbb{R})$ consisting of matrices of rank m. If A is an arbitrary such matrix, the fact that $\operatorname{rank} A = m$ means that A has some nonsingular $m\times m$ minor. By continuity of the determinant function, this same minor has nonzero determinant on some neighborhood of A in $\mathrm{M}(m\times n,\mathbb{R})$, which implies that A has a neighborhood contained in $\mathrm{M}_m(m\times n,\mathbb{R})$. Thus $\mathrm{M}_m(m\times n,\mathbb{R})$ is an open subset of $\mathrm{M}(m\times n,\mathbb{R})$, and therefore is itself a smooth mn-dimensional manifold. A similar argument shows that $\mathrm{M}_n(m\times n,\mathbb{R})$ is a smooth mn-manifold when $n<m$.

Example 1.20 (Spheres). We showed in Example 1.2 that the n-sphere $\mathbb{S}^n\subset\mathbb{R}^{n+1}$ is a topological n-manifold. Now we put a smooth structure on $\mathbb{S}^n$ as follows. For each $i=1,\dots,n+1$, let $(U_i^\pm,\varphi_i^\pm)$ denote the graph coordinate charts we constructed in Example 1.2. For any distinct indices i and j, the transition map $\varphi_i^\pm\circ(\varphi_j^\pm)^{-1}$ is easily computed. In the case $i<j$, we get

$$\varphi_i^\pm\circ(\varphi_j^\pm)^{-1}\left(u^1,\dots,u^n\right)=\left(u^1,\dots,\widehat{u^i},\dots,\pm\sqrt{1-|u|^2},\dots,u^n\right),$$

and a similar formula holds when $i>j$. When $i=j$, an even simpler computation gives $\varphi_i^\pm\circ(\varphi_i^\pm)^{-1}=\mathrm{Id}_{\mathbb{B}^n}$. Thus the collection of charts $\left\{\left(U_i^\pm,\varphi_i^\pm\right)\right\}$ is a smooth atlas, and so defines a smooth structure on $\mathbb{S}^n$. We call this its *standard smooth structure.*

Example 1.21 (Projective Spaces). The n-dimensional real projective space $\mathbb{RP}^n$ is a topological n-manifold by Example 1.3. We will show that the coordinate charts (U_i,φ_i) constructed in that example are all smoothly compatible. Assuming for convenience that $i>j$, it is straightforward to compute that

$$\varphi_j\circ\varphi_i^{-1}(u^1,\dots,u^n)=\left(\frac{u^1}{u^j},\dots,\frac{u^{j-1}}{u^j},\frac{u^{j+1}}{u^j},\dots,\frac{u^{i-1}}{u^j},\frac{1}{u^j},\frac{u^i}{u^j},\dots,\frac{u^n}{u^j}\right),$$

which is a diffeomorphism from $\varphi_i(U_i\cap U_j)$ to $\varphi_j(U_i\cap U_j)$.

Example 1.22 (Smooth Product Manifolds). If $M_1,\dots,M_k$ are smooth manifolds of dimensions $n_1,\dots,n_k$, respectively, we showed in Example 1.4 that the product space $M_1\times\cdots\times M_k$ is a topological manifold of dimension $n_1+\cdots+n_k$, with charts of the form $(U_1\times\cdots\times U_k,\varphi_1\times\cdots\times\varphi_k)$.

Any two such charts are smoothly compatible because, as is easily verified,

$$(\psi_1 \times \cdots \times \psi_k) \circ (\varphi_1 \times \cdots \times \varphi_k)^{-1} = (\psi_1 \circ \varphi_1^{-1}) \times \cdots \times (\psi_k \circ \varphi_k^{-1}),$$

which is a smooth map. This defines a natural smooth manifold structure on the product, called the *product smooth manifold structure*. For example, this yields a smooth manifold structure on the n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

In each of the examples we have seen so far, we have constructed a smooth manifold structure in two stages: We started with a topological space and checked that it was a topological manifold, and then we specified a smooth structure. It is often more convenient to combine these two steps into a single construction, especially if we start with a set that is not already equipped with a topology. The following lemma provides a shortcut.

