1300HF: Smooth manifolds

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Abstract

These are my (Marco Gualtieri) teaching notes for the year-long graduate core course in geometry and topology at the University of Toronto in 2009-10. They borrow without citation from many sources, including Bar-Natan, Godbillon, Guillemin-Pollack, Milnor, Sternberg, Lee, and Mrowka. If you spot any errors, please email me at mgualt@math.utoronto.ca

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1 Manifolds

A manifold is a space which looks like $\mathbb{R}^n$ at small scales (i.e. "locally"), but which may be very different from this at large scales (i.e. "globally"). In other words, manifolds are made up by gluing pieces of $\mathbb{R}^n$ together to make a more complicated whole. We would like to make this precise.

1.1 Topological manifolds

Definition 1. A real, n-dimensional *topological manifold* is a Hausdorff, second countable topological space which is locally homeomorphic to $\mathbb{R}^n$.

Note: "Locally homeomorphic to $\mathbb{R}^n$" simply means that each point p has an open neighbourhood U for which we can find a homeomorphism $\varphi : U \longrightarrow V$ to an open subset $V \in \mathbb{R}^n$. Such a homeomorphism φ is called a *coordinate chart* around p. A collection of charts which cover the manifold, i.e. whose union is the whole space, is called an *atlas*.

We now give a bunch of examples of topological manifolds. The simplest is, technically, the empty set. More simple examples include a countable set of points (with the discrete topology), and $\mathbb{R}^n$ itself, but there are more:

Example 1.1 (Circle)**.** *Define the circle $S^1 = \{z \in \mathbb{C} \ : \ |z| = 1\}$. Then for any fixed point $z \in S^1$, write it as $z = e^{2\pi ic}$ for a unique real number $0 \leq c < 1$, and define the map*

$$\nu_z : t \mapsto e^{2\pi it}. \tag{1}$$

We note that ν_z maps the interval $I_c = (c - \frac{1}{2}, c + \frac{1}{2})$ to the neighbourhood of z given by $S^1\backslash\{-z\}$, and it is a homeomorphism. Then $\varphi_z = \nu_z|_{I_c}^{-1}$ is a local coordinate chart near z.

By taking products of coordinate charts, we obtain charts for the Cartesian product of manifolds. Hence the Cartesian product is a manifold.

Example 1.2 (n-torus)**.** *$S^1 \times \cdots \times S^1$ is a topological manifold (of dimension given by the number n of factors), with charts $\{\varphi_{z_1} \times \cdots \times \varphi_{z_n} \ : \ z_i \in S^1\}$.*

Example 1.3 (open subsets)**.** *Any open subset $U \subset M$ of a topological manifold is also a topological manifold, where the charts are simply restrictions $\varphi|_U$ of charts φ for M.*

For example, the real $n \times n$ matrices $\mathrm{Mat}(n, \mathbb{R})$ form a vector space isomorphic to $\mathbb{R}^{n^2}$, and contain an open subset

$$GL(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) \ : \ \det A \neq 0\}, \tag{2}$$

known as the general linear group, which therefore forms a topological manifold.

Example 1.4 (Spheres)**.** *The n-sphere is defined as the subspace of unit vectors in $\mathbb{R}^{n+1}$:*

$$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \ : \ \sum x_i^2 = 1\}.$$

Let $N = (1, 0, \dots, 0)$ be the North pole and let $S = (-1, 0, \dots, 0)$ be the South pole in S^n. Then we may write S^n as the union $S^n = U_N \cup U_S$, where $U_N = S^n\backslash\{S\}$ and $U_S = S^n\backslash\{N\}$ are equipped with coordinate charts φ_N, φ_S into $\mathbb{R}^n$, given by the "stereographic projections" from the points S, N respectively

$$\varphi_N : (x_0, \vec{x}) \mapsto (1 + x_0)^{-1}\vec{x}, \tag{3}$$

$$\varphi_S : (x_0, \vec{x}) \mapsto (1 - x_0)^{-1}\vec{x}. \tag{4}$$

We have endowed the sphere S^n with a certain topology, but is it possible for another topological manifold $\tilde{S}^n$ to be homotopic to S^n without *being homeomorphic to it? The answer is no, and this is known as the topological Poincaré conjecture, and is usually stated as follows: any homotopy n-sphere is homeomorphic to the n-sphere. It was proven for $n > 4$ by Smale, for $n = 4$ by Freedman, and for $n = 3$ is equivalent to the smooth Poincaré conjecture which was proved by Hamilton-Perelman. In dimensions $n = 1, 2$ it is a consequence of the (easy) classification of topological 1- and 2-manifolds.*

Example 1.5 (Projective spaces)**.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $\mathbb{K}P^n$ is defined to be the space of lines through $\{0\}$ in $\mathbb{K}^{n+1}$, and is called the projective space over $\mathbb{K}$ of dimension n.*

More precisely, let $X = \mathbb{K}^{n+1}\backslash\{0\}$ and define an equivalence relation on X via $x \sim y$ iff $\exists\lambda \in \mathbb{K}^* = \mathbb{K}\backslash\{0\}$ such that $\lambda x = y$, i.e. x, y lie on the same line through the origin. Then

$$\mathbb{K}P^n = X/\sim,$$

and it is equipped with the quotient topology.

The projection map $\pi : X \longrightarrow \mathbb{K}P^n$ is an open *map, since if $U \subset X$ is open, then tU is also open $\forall t \in \mathbb{K}^*$, implying that $\cup_{t\in\mathbb{K}^*} tU = \pi^{-1}(\pi(U))$ is open, implying $\pi(U)$ is open. This immediately shows, by the way, that $\mathbb{K}P^n$ is second countable.*

To show $\mathbb{K}P^n$ is Hausdorff (which we must do, since Hausdorff is preserved by subspaces and products, but not *quotients), we would like to show that the diagonal in $\mathbb{K}P^n \times \mathbb{K}P^n$ is closed. We show this by showing that the graph of the equivalence relation is closed in $X \times X$ (this, together with the openness of π, gives us the result). This graph is simply*

$$\Gamma_\sim = \{(x, y) \in X \times X \ : \ x \sim y\},$$

and we notice that $\Gamma_\sim$ is actually the common zero set of the following continuous functions

$$f_{ij}(x, y) = (x_i y_j - x_j y_i) \quad i \neq j.$$

An atlas for $\mathbb{K}P^n$ is given by the open sets $U_i = \pi(\tilde{U}_i)$, where

$$\tilde{U}_i = \{(x_0, \ldots, x_n) \in X \ : \ x_i \neq 0\},$$

and these are equipped with charts to $\mathbb{K}^n$ given by

$$\varphi_i([x_0, \ldots, x_n]) = x_i^{-1}(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \tag{5}$$

which are indeed invertible by $(y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n)$.

Sometimes one finds it useful to simply use the "coordinates" $(x_0, \ldots, x_n)$ for $\mathbb{K}P^n$, with the understanding that the x_i are well-defined only up to overall rescaling. This is called using "projective coordinates" and in this case a point in $\mathbb{K}P^n$ is denoted by $[x_0 : \cdots : x_n]$.

Example 1.6 (Connected sum)**.** *Let $p \in M$ and $q \in N$ be points in topological manifolds and let (U, φ) and (V, ψ) be charts around p, q such that $\varphi(p) = 0$ and $\psi(q) = 0$.*

Choose ϵ small enough so that $B(0, 2\epsilon) \subset \varphi(U)$ and $B(0, 2\epsilon) \subset \varphi(V)$, and define the map of annuli

$$\phi : B(0, 2\epsilon)\backslash\overline{B(0, \epsilon)} \longrightarrow B(0, 2\epsilon)\backslash\overline{B(0, \epsilon)} \tag{6}$$

$$x \mapsto \tfrac{2\epsilon^2}{|x|^2} x. \tag{7}$$

This is a homeomorphism of the annulus to itself, exchanging the boundaries. Now we define a new topological manifold, called the connected sum $M\sharp N$, *as the quotient $X/\sim$, where*

$$X = (M\backslash\overline{\varphi^{-1}(B(0, \epsilon))}) \sqcup (N\backslash\overline{\psi^{-1}(B(0, \epsilon))}),$$

and we define an identification $x \sim \psi^{-1}\phi\varphi(x)$ for $x \in \varphi^{-1}(B(0, 2\epsilon))$. If $\mathcal{A}_M$ and $\mathcal{A}_N$ are atlases for M, N respectively, then a new atlas for the connect sum is simply

$$\mathcal{A}_M|_{M\backslash\overline{\varphi^{-1}(B(0,\epsilon))}} \cup \mathcal{A}_N|_{N\backslash\overline{\psi^{-1}(B(0,\epsilon))}}$$

Two important remarks concerning the connect sum: first, the connect sum of a sphere with itself is homeomorphic to the same sphere:

$$S^n\sharp S^n \cong S^n.$$

Second, by taking repeated connect sums of T^2 and $\mathbb{R}P^2$, we may obtain all compact 2-dimensional manifolds.

Example 1.7. *Let F be a topological space. A fiber bundle with fiber F is a triple (E, p, B), where E, B are topological spaces called the "total space" and "base", respectively, and $p : E \longrightarrow B$ is a continuous surjective map called the "projection map", such that, for each point $b \in B$, there is a neighbourhood U of b and a homeomorphism*

$$\Phi : p^{-1}U \longrightarrow U \times F,$$

such that $p_U \circ \Phi = p$, where $p_U : U \times F \longrightarrow U$ is the usual projection. The submanifold $p^{-1}(b) \cong F$ is called the "fiber over b".

When B, F are topological manifolds, then clearly E becomes one as well. We will often encounter such manifolds.

1.2 Smooth manifolds

Given coordinate charts (U_i, φ_i) and (U_j, φ_j) on a topological manifold, if we compare coordinates on the intersection $U_{ij} = U_i \cap U_j$, we see that the map

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij})$$

is a homeomorphism, simply because it is a composition of homeomorphisms. We can say this another way: topological manifolds are glued together by homeomorphisms.

This means that we may be able to differentiate a function in one coordinate chart but not in another, i.e. there is no way to make sense of calculus on topological manifolds. This is why we introduce smooth manifolds, which is simply a topological manifold where the gluing maps are required to be *smooth*.

First we recall the notion of a smooth map of finite-dimensional vector spaces.

Remark 1 (Aside on smooth maps of vector spaces)**.** *Let $U \subset V$ be an open set in a finite-dimensional vector space, and let $f : U \longrightarrow W$ be a function with values in another vector space W. The function f is said to be differentiable at $p \in U$ if there exists a linear map $Df(p) : V \longrightarrow W$ such that*

$$\lim_{||x|| \to 0} \frac{||f(p + x) - f(p) - Df(p)(x)||}{||x||} = 0.$$

Here we choose any norm[1] $|| \cdot ||$ *on U, V since such norms are all equivalent for finite-dimensional vector spaces. For infinite-dimensional vector spaces, the topology is highly sensitive to which norm is chosen, but we will work in finite dimensions.*

Given linear coordinates $(x_1, \ldots, x_n)$ on V, and $(y_1, \ldots, y_m)$ on W, we may express f in terms of its m components $f_j = y_j \circ f$, and then the linear map $Df(p)$ may be written as an $m \times n$ matrix, called the Jacobian matrix *of f at p.*

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{8}$$

We say that f is differentiable on U when it is differentiable at all $p \in U$ and we say it is continuously differentiable when

$$Df : U \longrightarrow \mathrm{Hom}(V, W)$$

is continuous. The vector space of continuously differentiable functions on U with values in W is called $C^1(U, W)$.

The first derivative Df is also a map from U to a vector space ($\mathrm{Hom}(V, W)$*), therefore if its derivative exists, we obtain a map*

$$D^2 f : U \longrightarrow \mathrm{Hom}(V, \mathrm{Hom}(V, W)),$$

and so on. The vector space of k times continuously differentiable functions on U with values in W is called $C^k(U, W)$. We are most interested in C^∞ or "smooth" maps, all of whose derivatives exist; the space of these is denoted $C^\infty(U, W)$, and hence we have

$$C^\infty(U, W) = \bigcap_k C^k(U, W).$$

[1] A norm on a vector space V is a map $| \cdot | : V \longrightarrow \mathbb{R}$ such that $||av|| = |a|||v||$ for $a \in \mathbb{R}$, $||v|| = 0$ iff $v = 0$, and satisfying the triangle inequality.

Note: for a C^2 function, $D^2 f$ actually has values in a smaller subspace of $V^* \otimes V^* \otimes W$, namely in $S^2V^* \otimes W$, since "mixed partials are equal".

After this aside, we can define a smooth manifold.

Definition 2. A *smooth manifold* is a topological manifold equipped with an equivalence class of smooth atlases, explained below.

Definition 3. An atlas $\mathcal{A} = \{U_i, \varphi_i\}$ for a topological manifold is called *smooth* when all gluing maps

$$\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij})$$

are smooth maps, i.e. lie in $C^\infty(\varphi_i(U_{ij}), \mathbb{R}^n)$. Two atlases $\mathcal{A}, \mathcal{A}'$ are *equivalent* if $\mathcal{A} \cup \mathcal{A}'$ is itself a smooth atlas.

Note: Instead of requiring an atlas to be smooth, we could ask for it to be C^k, or real-analytic, or even holomorphic (this makes sense for a $2n$-dimensional topological manifold when we identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

We may now verify that all the examples from section 1.1 are actually smooth manifolds:

Example 1.8 (Circle)**.** *For Example 1.1, only two charts, e.g. $\varphi_{\pm 1}$, suffice to define an atlas, and we have*

$$\varphi_{-1} \circ \varphi_1^{-1} = \begin{cases} t+1 & -\frac{1}{2} < t < 0 \\ t & 0 < t < \frac{1}{2}, \end{cases}$$

which is clearly C^∞. In fact all the charts φ_z are smoothly compatible. Hence the circle is a smooth manifold.

The Cartesian product of smooth manifolds inherits a natural smooth structure from taking the Cartesian product of smooth atlases. Hence the n-torus, for example, equipped with the atlas we described in Example 1.2, is smooth. Example 1.3 is clearly defining a smooth manifold, since the restriction of a smooth map to an open set is always smooth.

Example 1.9 (Spheres)**.** *The charts for the n-sphere given in Example 1.4 form a smooth atlas, since*

$$\varphi_N \circ \varphi_S^{-1} : \vec{z} \mapsto \tfrac{1-x_0}{1+x_0}\vec{z} = \tfrac{(1-x_0)^2}{|\vec{x}|^2}\vec{z} = |\vec{z}|^{-2}\vec{z},$$

which is smooth on $\mathbb{R}^n \backslash \{0\}$, as required.

Example 1.10 (Projective spaces)**.** *The charts for projective spaces given in Example 1.5 form a smooth atlas, since*

$$\varphi_1 \circ \varphi_0^{-1}(z_1, \ldots, z_n) = (z_1^{-1}, z_1^{-1}z_2, \ldots, z_1^{-1}z_n), \tag{9}$$

which is smooth on $\mathbb{R}^n \backslash \{z_1 = 0\}$, as required, and similarly for all φ_i, φ_j.

The connected sum in Example 1.6 is clearly smooth since ϕ was chosen to be a smooth map.

1.3 Smooth maps

For topological manifolds M, N of dimension m, n, the natural notion of morphism from M to N is that of a continuous map. A continuous map with continuous inverse is then a homeomorphism from M to N, which is the natural notion of equivalence for topological manifolds. Since the composition of continuous maps is continuous and associative, we obtain a category C^0-**Man** of topological manifolds and continuous maps. Recall that a category is simply a class of objects $\mathcal{C}$ (in our case, topological manifolds) and an associative class of arrows $\mathcal{A}$ (in our case, continuous maps) with source and target maps $\mathcal{A} \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{C}$ and an identity arrow for each object, given by a map $\mathrm{Id} : \mathcal{C} \longrightarrow \mathcal{A}$ (in our case, the identity map of any manifold to itself). Conventionally we write the set of arrows $\{a \in \mathcal{A} \ : \ s(a) = x \text{ and } t(a) = y\}$ as $\mathrm{Hom}(x, y)$. Also note that the associative composition of arrows mentioned above then becomes a map

$$\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \longrightarrow \mathrm{Hom}(x, z).$$

So, the category C^0-**Man** has objects which are topological manifolds, and $\mathrm{Hom}(M, N) = C^0(M, N)$ is the set of continuous maps $M \longrightarrow N$. We now describe the morphisms between smooth manifolds, completing the definition of the category of smooth manifolds.

Definition 4. A map $f : M \longrightarrow N$ is called smooth when for each chart (U, φ) for M and each chart (V, ψ) for N, the composition $\psi \circ f \circ \varphi^{-1}$ is a smooth map, i.e. $\psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n)$. The set of smooth maps (i.e. morphisms) from M to N is denoted $C^\infty(M, N)$. A smooth map with a smooth inverse is called a *diffeomorphism*.

If $g : L \longrightarrow M$ and $f : M \longrightarrow N$ are smooth maps, then so is the composition $f \circ g$, since if charts φ, χ, ψ for L, M, N are chosen near $p \in L$, $g(p) \in M$, and $(fg)(p) \in N$, then $\psi \circ (f \circ g) \circ \varphi^{-1} = A \circ B$, for $A = \psi f \chi^{-1}$ and $B = \chi g \varphi^{-1}$ both smooth mappings $\mathbb{R}^n \longrightarrow \mathbb{R}^n$. By the chain rule, $A \circ B$ is differentiable at p, with derivative $D_p(A \circ B) = (D_{g(p)}A)(D_p B)$ (matrix multiplication).

Now we have a new category, which we may call C^∞-**Man**, the category of smooth manifolds and smooth maps; two manifolds are considered isomorphic when they are diffeomorphic.

Example 1.11. *We show that the complex projective line $\mathbb{C}P^1$ is diffeomorphic to the 2-sphere S^2. Consider the maps $f_+(x_0, x_1, x_2) = [1 + x_0 : x_1 + ix_2]$ and $f_-(x_0, x_1, x_2) = [x_1 - ix_2 : 1 - x_0]$. Since $f_\pm$ is continuous on $x_0 \neq \pm 1$, and since $f_- = f_+$ on $|x_0| < 1$, the pair (f_-, f_+) defines a continuous map $f : S^2 \longrightarrow \mathbb{C}P^1$. To check smoothness, we compute the compositions*

$$\varphi_0 \circ f_+ \circ \varphi_N^{-1} : (y_1, y_2) \mapsto y_1 + iy_2, \tag{10}$$

$$\varphi_1 \circ f_- \circ \varphi_S^{-1} : (y_1, y_2) \mapsto y_1 - iy_2, \tag{11}$$

both of which are obviously smooth maps.

Remark 2 (Exotic smooth structures)**.** *The topological Poincaré conjecture, now proven, states that any topological manifold homotopic to the n-sphere is in fact homeomorphic to it. We have now seen how to put a differentiable structure on this n-sphere. Remarkably, there are other differentiable structures on the n-sphere which are not diffeomorphic to the standard one we gave; these are called* exotic *spheres.*

Since the connected sum of spheres is homeomorphic to a sphere, and since the connected sum operation is well-defined as a smooth manifold, it follows that the connected sum defines a monoid *structure on the set of smooth n-spheres. In fact, Kervaire and Milnor showed that for $n \neq 4$, the set of (oriented) diffeomorphism classes of smooth n-spheres forms a finite abelian group under the connected sum operation. This is not known to be the case in four dimensions. Kervaire and Milnor also compute the order of this group, and the first dimension where there is more than one smooth sphere is $n = 7$, in which case they show there are* 28 *smooth spheres, which we will encounter later on.*

The situation for spheres may be contrasted with that for the Euclidean spaces: any differentiable manifold homeomorphic to $\mathbb{R}^n$ for $n \neq 4$ must be diffeomorphic to it. On the other hand, by results of Donaldson, Freedman, Taubes, and Kirby, we know that there are uncountably many non-diffeomorphic smooth structures on the topological manifold $\mathbb{R}^4$; these are called fake $\mathbb{R}^4$*s.*

Example 1.12 (Lie groups)**.** *A group is a set G with an associative multiplication $G \times G \xrightarrow{m} G$, an identity element $e \in G$, and an inversion map $\iota : G \longrightarrow G$, usually written $\iota(g) = g^{-1}$.*

If we endow G with a topology for which G is a topological manifold and m, ι are continuous maps, then the resulting structure is called a topological group. *If G is a given a smooth structure and m, ι are smooth maps, the result is a* Lie group.

The real line (where m is given by addition), the circle (where m is given by complex multiplication), and their cartesian products give simple but important examples of Lie groups. We have also seen the general linear group $GL(n, \mathbb{R})$, which is a Lie group since matrix multiplication and inversion are smooth maps.

Since $m : G \times G \longrightarrow G$ is a smooth map, we may fix $g \in G$ and define smooth maps $L_g : G \longrightarrow G$ and $R_g : G \longrightarrow G$ via $L_g(h) = gh$ and $R_g(h) = hg$. These are called left multiplication *and* right multiplication. *Note that the group axioms imply that $R_g L_h = L_h R_g$.*

1.4 Manifolds with boundary

The concept of *manifold with boundary* is important for relating manifolds of different dimension. Our manifolds are defined intrinsically, meaning that they are not defined as subsets of another topological space; therefore, the notion of boundary will differ from the usual boundary of a subset.

To introduce boundaries in our manifolds, we need to change the local model which they are based on. For this reason, we introduce the half-space $H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_n \geq 0\}$, equip it with the induced topology from $\mathbb{R}^n$, and model our spaces on this one.

Definition 5. A topological manifold with boundary M is a second countable Hausdorff topological space which is locally homeomorphic to H^n. Its *boundary* ∂M is the $(n-1)$ manifold consisting of all points mapped to $x_n = 0$ by a chart, and its *interior* $\operatorname{Int} M$ is the set of points mapped to $x_n > 0$ by some chart.

A smooth structure on such a manifold *with boundary* is an equivalence class of smooth atlases, where smoothness is defined below.

Definition 6. Let V, W be finite-dimensional vector spaces, as before. A function $f : A \longrightarrow W$ from an arbitrary subset $A \subset V$ is smooth when it admits a smooth extension to an open neighbourhood $U_p \subset W$ of every point $p \in A$.

For example, the function $f(x, y) = y$ is smooth on H^2 but $f(x, y) = \sqrt{y}$ is not, since its derivatives do not extend to $y \leq 0$.

Note the important fact that if M is an n-manifold with boundary, $\operatorname{Int} M$ is a usual n-manifold, without boundary. Also, even more importantly, ∂M is an $n-1$-manifold without boundary, i.e. $\partial(\partial M) = \emptyset$. This is sometimes phrased as the equation

$$\partial^2 = 0.$$

Example 1.13 (Möbius strip)**.** *The mobius strip E is a compact 2-manifold with boundary. As a topological space it is the quotient of $\mathbb{R} \times [0, 1]$ by the identification $(x, y) \sim (x+1, 1-y)$. The map $\pi : [(x, y)] \mapsto e^{2\pi i x}$ is a continuous surjective map to S^1, called a projection map. We may choose charts $[(x, y)] \mapsto e^{x+i\pi y}$ for $x \in (x_0 - \epsilon, x_0 + \epsilon)$, and for any $\epsilon < \frac{1}{2}$.*

Note that ∂E is diffeomorphic to S^1. This actually provides us with our first example of a non-trivial fiber bundle. In this case, E is a bundle of intervals over a circle. It is nontrivial, since the trivial fiber bundle $S^1 \times [0, 1]$ has boundary $S^1 \sqcup S^1$.

1.5 Cobordism

$(n+1)$-Manifolds with boundary provide us with a natural equivalence relation on n-manifolds, called *cobordism*.

Definition 7. Compact n-manifolds M_1, M_2 are *cobordant* when there exists a compact $n+1$-manifold with boundary N such that ∂N is diffeomorphic to $M_1 \sqcup M_2$. If N is cobordant to M, then we say that M, N are in the same *cobordism class*.

- We use compact manifolds since any manifold M is the boundary of a noncompact manifold with boundary $M \times [0, 1)$.
- The set of cobordism classes of k-dimensional manifolds is called Ω_k, and forms an abelian group under the operation $[M_1] + [M_2] = [M_1 \sqcup M_2]$. The additive identity element is $0 = [\emptyset]$. Note that $\emptyset$ is a manifold of dimension k for all k.
- The direct sum $\Omega_\bullet = \bigoplus_{k \geq 0} \Omega_k$ then forms a commutative ring called the *Cobordism ring*, where the product is

  $$[M_1] \cdot [M_2] = [M_1 \times M_2].$$

  Note that while the Cartesian product of manifolds is a manifold, the Cartesian product of two manifolds with boundary is *not* a manifold with boundary. On the other hand, the Cartesian product of manifolds only one of which has boundary, is a manifold with boundary (why?)