Lemma 1.23 (Smooth Manifold Construction Lemma). *Let M be a set, and suppose we are given a collection $\{U_\alpha\}$ of subsets of M, together with an injective map $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$ for each α, such that the following properties are satisfied:*

- (*i*) *For each α, $\varphi_\alpha(U_\alpha)$ is an open subset of $\mathbb{R}^n$.*
- (*ii*) *For each α and β, $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.*
- (*iii*) *Whenever $U_\alpha \cap U_\beta \neq \varnothing$, $\varphi_\alpha \circ \varphi_\beta^{-1} \colon \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism.*
- (*iv*) *Countably many of the sets U_α cover M.*
- (*v*) *Whenever p, q are distinct points in M, either there exists some U_α containing both p and q or there exist disjoint sets U_α, U_β with $p \in U_\alpha$ and $q \in U_\beta$.*

Then M has a unique smooth manifold structure such that each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.

Proof. We define the topology by taking all sets of the form $\varphi_\alpha^{-1}(V)$, with V an open subset of $\mathbb{R}^n$, as a basis. To prove that this is a basis for a topology, we need to show that for any point p in the intersection of two basis sets $\varphi_\alpha^{-1}(V)$ and $\varphi_\beta^{-1}(W)$, there is a third basis set containing p and contained in the intersection. It suffices to show that $\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W)$ is itself a basis set (Figure 1.9). To see this, observe that (iii) implies that $\varphi_\alpha \circ \varphi_\beta^{-1}(W)$ is an open subset of $\varphi_\alpha(U_\alpha \cap U_\beta)$, and (ii) implies that this set is also open in $\mathbb{R}^n$. It follows that

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) = \varphi_\alpha^{-1}\left(V \cap \varphi_\alpha \circ \varphi_\beta^{-1}(W)\right)$$

is also a basis set, as claimed.

Each of the maps φ_α is then a homeomorphism (essentially by definition), so M is locally Euclidean of dimension n. If $\{U_{\alpha_i}\}$ is a countable collection of the sets U_α covering M, each of the sets U_{α_i} has a countable basis, and

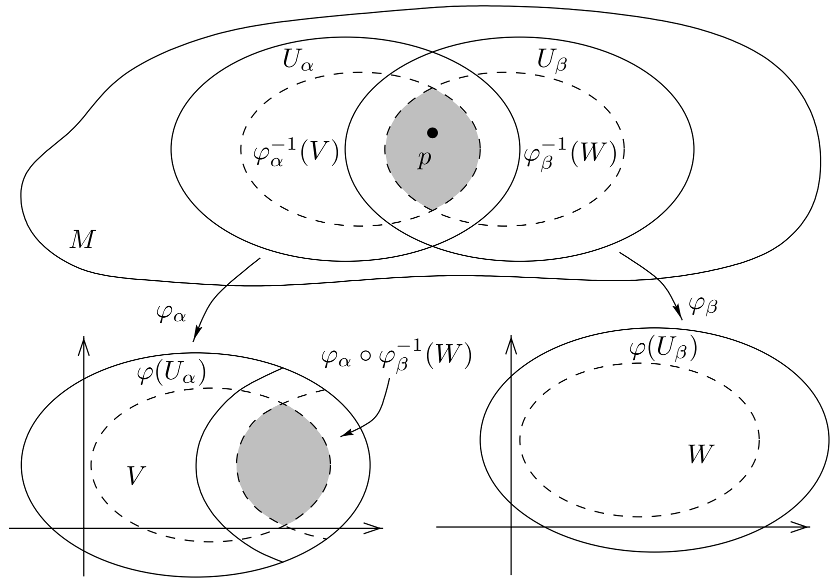

Figure 1.9. The smooth manifold construction lemma.

the union of all these is a countable basis for M, so M is second countable, and the Hausdorff property follows easily from (v). Finally, (iii) guarantees that the collection $\{(U_\alpha, \varphi_\alpha)\}$ is a smooth atlas. It is clear that this topology and smooth structure are the unique ones satisfying the conclusions of the lemma. □

Example 1.24 (Grassmann Manifolds). Let V be an n-dimensional real vector space. For any integer $0 \le k \le n$, we let $\mathrm{G}_k(V)$ denote the set of all k-dimensional linear subspaces of V. We will show that $\mathrm{G}_k(V)$ can be naturally given the structure of a smooth manifold of dimension $k(n-k)$. The construction is somewhat more involved than the ones we have done so far, but the basic idea is just to use linear algebra to construct charts for $\mathrm{G}_k(V)$, and then apply the smooth manifold construction lemma (Lemma 1.23). Since we will give a more straightforward proof that $\mathrm{G}_k(V)$ is a smooth manifold in Chapter 9 (Example 9.32), you may wish to skip the hard part of this construction (the verification that the charts are smoothly compatible) on first reading.