Note that the 1-point space $[*]$ is not null-cobordant, meaning it is not the boundary of a compact 1-manifold. Compact 0-dimensional manifolds are boundaries only if they consist of an even number of points. This shows that there are compact manifolds which are *not* boundaries.

Proposition 1.14. *The cobordism ring is 2-torsion, i.e. $x + x = 0 \;\; \forall x$.*

Proof. The zero element of the ring is $[\emptyset]$ and the multiplicative unit is $[*]$, the class of the one-point manifold. For any manifold M, the manifold with boundary $M \times [0, 1]$ has boundary $M \sqcup M$. Hence $[M] + [M] = [\emptyset] = 0$, as required. □

Example 1.15. *The n-sphere S^n is null-cobordant (i.e. cobordant to $\emptyset$), since $\partial\overline{B_{n+1}(0, 1)} \cong S^n$, where $B_{n+1}(0, 1)$ denotes the unit ball in $\mathbb{R}^{n+1}$.*

Example 1.16. *Any oriented compact 2-manifold Σ_g is null-cobordant , since we may embed it in $\mathbb{R}^3$ and the "inside" is a 3-manifold with boundary given by Σ_g.*

We would like to state an amazing theorem of Thom, which is a complete characterization of the cobordism ring.

Theorem 1.17 (René Thom 1954)**.** *The cobordism ring is a (countably generated) polynomial ring over $\mathbb{F}_2$ with generators in every dimension $n \neq 2^k - 1$, i.e.*

$$\Omega^\bullet = \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \ldots].$$

This theorem implies that there are 3 nonzero cobordism classes in dimension 4, namely x_2^2, x_4, and $x_2^2 + x_4$. Can you find 4-manifolds representing these classes? Can you find *connected* representatives? What is the abelian group structure on Ω_4? In fact, there is a finite set of numbers associated to each manifold, called the Stiefel-Whitney characteristic numbers, which completely determine whether two manifolds are cobordant.

1.6 Smooth functions and partitions of unity

The set $C^\infty(M, \mathbb{R})$ of smooth functions on M inherits much of the structure of $\mathbb{R}$ by composition. $\mathbb{R}$ is a ring, having addition $+ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and multiplication $\times : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ which are both smooth. As a result, $C^\infty(M, \mathbb{R})$ is as well: One way of seeing why is to use the smooth diagonal map $\Delta : M \longrightarrow M \times M$, i.e. $\Delta(p) = (p, p)$.

Then, given functions $f, g \in C^\infty(M, \mathbb{R})$ we have the sum $f + g$, defined by the composition

$$M \xrightarrow{\Delta} M \times M \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R} \xrightarrow{+} \mathbb{R} .$$

We also have the product fg, defined by the composition

$$M \xrightarrow{\Delta} M \times M \xrightarrow{f \times g} \mathbb{R} \times \mathbb{R} \xrightarrow{\times} \mathbb{R} .$$

Given a smooth map $\varphi : M \longrightarrow N$ of manifolds, we obtain a natural operation $\varphi^* : C^\infty(N, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$, given by $f \mapsto f \circ \varphi$. This is called the pullback of functions, and defines a homomorphism of rings since $\Delta \circ \varphi = (\varphi \times \varphi) \circ \Delta$.

The association $M \mapsto C^\infty(M, \mathbb{R})$ and $\varphi \mapsto \varphi^*$ takes objects and arrows of C^∞-Man to objects and arrows of the category of rings, respectively, in such a way which respects identities and composition of morphisms. Such a map is called a functor. In this case, it has the peculiar property that it switches the source and target of morphisms. It is therefore a *contravariant* functor from the category of manifolds to the category of rings, and is the basis for algebraic geometry, the algebraic representation of geometrical objects.

It is easy to see from this that any diffeomorphism $\varphi : M \longrightarrow M$ defines an automorphism φ^* of $C^\infty(M, \mathbb{R})$, but actually all automorphisms are of this form (Why?). Also, if M is a compact manifold, then an ideal $\mathcal{I} \subset C^\infty(M, \mathbb{R})$ is maximal if and only if it is the vanishing ideal $\{f \in C^\infty(M, \mathbb{R}) \ : \ f(p) = 0\}$ of a point $p \in M$ (Why? Also, Why must M be compact?).

The key tool for understanding the ring $C^\infty(M, \mathbb{R})$ is the partition of unity. This will allow us to *go from local to global*, i.e. to glue together objects which are defined locally, creating objects with global meaning.

Definition 8. A collection of subsets $\{U_\alpha\}$ of the topological space M is called *locally finite* when each point $x \in M$ has a neighbourhood V intersecting only finitely many of the U_α.

Definition 9. A covering $\{V_\alpha\}$ is a *refinement* of the covering $\{U_\beta\}$ when each V_α is contained in some U_β.

Lemma 1.18. *Any open covering $\{A_\alpha\}$ of a topological manifold has a countable, locally finite refinement $\{(U_i, \varphi_i)\}$ by coordinate charts such that $\varphi_i(U_i) = B(0, 3)$ and $\{V_i = \varphi_i^{-1}(B(0, 1))\}$ is still a covering of M. We will call such a cover a* regular *covering. In particular, any topological manifold is* paracompact *(i.e. every open cover has a locally finite refinement)*

Proof. If M is compact, the proof is easy: choosing coordinates around any point $x \in M$, we can translate and rescale to find a covering of M by a refinement of the type desired, and choose a finite subcover, which is obviously locally finite.

For a general manifold, we note that by second countability of M, there is a countable basis of coordinate neighbourhoods and each of these charts is a countable union of open sets P_i with $\overline{P_i}$ compact. Hence M has a countable basis $\{P_i\}$ such that $\overline{P_i}$ is compact.

Using these, we may define an increasing sequence of compact sets which exhausts M: let $K_1 = \overline{P}_1$, and

$$K_{i+1} = \overline{P_1 \cup \cdots \cup P_r},$$

where $r > 1$ is the first integer with $K_i \subset P_1 \cup \cdots \cup P_r$.

Now note that M is the union of ring-shaped sets $K_i \backslash K_{i-1}^\circ$, each of which is compact. If $p \in A_\alpha$, then $p \in K_{i+2} \backslash K_{i-1}^\circ$ for some i. Now choose a coordinate neighbourhood $(U_{p,\alpha}, \varphi_{p,\alpha})$ with $U_{p,\alpha} \subset K_{i+2} \backslash K_{i-1}^\circ$ and $\varphi_{p,\alpha}(U_{p,\alpha}) = B(0,3)$ and define $V_{p,\alpha} = \varphi^{-1}(B(0,1))$.

Letting p, α vary, these neighbourhoods cover the compact set $K_{i+1} \backslash K_i^\circ$ without leaving the band $K_{i+2} \backslash K_{i-1}^\circ$. Choose a finite subcover $V_{i,k}$ for each i. Then $(U_{i,k}, \varphi_{i,k})$ is the desired locally finite refinement. □

Definition 10. A smooth partition of unity is a collection of smooth non-negative functions $\{f_\alpha : M \longrightarrow \mathbb{R}\}$ such that

i) $\{\text{supp} f_\alpha = \overline{f_\alpha^{-1}(\mathbb{R} \backslash \{0\})}\}$ is locally finite,

ii) $\sum_\alpha f_\alpha(x) = 1 \quad \forall x \in M$, hence the name.

A partition of unity is *subordinate* to an open cover $\{U_i\}$ when $\forall \alpha$, $\text{supp} f_\alpha \subset U_i$ for some i.

Theorem 1.19. *Given a regular covering $\{(U_i, \varphi_i)\}$ of a manifold, there exists a partition of unity $\{f_i\}$ subordinate to it with $f_i > 0$ on V_i and $\text{supp} f_i \subset \varphi_i^{-1}(\overline{B(0,2)})$.*

Proof. A *bump function* is a smooth non-negative real-valued function $\tilde{g}$ on $\mathbb{R}^n$ with $\tilde{g}(x) = 1$ for $||x|| \leq 1$ and $\tilde{g}(x) = 0$ for $||x|| \geq 2$. For instance, take

$$\tilde{g}(x) = \frac{h(2 - ||x||)}{h(2 - ||x||) + h(||x|| + 1)},$$

for $h(t)$ given by $e^{-1/t}$ for $t > 0$ and 0 for $t < 0$.

Having this bump function, we can produce non-negative bump functions on the manifold $g_i = \tilde{g} \circ \varphi_i$ which have support $\text{supp} g_i \subset \varphi_i^{-1}(\overline{B(0,2)})$ and take the value $+1$ on $\overline{V_i}$. Finally we define our partition of unity via

$$f_i = \frac{g_i}{\sum_j g_j}, \quad i = 1, 2, \ldots.$$

□

Corollary 1.20 (Existence of bump functions)**.** *Let $A \subset M$ be any closed subset of a manifold, and let U be any open neighbourhood of A. Then there exists a smooth function $f_U : M \longrightarrow \mathbb{R}$ with $f_U \equiv 1$ on A and $\text{supp} f_U \subset U$.*

Proof. Consider the open cover $\{U, M \backslash A\}$ of M. Choose a regular subcover (U_i, φ_i) with subordinate partition of unity f_i. Then let f_U be the sum of all f_i with support contained in U. □

One interesting application of partitions of unity is to the extension of any chart $\varphi_i : U_i \subset M \longrightarrow \mathbb{R}^n$ to a smooth mapping $\varphi : M \longrightarrow \mathbb{R}^n$. If ψ is a bump function supported in U_i, then take

$$\varphi(x) = \begin{cases} 0 \text{ for } x \in M \backslash U_i \\ \psi(x)\varphi_i(x) \text{ for } x \in U_i \end{cases}$$

2 The tangent functor

The tangent bundle of an n-manifold M is a $2n$-manifold, called TM, naturally constructed in terms of M, which is made up of the disjoint union of all tangent spaces to all points in M. Usually we think of tangent spaces as subspaces of Euclidean space which approximate a curved subset, but interestingly, the tangent space does not require an ambient space in order to be defined. In other words, the tangent space is "intrinsic" to the manifold and does not depend on any embedding.

As a set, it is fairly easy to describe, as simply the disjoint union of all tangent spaces. However we must explain precisely what we mean by the tangent space T_pM to $p \in M$.

Definition 11. Let (U, φ), (V, ψ) be coordinate charts around $p \in M$. Let $u \in T_{\varphi(p)}\varphi(U)$ and $v \in T_{\psi(p)}\psi(V)$. Then the triples (U, φ, u), (V, ψ, v) are called equivalent when $D(\psi \circ \varphi^{-1})(\varphi(p)) : u \mapsto v$. The chain rule for derivatives $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ guarantees that this is indeed an equivalence relation.

The set of equivalence classes of such triples is called the tangent space to p of M, denoted T_pM, and forms a real vector space of dimension $\dim M$.

As a set, the tangent bundle is defined by

$$TM = \bigsqcup_{p\in M} T_pM,$$

and it is equipped with a natural surjective map $\pi : TM \longrightarrow M$, which is simply $\pi(X) = x$ for $X \in T_xM$.

We now give it a manifold structure in a natural way.