Let P and Q be any complementary subspaces of V of dimensions k and $(n-k)$, respectively, so that V decomposes as a direct sum: $V = P \oplus Q$. The graph of any linear map $A\colon P \to Q$ is a k-dimensional subspace $\Gamma(A) \subset V$, defined by

$$\Gamma(A) = \{x + Ax : x \in P\}.$$

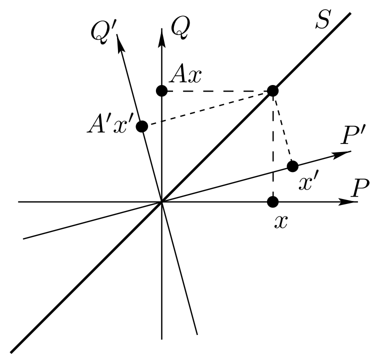

Figure 1.10. Smooth compatibility of coordinates on $\mathrm{G}_k(V)$.

Any such subspace has the property that its intersection with Q is the zero subspace. Conversely, any subspace with this property is easily seen to be the graph of a unique linear map $A\colon P \to Q$.

Let $L(P,Q)$ denote the vector space of linear maps from P to Q, and let U_Q denote the subset of $\mathrm{G}_k(V)$ consisting of k-dimensional subspaces whose intersection with Q is trivial. Define a map $\psi\colon L(P,Q) \to U_Q$ by

$$\psi(A) = \Gamma(A).$$

The discussion above shows that ψ is a bijection. Let $\varphi = \psi^{-1}\colon U_Q \to L(P,Q)$. By choosing bases for P and Q, we can identify $L(P,Q)$ with $\mathrm{M}((n-k)\times k, \mathbb{R})$ and hence with $\mathbb{R}^{k(n-k)}$, and thus we can think of (U_Q, φ) as a coordinate chart. Since the image of each chart is all of $L(P,Q)$, condition (i) of Lemma 1.23 is clearly satisfied.

Now let (P', Q') be any other such pair of subspaces, and let ψ', φ' be the corresponding maps. The set $\varphi(U_Q \cap U_{Q'}) \subset L(P,Q)$ consists of all $A \in L(P,Q)$ whose graphs intersect Q' trivially, which is easily seen to be an open set, so (ii) holds. We need to show that the transition map $\varphi' \circ \varphi^{-1} = \varphi' \circ \psi$ is smooth on this set. This is the trickiest part of the argument.

Suppose $A \in \varphi(U_Q \cap U_{Q'}) \subset L(P,Q)$ is arbitrary, and let S denote the subspace $\psi(A) = \Gamma(A) \subset V$. If we put $A' = \varphi' \circ \psi(A)$, then by definition A' is the unique linear map from P' to Q' whose graph is equal to S. To identify this map, let $x' \in P'$ be arbitrary, and note that $A'x'$ is the unique element of Q' such that $x' + A'x' \in S$, which is to say that

$$x' + A'x' = x + Ax \quad \text{for some } x \in P. \tag{1.2}$$

(See Figure 1.10.) There is in fact a unique $x \in P$ for which this holds, characterized by the property that

$$x + Ax - x' \in Q'.$$

If we let $I_A\colon P \to V$ denote the map $I_A(x) = x + Ax$ and let $\pi_{P'}\colon V \to P'$ be the projection onto P' with kernel Q', then x satisfies

$$0 = \pi_{P'}(x + Ax - x') = \pi_{P'} \circ I_A(x) - x'.$$

As long as A stays in the open subset of linear maps whose graphs intersect Q' trivially, $\pi_{P'} \circ I_A\colon P \to P'$ is invertible, and thus we can solve this last equation for x to obtain $x = (\pi_{P'} \circ I_A)^{-1}(x')$. Therefore, A' is given in terms of A by

$$A'x' = I_A x - x' = I_A \circ (\pi_{P'} \circ I_A)^{-1}(x') - x'. \tag{1.3}$$

If we choose bases (E'_i) for P' and (F'_j) for Q', the columns of the matrix representation of A' are the components of $A'E'_i$. By (1.3), this can be written

$$A'E'_i = I_A \circ (\pi_{P'} \circ I_A)^{-1}(E'_i) - E'_i.$$

The matrix entries of I_A clearly depend smoothly on those of A, and thus so also do those of $\pi_{P'} \circ I_A$. By Cramer's rule, the components of the inverse of a matrix are rational functions of the matrix entries, so the expression above shows that the components of $A'E'_i$ depend smoothly on the components of A. This proves that $\varphi' \circ \varphi^{-1}$ is a smooth map, so the charts we have constructed satisfy condition (iii) of Lemma 1.23.

To check the countability condition (iv), we just note that $\mathrm{G}_k(V)$ can in fact be covered by *finitely* many of the sets U_Q: For example, if $(E_1, \dots, E_n)$ is any fixed basis for V, any partition of the basis elements into two subsets containing k and $n-k$ elements determines appropriate subspaces P and Q, and any subspace S must have trivial intersection with Q for at least one of these partitions (see Exercise A.34). Thus $\mathrm{G}_k(V)$ is covered by the finitely many charts determined by all possible partitions of a fixed basis. Finally, the Hausdorff condition (v) is easily verified by noting that for any two k-dimensional subspaces $P, P' \subset V$, it is possible to find a subspace Q of dimension $n-k$ whose intersections with both P and P' are trivial, and then P and P' are both contained in the domain of the chart determined by, say, (P, Q).

The smooth manifold $\mathrm{G}_k(V)$ is called the *Grassmann manifold* of k-planes in V, or simply a *Grassmannian*. In the special case $V = \mathbb{R}^n$, the Grassmannian $\mathrm{G}_k(\mathbb{R}^n)$ is often denoted by some simpler notation such as $\mathrm{G}_{k,n}$ or $\mathrm{G}(k,n)$. Note that $\mathrm{G}_1(\mathbb{R}^{n+1})$ is exactly the n-dimensional projective space $\mathbb{RP}^n$.

Manifolds with Boundary

In many important applications of manifolds, most notably those involving integration, we will encounter spaces that would be smooth manifolds except that they have a "boundary" of some sort. Simple examples of such

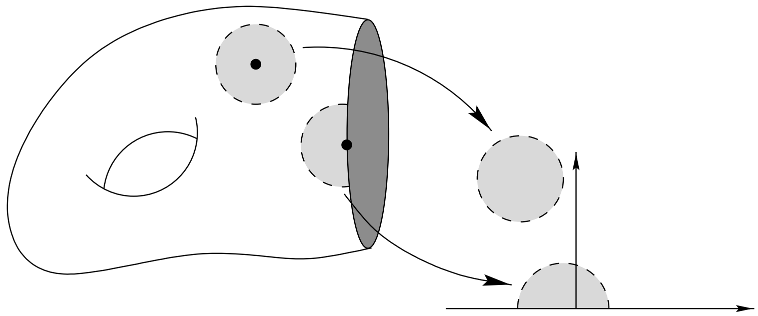

Figure 1.11. A manifold with boundary.

spaces include the closed unit ball in $\mathbb{R}^n$ and the closed upper hemisphere in $\mathbb{S}^n$. To accommodate such spaces, we need to generalize our definition of manifolds.