Proposition 2.1. *For an n-manifold M, the set TM has a natural topology and smooth structure which make it a $2n$-manifold, and make $\pi : TM \longrightarrow M$ a smooth map.*

Proof. Any chart (U, φ) for M defines a bijection

$$T\varphi(U) \cong U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$$

via $(p, v) \mapsto (U, \varphi, v)$. Using this, we induce a smooth manifold structure on $\pi^{-1}(U)$, and view the inverse of this map as a chart $(\pi^{-1}(U), \Phi)$ to $\varphi(U) \times \mathbb{R}^n$.

given another chart (V, ψ), we obtain another chart $(\pi^{-1}(V), \Psi)$ and we may compare them via

$$\Psi \circ \Phi^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \longrightarrow \psi(U \cap V) \times \mathbb{R}^n,$$

which is given by $(p, u) \mapsto ((\psi \circ \varphi^{-1})(p), D(\psi \circ \varphi^{-1})_p u)$, which is smooth. Therefore we obtain a topology and smooth structure on all of TM (by defining W to be open when $W \cap \pi^{-1}(U)$ is open for every U in an atlas for M; all that remains is to verify the Hausdorff property, which holds since points x, y are either in the same chart (in which case it is obvious) or they can be separated by the given type of charts. □

A more constructive way of looking at the tangent bundle: We choose a countable, locally finite atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ for M with gluing maps $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$, and glue $U_i \times \mathbb{R}^n$ to $U_j \times \mathbb{R}^n$ via an equivalence

$$(x, u) \sim (y, v) \;\Leftrightarrow\; y = \varphi_{ij}(x) \;\text{ and }\; v = (D\varphi_{ij})_x u,$$

and verify the conditions of the general gluing construction (Assignment 1), obtaining a manifold $TM_{\mathcal{A}}$. Then show that the result is independent of the chosen atlas in the smooth structure: for a different atlas $\mathcal{A}'$, one obtains a diffeomorphism $\varphi_{\mathcal{A}\mathcal{A}'} : TM_{\mathcal{A}} \longrightarrow TM_{\mathcal{A}'}$ which itself satisfies $\varphi_{\mathcal{A}'\mathcal{A}''} \circ \varphi_{\mathcal{A}\mathcal{A}'} = \varphi_{\mathcal{A}\mathcal{A}''}$.

It is easy to see from the definition that (TM, π_M, M) is a fiber bundle with fiber type $\mathbb{R}^n$; in fact there is slightly more structure involved: the tangent spaces T_pM have a natural vector space structure and the given local trivializations $\Phi : \pi_M^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ preserve the vector space structure on each fiber, i.e. $\Phi|_{T_pM} : T_pM \longrightarrow \{p\} \times \mathbb{R}^n$ is a linear map for all p. This makes (TM, π_M, M) into a vector bundle.

2.1 Tangent morphism

The tangent bundle itself is only the result of applying the tangent functor to a manifold. We must explain how to apply the tangent functor to a morphism of manifolds. This is otherwise known as taking the "derivative" of a smooth map $f : M \longrightarrow N$. Such a map may be defined locally in charts (U_i, φ_i) for M and (V_α, ψ_α) for N as a collection of vector-valued functions $\psi_\alpha \circ f \circ \varphi_i^{-1} = f_{i\alpha} : \varphi_i(U_i) \longrightarrow \psi_\alpha(V_\alpha)$ which satisfy

$$\psi_{\alpha\beta} \circ f_{i\alpha} = f_{j\beta} \circ \varphi_{ij}.$$

Differentiating, we obtain

$$D\psi_{\alpha\beta} \circ Df_{i\alpha} = Df_{j\beta} \circ D\varphi_{ij},$$

and hence we obtain a map $TM \longrightarrow TN$. This map is called the derivative of f and is denoted $Tf : TM \longrightarrow TN$ (or sometimes just $Df : TM \longrightarrow TN$). Sometimes it is called the "push-forward" of vectors and is denoted f_*. The map fits into the commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow[f]{} & N \end{array}$$

Just as $\pi^{-1}(x) = T_xM \subset TM$ is a vector space for all x, making TM into a "bundle of vector spaces", the map $Tf : T_xM \longrightarrow T_{f(x)}N$ is a linear map and hence Tf is a "bundle of linear maps". The pair (f, Tf) is a morphism of vector bundles $(TM, \pi_M, M) \longrightarrow (TN, \pi_N, N)$.

The usual chain rule for derivatives then implies that if $f \circ g = h$ as maps of manifolds, then $Tf \circ Tg = Th$. As a result, we obtain the following category-theoretic statement.

Proposition 2.2. *The map T which takes a manifold M to its tangent bundle TM, and which takes maps $f : M \longrightarrow N$ to the derivative $Tf : TM \longrightarrow TN$, is a functor from the category of manifolds and smooth maps to itself.*

The tangent bundle allows us to make sense of the notion of vector field in a global way. Locally, in a chart (U_i, φ_i), we would say that a vector field X_i is simply a vector-valued function on U_i, i.e. a function $X_i : \varphi(U_i) \longrightarrow \mathbb{R}^n$. Of course if we had another vector field X_j on (U_j, φ_j), then the two would agree as vector fields on the overlap $U_i \cap U_j$ when $D\varphi_{ij} : X_i(p) \mapsto X_j(\varphi_{ij}(p))$. So, if we specify a collection $\{X_i \in C^\infty(U_i, \mathbb{R}^n)\}$ which glue on overlaps, this would define a global vector field. This leads precisely to the following definition.

Definition 12. A smooth vector field on the manifold M is a smooth map $X : M \longrightarrow TM$ such that $\pi \circ X : M \longrightarrow M$ is the identity. Essentially it is a smooth assignment of a unique tangent vector to each point in M.

Such maps X are also called *cross-sections* or simply *sections* of the tangent bundle TM, and the set of all such sections is denoted $C^\infty(M, TM)$ or sometimes $\Gamma^\infty(M, TM)$, to distinguish them from simply smooth maps $M \longrightarrow TM$.

Example 2.3. *From a computational point of view, given an atlas $(\tilde{U}_i, \varphi_i)$ for M, let $U_i = \varphi_i(\tilde{U}_i) \subset \mathbb{R}^n$ and let $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$. Then a global vector field $X \in \Gamma^\infty(M, TM)$ is specified by a collection of vector-valued functions $X_i : U_i \longrightarrow \mathbb{R}^n$ such that $D\varphi_{ij}(X_i(x)) = X_j(\varphi_{ij}(x))$ for all $x \in \varphi_i(\tilde{U}_i \cap \tilde{U}_j)$.*

For example, if $S^1 = U_0 \sqcup U_1 / \sim$, with $U_0 = \mathbb{R}$ and $U_1 = \mathbb{R}$, with $x \in U_0 \backslash \{0\} \sim y \in U_1 \backslash \{0\}$ whenever $y = x^{-1}$, then $\varphi_{01} : x \mapsto x^{-1}$ and $D\varphi_{01}(x) : (x, v) \mapsto (x^{-1}, -x^{-2}v)$.

If we choose the coordinate vector field $X_0 = \frac{\partial}{\partial x}$ (in coordinates this is simply $x \mapsto (x, 1)$), then we see that $D\varphi_{01}(X_0) = (x^{-1}, -x^{-2} \cdot 1)$, i.e. the vector field $y \mapsto (y, -y^2)$, in other words $X_1 = -y^2 \frac{\partial}{\partial y}$.

Hence the following local vector fields glue to form a global vector field on S^1:

$$X_0 = \frac{\partial}{\partial x}$$
$$X_1 = -y^2 \frac{\partial}{\partial y}.$$

This vector field does not vanish in U_0 but vanishes to order 2 at a single point in U_1. Find the local expression in these charts for the rotational vector field on S^1 given in polar coordinates by $\frac{\partial}{\partial \theta}$.

A word of warning: it may be tempting to think that the assignment $M \mapsto \Gamma^\infty(M, TM)$ is a functor from manifolds to vector spaces; it is not, because there is no way to push forward or pull back vector fields. Nevertheless, if $f : M \longrightarrow N$ is a smooth map, it does define an equivalence relation between vector fields on M and N:

Definition 13. if $f : M \longrightarrow N$ smooth, then $X \in \Gamma^\infty(M, TM)$ is called f-related to $Y \in \Gamma^\infty(N, TN)$ when $f_*(X(p)) = Y(f(p))$, i.e. the diagram commutes:

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ {\scriptstyle X}\uparrow & & \uparrow{\scriptstyle Y} \\ M & \xrightarrow[f]{} & N \end{array}$$

So, another way to phrase the definition of a vector field is that they are local vector-valued functions which are φ_{ij}-related on overlaps.

2.2 Properties of vector fields

The concept of derivation of an algebra A is the infinitesimal version of an automorphism of A. That is, if $\phi_t : A \longrightarrow A$ is a family of automorphisms of A starting at Id, so that $\phi_t(ab) = \phi_t(a)\phi_t(b)$, then the map $a \mapsto \frac{d}{dt}|_{t=0}\phi_t(a)$ is a derivation.

Definition 14. A derivation of the $\mathbb{R}$-algebra A is a $\mathbb{R}$-linear map $D : A \longrightarrow A$ such that $D(ab) = (Da)b + a(Db)$. The space of all derivations is denoted $\mathrm{Der}(A)$. Note that this makes sense for noncommutative algebras also.

In the following, we show that derivations of the algebra of functions actually correspond to vector fields.

The vector fields $\Gamma^\infty(M, TM)$ form a vector space over $\mathbb{R}$ of infinite dimension (unless $\dim M = 0$). They also form a module over the ring of smooth functions $C^\infty(M, \mathbb{R})$ via pointwise multiplication: for $f \in C^\infty(M, \mathbb{R})$ and $X \in \Gamma^\infty(M, TM)$, we claim that $fX : x \mapsto f(x)X(x)$ defines a smooth vector field: this is clear from local considerations: A vector field $V = \sum_i v^i(x^1, \dots x^n)\frac{\partial}{\partial x^i}$ is smooth precisely when the component functions v^i are smooth: the vector field fV then has components fv^i, still smooth.

The important property of vector fields which we are interested in is that they act as $\mathbb{R}$-derivations of the algebra of smooth functions. Locally, it is clear that a vector field $X = \sum_i a^i \frac{\partial}{\partial x^i}$ gives a derivation of the algebra of smooth functions, via the formula $X(f) = \sum_i a^i \frac{\partial f}{\partial x^i}$, since

$$X(fg) = \sum_i a^i (\tfrac{\partial f}{\partial x^i} g + f \tfrac{\partial g}{\partial x^i}) = X(f)g + fX(g).$$

We wish to verify that this local action extends to a well-defined global derivation on $C^\infty(M, \mathbb{R})$.

Proposition 2.4. *Let f be a smooth function on $U \subset \mathbb{R}^n$, and $X : U \longrightarrow T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ a vector field. Then*

$$X(f) = \pi_2 \circ Df \circ X,$$

where $\pi_2 : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is the second projection (i.e. projection to the fiber of $T\mathbb{R}^n$.) In local coordinates, we have $X(f) = \sum_i a^i \frac{\partial f}{\partial x^i}$ whereas $Df : X(x) \mapsto (f(x), \sum_i \frac{\partial f}{\partial x^i} a^i)$, so that we obtain the result by projection.

Proposition 2.5. *Local partial differentiation extends to an injective map $\Gamma^\infty(M, TM) \longrightarrow \mathrm{Der}(C^\infty(M, \mathbb{R}))$.*

Proof. A global function is given by $f_i = f_j \circ \varphi_{ij}$. We verify that

$$\begin{aligned} X_i(f_i) &= \pi_2 \circ Df_i \circ X_i && (12)\\ &= \pi_2 \circ Df_j \circ D\varphi_{ij} \circ X_i && (13)\\ &= \pi_2 \circ Df_j \circ X_j \circ \varphi_{ij} && (14)\\ &= X_j(f_j) \circ \varphi_{ij}, && (15) \end{aligned}$$

showing that $\{X_i(f_i)\}$ defines a global function. Injectivity follows from the local fact that $V(f) = 0$ for all f would imply, for $V = \sum_i v^i \frac{\partial}{\partial x^i}$, that $V(x^i) = v^i = 0$ for all i, i.e. $V = 0$. □

In fact, vector fields provide all possible derivations of the algebra $A = C^\infty(M, \mathbb{R})$:

Theorem 2.6. *The map $\Gamma^\infty(M, TM) \longrightarrow \mathrm{Der}(C^\infty(M, \mathbb{R}))$ is an isomorphism.*

Proof. First we prove the result for an open set $U \subset \mathbb{R}^n$. Let D be a derivation of $C^\infty(U, \mathbb{R})$ and define the smooth functions $a^i = D(x^i)$. Then we claim $D = \sum_i a^i \frac{\partial}{\partial x^i}$. We prove this by testing against smooth functions. Any smooth function f on $\mathbb{R}^n$ may be written

$$f(x) = f(0) + \sum_i x^i g_i(x),$$

with $g_i(0) = \frac{\partial f}{\partial x^i}(0)$ (simply take $g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx)dt$). Translating the origin to $y \in U$, we obtain for any $z \in U$

$$f(z) = f(y) + \sum_i (x^i(z) - x^i(y))g_i(z), \qquad g_i(y) = \tfrac{\partial f}{\partial x^i}(y).$$

Applying D, we obtain

$$Df(z) = \sum_i (Dx^i)g_i(z) - \sum_i (x^i(z) - x^i(y))Dg_i(z).$$

Letting z approach y, we obtain

$$Df(y) = \sum_i a^i \tfrac{\partial f}{\partial x^i}(y) = X(f)(y),$$

as required.

To prove the global result, let $(V_i \subset U_i, \varphi_i)$ be a regular covering and θ_i the associated partition of unity. Then for each i, $\theta_i D : f \mapsto \theta_i D(f)$ is also a derivation of $C^\infty(M, \mathbb{R})$. This derivation defines a unique derivation D_i of $C^\infty(U_i, \mathbb{R})$ such that $D_i(f|_{U_i}) = (\theta_i Df)|_{U_i}$, since for any point $p \in U_i$, a given function $g \in C^\infty(U_i, \mathbb{R})$ may be replaced with a function $\tilde{g} \in C^\infty(M, \mathbb{R})$ which agrees with g on a small neighbourhood of p, and we define $(D_i g)(p) = \theta_i(p)D\tilde{g}(p)$. This definition is independent of $\tilde{g}$, since if $h_1 = h_2$ on an open set, $Dh_1 = Dh_2$ on that open set (let $\psi = 1$ in a neighbourhood of p and vanish outside U_i; then $h_1 - h_2 = (h_1 - h_2)(1 - \psi)$ and applying D we obtain zero).

The derivation D_i is then represented by a vector field X_i, which must vanish outside the support of θ_i. Hence it may be extended by zero to a global vector field which we also call X_i. Finally we observe that for $X = \sum_i X_i$, we have

$$X(f) = \sum_i X_i(f) = \sum_i D_i(f) = D(f),$$

as required. □

Since vector fields are derivations, we deduce that they have all the properties that derivationsdo, and we also have a natural source of examples, coming from infinitesimal automorphisms of M.

Definition 15. For any algebra A, the derivations $\mathrm{Der}(A)$ form a Lie algebra via the bracket $[X, Y](f) = X(Y(f)) - Y(X(f))$. For vector fields ($A = C^\infty(M, \mathbb{R})$), this bracket is called the Lie bracket.

Example 2.7. *Let φ_t : be a smooth family of diffeomorphisms of M with $\varphi_0 = \mathrm{Id}$. That is, let $\varphi : (-\epsilon, \epsilon) \times M \longrightarrow M$ be a smooth map and $\varphi_t : M \longrightarrow M$ a diffeomorphism for each t. Then $X(f)(p) = \frac{d}{dt}|_{t=0}(\varphi_t^* f)(p)$ defines a smooth vector field. A better way of seeing that it is smooth is to rewrite it as follows: Let $\frac{\partial}{\partial t}$ be the coordinate vector field on $(-\epsilon, \epsilon)$ and observe $X(f)(p) = \frac{\partial}{\partial t}(\varphi^* f)(0, p)$.*

In many cases, a smooth vector field may be expressed as above, i.e. as an infinitesimal automorphism of M, but this is not always the case. In general, it gives rise to a "local 1-parameter group of diffeomorphisms", as follows:

Definition 16. A local 1-parameter group of diffeomorphisms is an open set $U \subset \mathbb{R} \times M$ containing $\{0\} \times M$ and a smooth map

$$\begin{aligned} \Phi : & U \longrightarrow M \\ & (t, x) \mapsto \varphi_t(x) \end{aligned}$$

such that $\mathbb{R} \times \{x\} \cap U$ is connected, $\varphi_0(x) = x$ for all x and if $(t, x), (t + t', x), (t', \varphi_t(x))$ are all in U then $\varphi_{t'}(\varphi_t(x)) = \varphi_{t+t'}(x)$ (note that this last fact indicates that φ_t are all diffeomorphisms, having inverses φ_{-t}).

Then the local existence and uniqueness of solutions to systems of ODE implies that every smooth vector field $X \in \Gamma^\infty(M, TM)$ gives rise to a local 1-parameter group of diffeomorphisms (U, Φ) such that the curve $\gamma_x : t \mapsto \varphi_t(x)$ is such that $(\gamma_x)_*(\frac{d}{dt}) = X(\gamma_x(t))$ (this means that γ_x is an integral curve or "trajectory" of the "dynamical system" defined by X). Furthermore, if (U', Φ') are another such data, then $\Phi = \Phi'$ on $U \cap U'$.

Definition 17. A vector field $X \in \Gamma^\infty(M, TM)$ is called *complete* when it has a local 1-parameter group of diffeomorphisms with $U = \mathbb{R} \times M$.

Theorem 2.8. *If M is compact, then every smooth vector field is complete. Similarly any compactly-supported vector field is complete.*

Example 2.9. *The vector field $X = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$ is not complete. For initial condition x_0, have integral curve $\gamma(t) = x_0(1 - tx_0)^{-1}$, which gives $\Phi(t, x_0) = x_0(1 - tx_0)^{-1}$, which is well-defined on $\{1 - tx > 0\}$.*

Remark 3. *If ϕ_t and ψ_t are families of automorphisms of A with $\phi_0 = \psi_0 = \mathrm{Id}$, then they correspond to derivations $X = \frac{d}{dt}|_{t=0}\phi_t$ and $Y = \frac{d}{dt}|_{t=0}\psi_t$, and the family of automorphisms $\gamma_t = \phi_t\psi_t\phi_t^{-1}\psi_t^{-1}$ has $\frac{d}{dt}|_{t=0}\gamma_t = 0$ and $\frac{d^2}{dt^2}|_{t=0}\gamma_t = [X, Y]$.*

2.3 Local structure of smooth maps

In some ways, smooth manifolds are easier to produce or find than general topological manifolds, because of the fact that smooth maps have linear approximations. Therefore smooth maps often behave like linear maps of vector spaces, and we may gain inspiration from vector space constructions (e.g. subspace, kernel, image, cokernel) to produce new examples of manifolds.

In charts (U, φ), (V, ψ) for the smooth manifolds M, N, a smooth map $f : M \longrightarrow N$ is represented by a smooth map $\psi \circ f \circ \varphi^{-1} \in C^\infty(\varphi(U), \mathbb{R}^n)$. We shall give a general local classification of such maps, based on the behaviour of the derivative. The fundamental result which provides information about the map based on its derivative is the *inverse function theorem*.

Theorem 2.10 (Inverse function theorem)**.** *Let $U \subset \mathbb{R}^m$ an open set and $f : U \longrightarrow \mathbb{R}^m$ a smooth map such that $Df(p)$ is an invertible linear operator. Then there is a neighbourhood $V \subset U$ of p such that $f(V)$ is open and $f : V \longrightarrow f(V)$ is a diffeomorphism. furthermore, $D(f^{-1})(f(p)) = (Df(p))^{-1}$.*

Proof not given in class – this is the standard proof seen in first analysis course. Without loss of generality, assume that U contains the origin, that $f(0) = 0$ and that $Df(p) = \mathrm{Id}$ (for this, replace f by $(Df(0))^{-1} \circ f$. We are trying to invert f, so solve the equation $y = f(x)$ uniquely for x. Define g so that $f(x) = x + g(x)$. Hence $g(x)$ is the nonlinear part of f.

The claim is that if y is in a sufficiently small neighbourhood of the origin, then the map $h_y : x \mapsto y - g(x)$ is a contraction mapping on some closed ball; it then has a unique fixed point $\phi(y)$ by the Banach fixed point theorem (Look it up!), and so $y - g(\phi(y)) = \phi(y)$, i.e. ϕ is an inverse for f.

Why is h_y a contraction mapping? Note that $Dh_y(0) = 0$ and hence there is a ball $B(0, r)$ where $||Dh_y|| \leq \frac{1}{2}$. This then implies (mean value theorem) that for $x, x' \in B(0, r)$,

$$||h_y(x) - h_y(x')|| \leq \tfrac{1}{2}||x - x'||.$$

Therefore h_y does look like a contraction, we just have to make sure it's operating on a complete metric space. Let's estimate the size of $h_y(x)$:

$$||h_y(x)|| \leq ||h_y(x) - h_y(0)|| + ||h_y(0)|| \leq \tfrac{1}{2}||x|| + ||y||.$$

Therefore by taking $y \in B(0, \frac{r}{2})$, the map h_y is a contraction mapping on $\overline{B(0, r)}$. Let $\phi(y)$ be the unique fixed point of h_y guaranteed by the contraction mapping theorem.

To see that ϕ is continuous (and hence f is a homeomorphism), we compute

$$\begin{aligned} ||\phi(y) - \phi(y')|| &= ||h_y(\phi(y)) - h_{y'}(\phi(y'))|| \\ &\leq ||g(\phi(y)) - g(\phi(y'))|| + ||y - y'|| \\ &\leq \tfrac{1}{2}||\phi(y) - \phi(y')|| + ||y - y'||, \end{aligned}$$

so that we have $||\phi(y) - \phi(y')|| \leq 2||y - y''||$, as required.

To see that ϕ is differentiable, we guess the derivative $(Df)^{-1}$ and compute. Let $x = \phi(y)$ and $x' = \phi(y')$. For this to make sense we must have chosen r small enough so that Df is nonsingular on $\overline{B(0,r)}$, which is not a problem.

$$\begin{aligned}
||\phi(y) - \phi(y') - (Df(x))^{-1}(y - y')|| &= ||x - x' - (Df(x))^{-1}(f(x) - f(x'))|| \\
&\le ||(Df(x))^{-1}||||(Df(x))(x - x') - (f(x) - f(x'))|| \\
&\le o(||x - x'||), \text{ using differentiability of } f \\
&\le o(||y - y'||), \text{ using continuity of } \phi.
\end{aligned}$$

Now that we have shown ϕ is differentiable with derivative $(Df)^{-1}$, we use the fact that Df is C^∞ and inversion is C^∞, implying that $D\phi$ is C^∞ and hence ϕ also. □

This theorem immediately provides us with a local normal form for a smooth map with $Df(p)$ invertible: we may choose coordinates on sufficiently small neighbourhoods of $p, f(p)$ so that f is represented by the identity map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

In fact, the inverse function theorem leads to a normal form theorem for a more general class of maps:

Theorem 2.11 (Constant rank theorem). *If $f : M \longrightarrow N$ is a smooth map of manifolds of dimension m, n respectively, and if Tf has constrant rank k in some open set $\tilde{U} \subset M$ then for each point $p \in \tilde{U}$ there are charts (U, φ) and (V, ψ) containing $p, f(p)$ such that*

$$\psi \circ f \circ \varphi^{-1} : (x_1, \dots, x_m) \mapsto (x_1, \dots, x_k, 0, \dots, 0).$$

Proof. Begin by choosing coordinates near $p, f(p)$ on M and N. Since rk $(Tf) = k$ at p, there is a $k \times k$ minor of $Df(p)$ with nonzero determinant. Reorder the coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$ so that this minor is top left, and translate coordinates so that $f(0) = 0$. label the coordinates $(x_1, \dots, x_k, y_1, \dots y_{m-k})$ on V and $(u_1, \dots u_k, v_1, \dots, v_{n-k})$ on W.

Then we may write $f(x, y) = (Q(x, y), R(x, y))$, where Q is the projection to $u = (u_1, \dots, u_k)$ and R is the projection to v. with $\frac{\partial Q}{\partial x}$ nonsingular. First we wish to put Q into normal form. Consider the map $\phi(x, y) = (Q(x, y), y)$, which has derivative

$$D\phi = \begin{pmatrix} \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \\ 0 & 1 \end{pmatrix}$$

As a result we see $D\phi(0)$ is nonsingular and hence there exists a local inverse $\phi^{-1}(x, y) = (A(x, y), B(x, y))$. Since it's an inverse this means $(x, y) = \phi(\phi^{-1}(x, y)) = (Q(A, B), B)$, which implies that $B(x, y) = y$.