The model for these spaces will be the closed n-dimensional *upper half-space* $\mathbb{H}^n \subset \mathbb{R}^n$, defined as

$$\mathbb{H}^n = \left\{ \left(x^1, \dots, x^n\right) \in \mathbb{R}^n : x^n \geq 0 \right\}.$$

We will use $\operatorname{Int} \mathbb{H}^n$ and $\partial \mathbb{H}^n$ to denote the interior and boundary of $\mathbb{H}^n$, respectively, as a subset of $\mathbb{R}^n$:

$$\begin{aligned} \operatorname{Int} \mathbb{H}^n &= \left\{ \left(x^1, \dots, x^n\right) \in \mathbb{R}^n : x^n > 0 \right\}, \\ \partial \mathbb{H}^n &= \left\{ \left(x^1, \dots, x^n\right) \in \mathbb{R}^n : x^n = 0 \right\}. \end{aligned}$$

An n-dimensional *topological manifold with boundary* is a second-countable Hausdorff space M in which every point has a neighborhood homeomorphic to a (relatively) open subset of $\mathbb{H}^n$ (Figure 1.11). An open subset $U \subset M$ together with a homeomorphism φ from U to an open subset of $\mathbb{H}^n$ will be called a chart, just as in the case of manifolds. When it is necessary to make the distinction, we will call (U, φ) an *interior chart* if $\varphi(U) \subset \operatorname{Int} \mathbb{H}^n$, and a *boundary chart* if $\varphi(U) \cap \partial \mathbb{H}^n \neq \varnothing$.

To see how to define a smooth structure on a manifold with boundary, recall that a smooth map from an arbitrary subset $A \subset \mathbb{R}^n$ to $\mathbb{R}^k$ is defined to be a map that admits a smooth extension to an open neighborhood of each point (see the Appendix, page 587). Thus if U is an open subset of $\mathbb{H}^n$, a map $F\colon U \to \mathbb{R}^k$ is smooth if for each $x \in U$, there exists an open set $V \subset \mathbb{R}^n$ and a smooth map $\widetilde{F}\colon V \to \mathbb{R}^k$ that agrees with F on $V \cap \mathbb{H}^n$ (Figure 1.12). If F is such a map, the restriction of F to $U \cap \operatorname{Int} \mathbb{H}^n$ is smooth in the usual sense. By continuity, all the partial derivatives of F at points of $U \cap \partial \mathbb{H}^n$ are determined by their values in $\operatorname{Int} \mathbb{H}^n$, and therefore in particular are independent of the choice of extension. It is a fact (which we will neither prove nor use) that $F\colon U \to \mathbb{R}^k$ is smooth in this sense if and

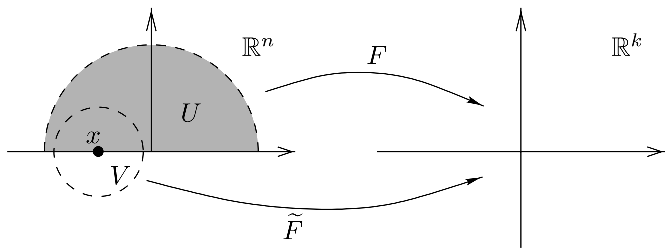

Figure 1.12. Smoothness of maps on open subsets of $\mathbb{H}^n$.

only if F is continuous, $F|_{U\cap \operatorname{Int}\mathbb{H}^n}$ is smooth, and the partial derivatives of $F|_{U\cap \operatorname{Int}\mathbb{H}^n}$ of all orders have continuous extensions to all of U.

For example, let $\mathbb{B}^2 \subset \mathbb{R}^2$ be the open unit disk, let $U = \mathbb{B}^2 \cap \mathbb{H}^2$, and define $f\colon U \to \mathbb{R}$ by $f(x,y) = \sqrt{1-x^2-y^2}$. Because f extends smoothly to all of $\mathbb{B}^2$ (by the same formula), f is a smooth function on U. On the other hand, although $g(x,y) = \sqrt{y}$ is continuous on U and smooth in $U \cap \operatorname{Int}\mathbb{H}^2$, it has no smooth extension to any neighborhood of the origin in $\mathbb{R}^2$ because $\partial g/\partial y \to \infty$ as $y \to 0$. Thus g is not smooth on U.

Now let M be a topological manifold with boundary. Just as in the manifold case, a smooth structure for M is defined to be a maximal smooth atlas—a collection of charts whose domains cover M and whose transition maps (and their inverses) are smooth in the sense just described. With such a structure, M is called a *smooth manifold with boundary.* A point $p \in M$ is called a *boundary point* if its image under some smooth chart is in $\partial\mathbb{H}^n$, and an *interior point* if its image under some smooth chart is in $\operatorname{Int}\mathbb{H}^n$. The *boundary* of M (the set of all its boundary points) is denoted by ∂M; similarly, its *interior*, the set of all its interior points, is denoted by $\operatorname{Int} M$. Once we have developed a bit more machinery, you will be able to show that M is the disjoint union of ∂M and $\operatorname{Int} M$ (see Problem 7-7).