Then $f \circ \phi^{-1} : (x, y) \mapsto (x, \tilde{R} = R(A, y))$, and must still be of rank k. Since its derivative is

$$D(f \circ \phi^{-1}) = \begin{pmatrix} I_{k\times k} & 0 \\ \frac{\partial \tilde{R}}{\partial x} & \frac{\partial \tilde{R}}{\partial y} \end{pmatrix}$$

and since we know that Tf must have rank k *in a neighbourhood of* p, we conclude that $\frac{\partial \tilde{R}}{\partial y} = 0$ in a neighbourhood of p, meaning that $\tilde{R}$ is a function $S(x)$ only of the variables x.

$$f \circ \phi^{-1} : (x, y) \mapsto (x, S(x)).$$

We now postcompose by the diffeomorphism $\sigma : (u, v) \mapsto (u, v - S(u))$, to obtain

$$\sigma \circ f \circ \phi^{-1} : (x, y) \mapsto (x, 0),$$

as required. □

Some special cases of the above theorem have special names:

$$\begin{aligned}
\text{local immersion: } & (x^1, \dots, x^m) \mapsto (x^1, \dots, x^m, 0, \dots, 0) \\
\text{local submersion: } & (x^1, \dots, x^m) \mapsto (x^1, \dots, x^k) \\
\text{local diffeomorphism: } & (x^1, \dots, x^m) \mapsto (x^1, \dots, x^m)
\end{aligned}$$

Definition 18. A smooth map $f : M \longrightarrow N$ is called a *submersion* when $Tf(p)$ is surjective at all points $p \in M$, and is called an *immersion* when $Tf(p)$ is injective at all points $p \in M$.

For linear maps $A : V \longrightarrow W$, we obtain new vector spaces as subspaces $\ker(A) \subset V$ and $\operatorname{im}(A) \subset W$. The same thing occurs for smooth maps, assuming that they satisfy the conditions of the theorem above.

Definition 19. An *embedded submanifold* (sometimes called regular submanifold) of dimension k in an n-manifold M is a subspace $S \subset M$ such that $\forall s \in S$, there exists a chart (U, φ) for M, containing s, and with

$$S \cap U = \varphi^{-1}(x_{k+1} = \cdots = x_n = 0).$$

In other words, the inclusion $S \subset M$ is locally isomorphic to the vector space inclusion $\mathbb{R}^k \subset \mathbb{R}^n$.

Of course, the remaining coordinates $\{x_1, \ldots, x_k\}$ define a smooth manifold structure on S itself, justifying the terminology.

Proposition 2.12 (analog of kernel)**.** *If $f : M \longrightarrow N$ is a smooth map of manifolds, and if $Tf(p)$ has constant rank on M, then for any $q \in f(M)$, the inverse image $f^{-1}(q) \subset M$ is an embedded submanifold.*

Proof. Let $x \in f^{-1}(q)$. Then there exist charts ψ, φ such that $\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$ and $f^{-1}(q) \cap U = \{x_1 = \cdots = x_k = 0\}$. Hence we obtain that $f^{-1}(q)$ is a codimension k embedded submanifold. □

Example 2.13. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be given by $(x_1, \ldots, x_n) \mapsto \sum x_i^2$. Then $Df(x) = (2x_1, \ldots, 2x_n)$, which has rank 1 at all points in $\mathbb{R}^n \backslash \{0\}$. Hence since $f^{-1}(q)$ contains $\{0\}$ iff $q = 0$, we see that $f^{-1}(q)$ is an embedded submanifold for all $q \neq 0$. Exercise: show that this manifold structure is compatible with that obtained in Example 1.9.*

If Tf has maximal rank at a point $p \in M$, this is a special case, because then it will have maximal rank in a neighbourhood of p, and the local normal form will hold.

Definition 20. A point $p \in M$ for which $Tf(p)$ has maximal rank is called a *regular* point. Otherwise it is called a critical point. Values $q \in N$ for which $f^{-1}(q)$ are all regular points are called *regular values* (including points for which $f^{-1}(q) = \emptyset$). Other values are called critical values. Warning: even if q is a critical value, $f^{-1}(q)$ may contain regular points.

Proposition 2.14 (maximal rank special case)**.** *If $f : M \longrightarrow N$ is a smooth map of manifolds and $q \in N$ is a regular value, then $f^{-1}(q)$ is an embedded submanifold of M.*

Proof. Since the rank is maximal along $f^{-1}(q)$, it must be maximal in an open neighbourhood $U \subset M$ containing $f^{-1}(q)$, and hence $f : U \longrightarrow N$ is of constant rank. □

Warning: An immersion locally defines an embedded submanifold. But globally, it may not be injective, and it also may not be a homeomorphism onto its image (examples: figure 8 embedding of S^1 in $\mathbb{R}^2$ and number 9 immersion of $\mathbb{R}$ in $\mathbb{R}^2$.)

Definition 21. If f is an injective immersion which is a homeomorphism onto its image (when the image is equipped with subspace topology), then we call f an *embedding*

Proposition 2.15. *If $f : M \longrightarrow N$ is an embedding, then $f(M)$ is a regular submanifold.*

Proof. Let $f : M \longrightarrow N$ be an embedding. Then for all $m \in M$, we have charts (U, φ), (V, ψ) where $\psi \circ f \circ \varphi^{-1} : (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$. If $f(U) = f(M) \cap V$, we're done. To make sure that some other piece of M doesn't get sent into the neighbourhood, use the fact that $fx(U)$ is open in the subspace topology. This means we can find a smaller open set $V' \subset V$ such that $V' \cap f(M) = f(U)$. Then we can restrict the charts $(V', \psi|_{V'})$, $(U' = f^{-1}(V'), \varphi_{U'})$ so that we see the embedding. □

Remark 4. *If $\iota : M \longrightarrow N$ is an embedding of M into N, then $T\iota : TM \longrightarrow TN$ is also an embedding, and hence $T^k\iota : T^kM \longrightarrow T^kN$ are all embeddings.*

Having the constant rank theorem in hand, we may also apply it to study manifolds *with boundary*. The following two results illustrate how this may easily be done.

Proposition 2.16. *Let M be a smooth n-manifold and $f : M \longrightarrow \mathbb{R}$ a smooth real-valued function, and let a, b, with $a < b$, be regular values of f. Then $f^{-1}([a, b])$ is a cobordism between the $n-1$-manifolds $f^{-1}(a)$ and $f^{-1}(b)$.*

Proof. The pre-image $f^{-1}((a, b))$ is an open subset of M and hence a submanifold of M. Since p is regular for all $p \in f^{-1}(a)$, we may (by the constant rank theorem) find charts such that f is given near p by the linear map

$$(x_1, \ldots, x_m) \mapsto x_m.$$

Possibly replacing x_m by $-x_m$, we therefore obtain a chart near p for $f^{-1}([a, b])$ into H^m, as required. Proceed similarly for $p \in f^{-1}(b)$. □

Example 2.17. *Using $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $(x_1, \ldots, x_n) \mapsto \sum x_i^2$, this gives a simple proof for the fact that the closed unit ball $\overline{B(0, 1)} = f^{-1}([-1, 1])$ is a manifold with boundary.*

Example 2.18. *Consider the C^∞ function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $(x, y, z) \mapsto x^2 + y^2 - z^2$. Both $+1$ and -1 are regular values for this map, with pre-images given by 1- and 2-sheeted hyperboloids, respectively. Hence $f^{-1}([-1, 1])$ is a cobordism between hyperboloids of 1 and 2 sheets. In other words, it defines a cobordism between the disjoint union of two closed disks and the closed cylinder (each of which has boundary $S^1 \sqcup S^1$). Does this cobordism tell us something about the cobordism class of a connected sum?*

Proposition 2.19. *Let $f : M \longrightarrow N$ be a smooth map from a manifold with boundary to the manifold N. Suppose that $q \in N$ is a regular value of f and also of $f|_{\partial M}$. Then the pre-image $f^{-1}(q)$ is a regular submanifold with boundary (i.e. locally modeled on $\mathbb{R}^k \subset \mathbb{R}^n$ or the inclusion $H^k \subset H^n$ given by $(x_1, \ldots x_k) \mapsto (0, \ldots, 0, x_1, \ldots x_k)$.) Furthermore, the boundary of $f^{-1}(q)$ is simply its intersection with ∂M.*

Proof. If $p \in f^{-1}(q)$ is not in ∂M, then as before $f^{-1}(q)$ is a regular submanifold in a neighbourhood of p. Therefore suppose $p \in \partial M \cap f^{-1}(q)$. Pick charts φ, ψ so that $\varphi(p) = 0$ and $\psi(q) = 0$, and $\psi f \varphi^{-1}$ is a map $U \subset H^m \longrightarrow \mathbb{R}^n$. Extend this to a smooth function $\tilde{f}$ defined in an open set $\tilde{U} \subset \mathbb{R}^m$ containing U. Shrinking $\tilde{U}$ if necessary, we may assume $\tilde{f}$ is regular on $\tilde{U}$. Hence $\tilde{f}^{-1}(0)$ is a regular submanifold of $\mathbb{R}^m$ of dimension $m - n$.

Now consider the real-valued function $\pi : \tilde{f}^{-1}(0) \longrightarrow \mathbb{R}$ given by the restriction of $(x_1, \ldots, x_m) \mapsto x_m$. $0 \in \mathbb{R}$ must be a regular value of π, since if not, then the tangent space to $\tilde{f}^{-1}(0)$ at 0 would lie completely in $x_m = 0$, which contradicts the fact that q is a regular point for $f|_{\partial M}$.

Hence, by Proposition 2.16, we have expressed $f^{-1}(q)$, in a neighbourhood of p, as a regular submanifold with boundary given by $\{\varphi^{-1}(x) \ : \ x \in \tilde{f}^{-1}(0) \text{ and } \pi(x) \geq 0\}$, as required. □

One important use of the above result is in a proof of the Brouwer fixed point theorem. But in order to use it, we need to know that most values are regular values, i.e. that regular values are generic. This is a result of transversality theory, known as Sard's theorem [next section].

Corollary 2.20. *Let M be a compact manifold with boundary. There is no smooth map $f : M \longrightarrow \partial M$ leaving ∂M pointwise fixed. Such a map is called a smooth retraction of M onto its boundary.*

Proof. Such a map f must have a regular value by Sard's theorem, let this value be $y \in \partial M$. Then y is obviously a regular value for $f|_{\partial M} = \mathrm{Id}$ as well, so that $f^{-1}(y)$ must be a compact 1-manifold with boundary given by $f^{-1}(y) \cap \partial M$, which is simply the point y itself. Since there is no compact 1-manifold with a single boundary point, we have a contradiction. □

For example, this shows that the identity map $S^n \longrightarrow S^n$ may not be extended to a smooth map $f : \overline{B(0, 1)} \longrightarrow S^n$.

Corollary 2.21. *Every smooth map of the closed n-ball to itself has a fixed point.*

Proof. Let $D^n = \overline{B(0, 1)}$. If $g : D^n \longrightarrow D^n$ had no fixed points, then define the function $f : D^n \longrightarrow S^{n-1}$ as follows: let $f(x)$ be the point nearer to x on the line joining x and $g(x)$.

This map is smooth, since $f(x) = x + tu$, where

$$u = ||x - g(x)||^{-1}(x - g(x)),$$

and t is the positive solution to the quadratic equation $(x + tu) \cdot (x + tu) = 1$, which has positive discriminant $b^2 - 4ac = 4(1 - |x|^2 + (x \cdot u)^2)$. Such a smooth map is therefore impossible by the previous corollary. □

Theorem 2.22 (Brouwer fixed point theorem)**.** *Any continuous self-map of D^n has a fixed point.*

Not given in class, won't use it in class. The Weierstrass approximation theorem says that any continuous function on $[0, 1]$ can be uniformly approximated by a polynomial function in the supremum norm $||f||_\infty = \sup_{x\in[0,1]} |f(x)|$. In other words, the polynomials are dense in the continuous functions with respect to the supremum norm. The Stone-Weierstrass is a generalization, stating that for any compact Hausdorff space X, if A is a subalgebra of $C^0(X, \mathbb{R})$ such that A separates points ($\forall x, y, \exists f \in A : f(x) \neq f(y)$) and contains a nonzero constant function, then A is dense in C^0.

Given this result, approximate a given continuous self-map g of D^n by a polynomial function p' so that $||p' - g||_\infty < \epsilon$ on D^n. To ensure p' sends D^n into itself, rescale it via

$$p = (1 + \epsilon)^{-1} p'.$$

Then clearly p is a D^n self-map while $||p - g||_\infty < 2\epsilon$. If g had no fixed point, then $|g(x) - x|$ must have a minimum value μ on D^n, and by choosing $2\epsilon = \mu$ we guarantee that for each x,

$$|p(x) - x| \geq |g(x) - x| - |g(x) - p(x)| > \mu - \mu = 0.$$

Hence p has no fixed point. Such a smooth function can't exist and hence we obtain the result. $\square$

3 Transversality

In this section, we continue to use the inverse and constant rank theorems to produce more manifolds, except now these are cut out only locally by functions. We ask when the *intersection* of two submanifolds yields a submanifold. You should think that intersecting a given submanifold with another is the local imposing of a certain number of constraints.

Two subspaces $K, L \subset V$ of a vector space V are called *transversal* when $K + L = V$, i.e. every vector in V may be written as a (possibly non-unique) linear combination of vectors in K and L. In this situation one can easily see that

$$\dim V = \dim K + \dim L - \dim K \cap L.$$

We may apply this to submanifolds as follows:

Definition 22. Let $K, L \subset M$ be regular submanifolds such that every point $p \in K \cap L$ satisfies

$$T_pK + T_pL = T_pM.$$

Then K, L are said to be *transverse* submanifolds and we write $K \pitchfork L$.

Proposition 3.1. *If $K, L \subset M$ are transverse regular submanifolds then $K \cap L$ is also a regular submanifold, of dimension* $\dim K + \dim L - \dim M$.

Proof. Let $p \in K \cap L$. Then there is a neighbourhood U of p for which $K \cap U = f^{-1}(0)$ for 0 a regular value of a function $f : U \longrightarrow \mathbb{R}^k$ and $L \cap U = g^{-1}(0)$ for 0 a regular value of a function $g : L \cap U \longrightarrow \mathbb{R}^l$, where K and L have codimension k, l respectively.

Now note that $K \cap L \cap U = (f, g)^{-1}(0)$, where $(f, g) : K \cap L \cap U \longrightarrow \mathbb{R}^{k+l}$. But 0 is a regular value for (f, g), since $\ker T(f, g) = \ker Tf \cap \ker Tg = T_pK \cap T_pL$, which has codimension $k + l$ by the transversality assumption. Hence the rank of $T(f, g)$ must be $k + l$, just because the rank of a linear map is always given by the codimension of its kernel. $\square$

Example 3.2 (Exotic spheres)**.** *Consider the following intersections in $\mathbb{C}^5\backslash 0$:*

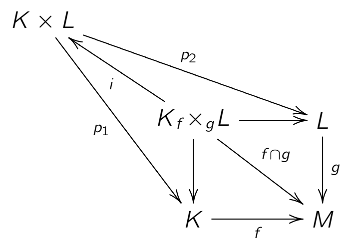
$$S^7_k = \{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}.$$

This is a transverse intersection, and for $k = 1, \ldots, 28$ the intersection is a smooth manifold homeomorphic to S^7. These exotic 7-spheres were constructed by Brieskorn and represent each of the 28 diffeomorphism classes on S^7.

We now phrase the previous transversality result in a slightly different way, in terms of the embedding maps k, l for K, L in M. Specifically, we say the maps k, l are transverse when, $\forall a \in K, b \in L$ such that $k(a) = l(b) = p$, we have $\mathrm{im}(Tk(a)) + \mathrm{im}(Tl(b)) = T_pM$. The advantage of this approach is that it makes sense for any maps, not necessarily embeddings.

Definition 23. Two maps $f : K \longrightarrow M$, $g : L \longrightarrow M$ of manifolds are called *transverse* when $Tf(T_aK) + Tg(T_bL) = T_pM$ for all a, b, p such that $f(a) = g(b) = p$.

Proposition 3.3. *If $f : K \longrightarrow M$, $g : L \longrightarrow M$ are transverse smooth maps, then $K_f\times_g L = \{(a, b) \in K \times L \ : \ f(a) = g(b)\}$ is naturally a smooth manifold equipped with commuting maps*



where i is the inclusion and $f \cap g : (a, b) \mapsto f(a) = g(b)$.

The manifold $K_f\times_g L$ of the previous proposition is called the *fiber product* of K with L over M, and is a generalization of the intersection of submanifolds.

Proof. Consider the graphs $\Gamma_f \subset K \times M$ and $\Gamma_g \subset L \times M$. Then we show that the following intersection of regular submanifolds is transverse:

$$\Gamma_{f\cap g} = (\Gamma_f \times \Gamma_g) \cap (K \times L \times \Delta_M),$$

where $\Delta_M = \{(p, p) \in M \times M \ : \ p \in M\}$ is the diagonal. To show this, let $f(k) = g(l) = m$ so that $x = (k, l, m, m) \in X$, and note that

$$T_x(\Gamma_f \times \Gamma_g) = \{((v, Df(v)), (w, Dg(w))), \ v \in T_kK, \quad w \in T_lL\} \tag{16}$$

whereas we also have

$$T_x(K \times L \times \Delta_M) = \{((v, m), (w, m)) \ : \ v \in T_kK, \quad w \in T_lL, \quad m \in T_pM\} \tag{17}$$

By transversality of f, g, any tangent vector $m_i \in T_pM$ may be written as $Df(v_i)+Dg(w_i)$ for some (v_i, w_i), $i = 1, 2$. In particular, we may decompose a general tangent vector to $M \times M$ as

$$(m_1, m_2) = (Df(v_2), Df(v_2)) + (Dg(w_1), Dg(w_1)) + (Df(v_1 - v_2), Dg(w_2 - w_1)),$$

leading directly to the transversality of the spaces (16), (17). This shows that $\Gamma_{f\cap g}$ is a regular submanifold of $K\times L\times M\times M$. Actually since it sits inside $K\times L\times\Delta_M$, we may compose with the projection diffeomorphism to view it as a regular submanifold in $K \times L \times M$. Then we observe that the restriction of the projection onto $K \times L$ to the submanifold $\Gamma_{f\cap g}$ is an embedding with image exactly $K_f\times_g L$. Hence the fiber product is a smooth manifold and $\Gamma_{f\cap g}$ may then be viewed as the graph of a smooth map $f \cap g : K_f\times_g L \longrightarrow M$ which makes the diagram above commute by definition. ☐

Example 3.4. *If $K_1 = M\times Z_1$ and $K_2 = M\times Z_2$, we may view both K_i as smooth fiber bundles over M with fibers Z_i. If p_i are the projections to M, then $K_1 \times_M K_2 = M \times Z_1 \times Z_2$, hence the name "fiber product".*

Example 3.5. *Consider the Hopf map $p : S^3 \longrightarrow S^2$ given by composing the embedding $S^3 \subset \mathbb{C}^2\backslash\{0\}$ with the projection $\pi : \mathbb{C}^2\backslash\{0\} \longrightarrow \mathbb{C}P^1 \cong S^2$. Then for any point $q \in S^2$, $p^{-1}(q) \cong S^1$. Since p is a submersion, it is obviously transverse to itself, hence we may form the fiber product*

$$S^3 \times_{S^2} S^3,$$

which is a smooth 4-manifold equipped with a map $p\cap p$ to S^2 with fibers $(p\cap p)^{-1}(q) \cong S^1 \times S^1$.

These are our first examples of nontrivial fiber bundles, which we shall explore later.

The following result is an exercise: just as we may take the product of a manifold with boundary K with a manifold without boundary L to obtain a manifold with boundary $K \times L$, we have a similar result for fiber products.

Proposition 3.6. *Let K be a manifold with boundary where L, M are without boundary. Assume that $f : K \longrightarrow M$ and $g : L \longrightarrow M$ are smooth maps such that both f and ∂f are transverse to g. Then the fiber product $K \times_M L$ is a manifold with boundary equal to $\partial K \times_M L$.*

3.1 Stability

We wish to understand the intuitive notion that "transversality is a stable condition", which in some sense means that if true, it remains so under small perturbations (of the submanifolds or maps involved). After this, we will go much further using Sard's theorem, and show that not only is it *stable*, it is actually *generic*, meaning that even if it is not true, it can be made true by a small perturbation. In this sense, stability says that transversal maps form an open set, and genericity says that this open set is dense in the space of maps. To make this precise, we would introduce a topology on the space of maps, something which we leave for another course.

A property of a smooth map $f_0 : M \longrightarrow N$ is *stable* under perturbations when for any smooth homotopy f_t of f_0, i.e. a smooth map $f : [0, 1] \times M \longrightarrow N$ with $f|_{\{0\}\times M} = f_0$, the property holds for all $f_t = f|_{\{t\}\times M}$ with $t < \epsilon$ for some $\epsilon > 0$.

Proposition 3.7. *Let M be a compact manifold and $f_0 : M \longrightarrow N$ a smooth map. Then the property of being an immersion or submersion are each stable under perturbations. If M' is compact, then the transversality of $f_0 : M \longrightarrow N$, $g_0 : M' \longrightarrow N$ is also stable under perturbations of f_0, g_0.*

As an exercise, show that local diffeomorphisms, diffeomorphisms, and embeddings are also stable.

Proof. Let $f_t, t \in [0, 1]$ be a smooth homotopy of f_0, and suppose that f_0 is an immersion. This means that at each point $p \in M$, the jacobian of f_0 in some chart has a $m\times m$ submatrix with nonvanishing determinant, for $m = \dim M$. By continuity, this $m\times m$ submatrix must have nonvanishing determinant in a neighbourhood around $(0, p) \in [0, 1] \times M$. $\{0\} \times M$ may be covered by a finite number of such neighbourhoods, since M is compact. Choose ϵ such that $[0, \epsilon) \times M$ is contained in the union of these intervals, giving the result.

The proof for submersions is identical. The condition that f_0 be transversal to g_0 is equivalent to the fact that $\Gamma_{f_0} \times \Gamma_{g_0}$ is transversal to $C = M \times M' \times \Delta_N$. Choosing coordinate charts adapted to C, we may express this locally as a submersion condition. Hence by the previous result we have stability. □

3.2 Genericity of transversality

The fundamental idea which allows us to prove that transversality is a generic condition is a the theorem of Sard showing that critical values of a smooth map $f : M \longrightarrow N$ (i.e. points $q \in N$ for which the map f and the inclusion $\iota : q \hookrightarrow N$ fail to be transverse maps) are *rare*. The following proof is taken from Milnor, based on Pontryagin.

The meaning of "rare" will be that the set of critical values is of *measure zero*, which means, in $\mathbb{R}^m$, that for any $\epsilon > 0$ we can find a sequence of balls in $\mathbb{R}^m$, containing $f(C)$ in their union, with total volume less than ϵ. Some easy facts about sets of measure zero: the countable union of measure zero sets is of measure zero, the complement of a set of measure zero is dense.

We begin with an elementary lemma which shows that "measure zero" is a property preserved by diffeomorphisms.

Lemma 3.8. *Let $A \subset \mathbb{R}^m$ have measure zero and let $F : A \longrightarrow \mathbb{R}^n$ be a C^1 map with $m \leq n$. Then $F(A)$ has measure zero.*

Proof. F has an extension to a neighbourhood W of A. Let $\overline{B}$ be a closed ball in W. We then show that $F(A\cap \overline{B})$ has measure zero, and since $F(A)$ is the union of countably many such sets, we obtain $F(A)$ of measure zero.

Since F is C^1, we have the mean value theorem stating for all $x, y \in \overline{B}$

$$f(y) - f(x) = [\int_0^1 F_*((1 - t)x + ty)dt](y - x),$$

where the integral of the matrix is done component-wise. Then we have the estimate

$$\begin{aligned} ||f(y) - f(x)|| &= ||[\int_0^1 F_*((1-t)x + ty)dt](y-x)|| \\ &\leq \int_0^1 ||F_*((1-t)x + ty)(y-x)||dt \\ &\leq \int_0^1 ||F_*((1-t)x + ty)|| \cdot ||y-x||dt \\ &= C||y-x||. \end{aligned}$$

Then the image of a ball of radius r contained in $\overline{B}$ would be contained in a ball of radius at most Cr, which would have volume proportional to r^n.