Be careful to observe the distinction between these new definitions of the terms "boundary" and "interior" and their usage to refer to the boundary and interior of a subset of a topological space. A manifold M with boundary may have nonempty boundary in this new sense, irrespective of whether it has a boundary as a subset of some other topological space. If we need to emphasize the difference between the two notions of boundary, we will use the terms *topological boundary* and *manifold boundary* as appropriate. For example, the closed unit disk $\overline{\mathbb{B}^2}$ is a smooth manifold with boundary (as you will be asked to show in Problem 1-9), whose manifold boundary is the circle. Its topological boundary as a subspace of $\mathbb{R}^2$ happens to be the circle as well. However, if we think of $\overline{\mathbb{B}^2}$ as a topological space in its own right, then as a subset of itself, it has empty topological boundary. And if we think of it as a subset of $\mathbb{R}^3$ (considering $\mathbb{R}^2$ as a subset of $\mathbb{R}^3$ in the obvious way), its topological boundary is all of $\overline{\mathbb{B}^2}$. Note that $\mathbb{H}^n$ is itself a smooth manifold with boundary, and its manifold boundary is the same as its topological boundary as a subset of $\mathbb{R}^n$.

Every smooth n-manifold can be considered as a smooth n-manifold with boundary in a natural way: By composing with a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{H}^n$ such as $\left(x^1, \dots, x^{n-1}, x^n\right) \mapsto \left(x^1, \dots, x^{n-1}, e^{x^n}\right)$, we can modify any manifold chart to take its values in $\operatorname{Int}\mathbb{H}^n$ without affecting the smooth compatibility condition. On the other hand, if M is a smooth n-manifold with boundary, any interior point $p \in \operatorname{Int} M$ is by definition in the domain of a smooth chart (U, φ) such that $\varphi(p) \in \operatorname{Int}\mathbb{H}^n$. Replacing U by the (possibly smaller) open set $\varphi^{-1}(\operatorname{Int}\mathbb{H}^n) \subset U$, we may assume that (U, φ) is an interior chart. Because open sets in $\operatorname{Int}\mathbb{H}^n$ are also open in $\mathbb{R}^n$, each interior chart is a chart in the ordinary manifold sense. Thus $\operatorname{Int} M$ is a topological n-manifold, and the set of all smooth interior charts is easily seen to be a smooth atlas, turning it into a smooth n-manifold. In particular, a smooth manifold with boundary whose boundary happens to be empty is a smooth manifold. However, manifolds with boundary are not manifolds in general.

Even though the term "manifold with boundary" encompasses manifolds as well, for emphasis we will sometimes use the phrase "manifold without boundary" when we are talking about manifolds in the original sense, and "manifold with or without boundary" when we are working in the broader class that includes both cases. In the literature, you will also encounter the terms *closed manifold* to mean a compact manifold without boundary, and *open manifold* to mean a noncompact manifold without boundary.

The topological properties of manifolds that we proved earlier in the chapter have natural extensions to manifolds with boundary. For the record, we state them here.

Proposition 1.25. *Let M be a topological manifold with boundary.*

(*a*) *M is locally path connected.*

(*b*) *M has at most countably many components, each of which is a connected topological manifold with boundary.*

(*c*) *The fundamental group of M is countable.*

◇ **Exercise 1.6.** Prove Proposition 1.25.

Many of the results that we will prove about smooth manifolds throughout the book have natural analogues for manifolds with boundary. We will mention the most important of these as we go along.

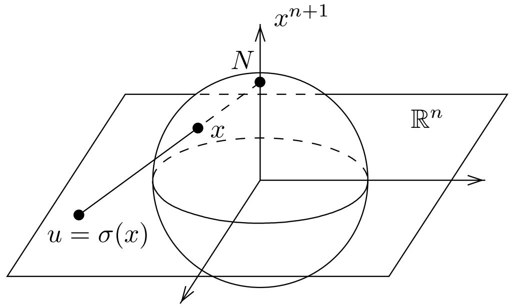

Figure 1.13. Stereographic projection.

Problems

1-1. Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second countable, but not Hausdorff. (This space is called the *line with two origins*.)

1-2. Show that the disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second countable.

1-3. Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: Begin by constructing homeomorphisms from $\mathbb{B}^n$ to itself that are smooth on $\mathbb{B}^n \smallsetminus \{0\}$.]

1-4. If k is an integer between 0 and $\min(m, n)$, show that the set of $m \times n$ matrices whose rank is at least k is an open submanifold of $\mathrm{M}(m \times n, \mathbb{R})$. Show that this is *not* true if "at least k" is replaced by "equal to k."

1-5. Let $N = (0, \ldots, 0, 1)$ be the "north pole" in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, and let $S = -N$ be the "south pole." Define *stereographic projection* $\sigma\colon \mathbb{S}^n \smallsetminus \{N\} \to \mathbb{R}^n$ by

$$\sigma\left(x^1, \ldots, x^{n+1}\right) = \frac{\left(x^1, \ldots, x^n\right)}{1 - x^{n+1}}.$$

Let $\widetilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \smallsetminus \{S\}$.

(a) For any $x \in \mathbb{S}^n \smallsetminus \{N\}$, show that $\sigma(x)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$, identified with $\mathbb{R}^n$ in the obvious way (Figure 1.13). Similarly, show that $\widetilde{\sigma}(x)$ is the point where the line through S

and x intersects the same subspace. (For this reason, $\widetilde{\sigma}$ is called *stereographic projection from the south pole*.)

(b) Show that σ is bijective, and

$$\sigma^{-1}\left(u^1,\dots,u^n\right)=\frac{\left(2u^1,\dots,2u^n,|u|^2-1\right)}{|u|^2+1}.$$

(c) Compute the transition map $\widetilde{\sigma}\circ\sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n\smallsetminus\{N\},\sigma)$ and $(\mathbb{S}^n\smallsetminus\{S\},\widetilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\widetilde{\sigma}$ are called *stereographic coordinates*.)

(d) Show that this smooth structure is the same as the one defined in Example 1.20.

1-6. By identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An *angle function* on a subset $U\subset\mathbb{S}^1$ is a continuous function $\theta\colon U\to\mathbb{R}$ such that $e^{i\theta(p)}=p$ for all $p\in U$. Show that there exists an angle function θ on an open subset $U\subset\mathbb{S}^1$ if and only if $U\neq\mathbb{S}^1$. For any such angle function, show that (U,θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.

1-7. *Complex projective n-space*, denoted by $\mathbb{CP}^n$, is the set of 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$, with the quotient topology inherited from the natural projection $\pi\colon\mathbb{C}^{n+1}\smallsetminus\{0\}\to\mathbb{CP}^n$. Show that $\mathbb{CP}^n$ is a compact $2n$-dimensional topological manifold, and show how to give it a smooth structure analogous to the one we constructed for $\mathbb{RP}^n$. (We identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ via $\left(x^1+iy^1,\dots,x^{n+1}+iy^{n+1}\right)\leftrightarrow\left(x^1,y^1,\dots,x^{n+1},y^{n+1}\right)$.)

1-8. Let k and n be integers such that $0<k<n$, and let $P,Q\subset\mathbb{R}^n$ be the subspaces spanned by $(e_1,\dots,e_k)$ and $(e_{k+1},\dots,e_n)$, respectively, where e_i is the ith standard basis vector. For any k-dimensional subspace $S\subset\mathbb{R}^n$ that has trivial intersection with Q, show that the coordinate representation $\varphi(S)$ constructed in Example 1.24 is the unique $(n-k)\times k$ matrix B such that S is spanned by the columns of the matrix $\binom{I_k}{B}$, where I_k denotes the $k\times k$ identity matrix.

1-9. Let $M=\overline{\mathbb{B}^n}$, the closed unit ball in $\mathbb{R}^n$. Show that M is a topological manifold with boundary, and that it can be given a natural smooth structure in which each point in $\mathbb{S}^{n-1}$ is a boundary point and each point in $\mathbb{B}^n$ is an interior point.