A is of measure zero, hence for each ϵ we have a countable covering of A by balls of radius r_k with total volume $c_m \sum_k r_k^m < \epsilon$. We deduce that $f(A_i)$ is covered by balls of radius Cr_k with total volume $\leq C^n c_n \sum_k r_k^n$ and since $n \geq m$ this is certainly arbitrarily small. We conclude that $f(A)$ is of measure zero. □

Remark 5. *If we considered the case $n < m$, the resulting sum of volumes may be larger in $\mathbb{R}^n$. For example, the projection map $\mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $(x, y) \mapsto x$ clearly takes the set of measure zero $y = 0$ to one of positive measure.*

A subset $A \subset M$ of a manifold is said to have measure zero when its image in any coordinate chart has measure zero. Since manifolds are second countable and we may choose a countable basis V_i such that $\overline{V}_i \subset U_i$ are compact subsets of coordinate charts (any coordinate neighbourhood is a countable union of closed balls), it follows that a subset $A \subset M$ of measure zero may be expressed as a countable union of subsets $A_k \subset \overline{V}_i$ with $\varphi_i(A_k)$ satisfying the Lemma. We therefore obtain

Proposition 3.9. *Let $f : M \longrightarrow N$ be a C^1 map of manifolds where* $\dim M \leq \dim N$. *Then the image $f(A)$ of a set $A \subset M$ of measure zero also has measure zero.*

Corollary 3.10 (Baby Sard)**.** *Let $f : M \longrightarrow N$ be a C^1 of manifolds where* $\dim M < \dim N$. *Then $f(M)$ (i.e. the set of critical values) has measure zero in N.*

Proof. We could form $\tilde{M} = M \times \mathbb{R}$ and consider $F : \tilde{M} \longrightarrow N$ given by $F(x, t) = f(x)$. Then $f(M) = F(M \times \{0\})$. Since $M \times \{0\} \subset M \times \mathbb{R}$ is measure zero, and $\dim \tilde{M} \leq \dim N$, so is the image. □

Now we investigate the measure of the critical values of a map $f : M \longrightarrow N$ where $\dim M = \dim N$. Of course the set of critical points need not have measure zero, but we shall see that because the values of f on the critical set do not vary much, the set of critical *values* will have measure zero.

Theorem 3.11 (Equidimensional Sard)**.** *Let $f : M \longrightarrow N$ be a C^1 map of n-manifolds, and let $C \subset M$ be the set of critical points. Then $f(C)$ has measure zero.*

Proof. It suffices to show result for the unit cube. Let $f : I^n \longrightarrow \mathbb{R}^n$ a C^1 map and let $C \subset I^n$ be the set of critical points.

Since $f \in C^1(I^n, \mathbb{R}^n)$, we have that the linear approximation to f at $x \in I^n$, namely

$$f_x^{lin}(y) = f(x) + f_*(x)(y - x),$$

approximates f to second order, i.e. there is a positive function $b(\epsilon)$ with $b \to 0$ as $\epsilon \to 0$ such that

$$||f(y) - f_x^{lin}(y)|| \leq b(|y - x|)||y - x||.$$

Of course f is still Lipschitz so that

$$||f(y) - f(x)|| \leq a||y - x|| \quad \forall x, y \in I^n$$

Since f is Lipschitz, we know that $||y - x|| < \epsilon$ implies $||f(y) - f(x)|| < a\epsilon$. But if x is a critical point, then f_x^{lin} has image contained in a hyperplane P_x, which is of lower dimension and hence measure zero. This means the distance of $f(y)$ to P_x is less than $\epsilon b(\epsilon)$.

Therefore $f(y)$ lies in the cube centered at $f(x)$ of edge $a\epsilon$, but if we choose the cube to have a face parallel to P_x, then the edge perpendicular to P_x can be shortened to only $2\epsilon b\epsilon$. Therefore $f(y)$ is in a region of volume $(a\epsilon)^{n-1}2\epsilon b(\epsilon)$.

Now partition I^n into h^n cubes each of edge h^{-1}. Any such cube containing a critical point x is certainly contained in a ball around x of radius $r = h^{-1}\sqrt{n}$. The image of this ball then has volume $\leq (ar)^{n-1}2rb(r) = Ar^n b(r)$ for $A = 2a^{n-1}$. The total volume of all the images is then less than

$$h^n A r^n b(r) = A n^{n/2} b(r).$$

Note that A and n are fixed, while $r = h^{-1}\sqrt{n}$ is determined by the number h of cubes. By increasing the number of cubes, we may decrease their radius arbitrarily, and hence the above total volume, as required. □

The argument above will not work for $\dim N < \dim M$; we need more control on the function f. In particular, one can find a C^1 function from $I^2 \longrightarrow \mathbb{R}$ which fails to have critical values of measure zero (hint: $C + C = [0, 2]$ where C is the Cantor set). As a result, Sard's theorem in general requires more differentiability of f.

Theorem 3.12 (Big Sard's theorem)**.** *Let $f : M \longrightarrow N$ be a C^k map of manifolds of dimension m, n, respectively. Let C be the set of critical points, i.e. points $x \in U$ with*

$$\text{rank } Df(x) < n.$$

Then $f(C)$ has measure zero if $k > \frac{m}{n} - 1$.

Do not give proof in class, no time. As before, it suffices to show for $f : I^m \longrightarrow \mathbb{R}^n$.

Define $C_1 \subset C$ to be the set of points x for which $Df(x) = 0$. Define $C_i \subset C_{i-1}$ to be the set of points x for which $D^j f(x) = 0$ for all $j \leq i$. So we have a descending sequence of closed sets:

$$C \supset C_1 \supset C_2 \supset \cdots \supset C_k.$$

We will show that $f(C)$ has measure zero by showing

1. $f(C_k)$ has measure zero,
2. each successive difference $f(C_i \backslash C_{i+1})$ has measure zero for $i \geq 1$,
3. $f(C \backslash C_1)$ has measure zero.

Step 1: For $x \in C_k$, Taylor's theorem gives the estimate

$$f(x + t) = f(x) + R(x, t), \quad \text{with } \; ||R(x, t)|| \leq c||t||^{k+1},$$

where c depends only on I^m and f, and t sufficiently small.

If we now subdivide I^m into h^m cubes with edge h^{-1}, suppose that x sits in a specific cube I_1. Then any point in I_1 may be written as $x + t$ with $||t|| \leq h^{-1}\sqrt{m}$. As a result, $f(I_1)$ lies in a cube of edge $ah^{-(k+1)}$, where $a = 2cm^{(k+1)/2}$ is independent of the cube size. There are at most h^m such cubes, with total volume less than

$$h^m(ah^{-(k+1)})^n = a^n h^{m-(k+1)n}.$$

Assuming that $k > \frac{m}{n} - 1$, this tends to 0 as we increase the number of cubes.

Step 2: For each $x \in C_i \backslash C_{i+1}$, $i \geq 1$, there is a $i + 1^{th}$ partial $\partial^{i+1} f_j / \partial x_{s_1} \cdots \partial x_{s_{i+1}}$ which is nonzero at x. Therefore the function

$$w(x) = \partial^k f_j / \partial x_{s_2} \cdots \partial x_{s_{i+1}}$$

vanishes at x but its partial derivative $\partial w / \partial x_{s_1}$ does not. WLOG suppose $s_1 = 1$, the first coordinate. Then the map

$$h(x) = (w(x), x_2, \ldots, x_m)$$

is a local diffeomorphism by the inverse function theorem (of class C^k) which sends a neighbourhood V of x to an open set V'. Note that $h(C_i \cap V) \subset \{0\} \times \mathbb{R}^{m-1}$. Now if we restrict $f \circ h^{-1}$ to $\{0\} \times \mathbb{R}^{m-1} \cap V'$, we obtain a map g whose critical points include $h(C_i \cap V)$. Hence we may prove by induction on m that $g(h(C_i \cap V)) = f(C_i \cap V)$ has measure zero. Cover by countably many such neighbourhoods V.

Step 3: Let $x \in C \backslash C_1$. Then there is some partial derivative, wlog $\partial f_1/\partial x_1$, which is nonzero at x. the map

$$h(x) = (f_1(x), x_2, \ldots, x_m)$$

is a local diffeomorphism from a neighbourhood V of x to an open set V' (of class C^k). Then $g = f \circ h^{-1}$ has critical points $h(V \cap C)$, and has critical values $f(V \cap C)$. The map g sends hyperplanes $\{t\} \times \mathbb{R}^{m-1}$ to hyperplanes $\{t\} \times \mathbb{R}^{n-1}$, call the restriction map g_t. A point in $\{t\} \times \mathbb{R}^{m-1}$ is critical for g_t if and only if it is critical for g, since the Jacobian of g is

$$\begin{pmatrix} 1 & 0 \\ * & \frac{\partial g_t^i}{\partial x_j} \end{pmatrix}$$

By induction on m, the set of critical values for g_t has measure zero in $\{t\} \times \mathbb{R}^{n-1}$. By Fubini, the whole set $g(C')$ (which is measurable, since it is the countable union of compact subsets (critical values not necessarily closed, but critical points are closed and hence a countable union of compact subsets, which implies the same of the critical values.) is then measure zero. To show this consequence of Fubini directly, use the following argument:

First note that for any covering of $[a, b]$ by intervals, we may extract a finite subcovering of intervals whose total length is $\leq 2|b-a|$. Why? First choose a minimal subcovering $\{I_1, \ldots, I_p\}$, numbered according to their left endpoints. Then the total overlap is at most the length of $[a, b]$. Therefore the total length is at most $2|b - a|$.

Now let $B \subset \mathbb{R}^n$ be compact, so that we may assume $B \subset \mathbb{R}^{n-1} \times [a, b]$. We prove that if $B \cap P_c$ has measure zero in the hyperplane $P_c = \{x^n = c\}$, for any constant $c \in [a, b]$, then it has measure zero in $\mathbb{R}^n$.

If $B \cap P_c$ has measure zero, we can find a covering by open sets $R_c^i \subset P_c$ with total volume $< \epsilon$. For sufficiently small α_c, the sets $R_c^i \times [c - \alpha_c, c + \alpha_c]$ cover $B \cap \bigcup_{z \in [c-\alpha_c, c+\alpha_c]} P_z$ (since B is compact). As we vary c, the sets $[c - \alpha_c, c + \alpha_c]$ form a covering of $[a, b]$, and we extract a finite subcover $\{I_j\}$ of total length $\leq 2|b - a|$.

Let R_j^i be the set R_c^i for $I_j = [c - \alpha_c, c + \alpha_c]$. Then the sets $R_j^i \times I_j$ form a cover of B with total volume $\leq 2\epsilon|b - a|$. We can make this arbitrarily small, so that B has measure zero. ☐

We now proceed with the first step towards showing that transversality is generic.

Theorem 3.13 (Transversality theorem)**.** *Let $F : X \times S \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps of manifolds where only X has boundary. Suppose that F and ∂F are transverse to g. Then for almost every $s \in S$, $f_s = F(\cdot, s)$ and ∂f_s are transverse to g.*

Proof. The fiber product $W = (X \times S) \times_Y Z$ is a regular submanifold (with boundary) of $X \times S \times Z$ and projects to S via the usual projection map π. We show that any $s \in S$ which is a regular value for both the projection map $\pi : W \longrightarrow S$ and its boundary map $\partial\pi$ gives rise to a f_s which is transverse to g. Then by Sard's theorem the s which fail to be regular in this way form a set of measure zero.

Suppose that $s \in S$ is a regular value for π. Suppose that $f_s(x) = g(z) = y$ and we now show that f_s is transverse to g there. Since $F(x, s) = g(z)$ and F is transverse to g, we know that

$$\mathrm{im} DF_{(x,s)} + \mathrm{im} Dg_z = T_y Y.$$

Therefore, for any $a \in T_y Y$, there exists $b = (w, e) \in T(X \times S)$ with $DF_{(x,s)} b - a$ in the image of Dg_z. But since $D\pi$ is surjective, there exists $(w', e, c') \in T_{(x,y,z)} W$. Hence we observe that

$$(Df_s)(w - w') - a = DF_{(x,s)}[(w, e) - (w', e)] - a = (DF_{(x,s)} b - a) - DF_{(x,s)}(w', e),$$

where both terms on the right hand side lie in $\mathrm{im} Dg_z$.

Precisely the same argument (with X replaced with ∂X and F replaced with ∂F) shows that if s is regular for $\partial\pi$ then ∂f_s is transverse to g. This gives the result. ☐

The previous result immediately shows that transversal maps to $\mathbb{R}^n$ are generic, since for any smooth map $f : M \longrightarrow \mathbb{R}^n$ we may produce a family of maps

$$F : M \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

via $F(x,s) = f(x) + s$. This new map F is clearly a submersion and hence is transverse to any smooth map $g : Z \longrightarrow \mathbb{R}^n$. For arbitrary target manifolds, we will imitate this argument, but we will require a (weak) version of Whitney's embedding theorem for manifolds into $\mathbb{R}^n$.

3.3 Whitney embedding

We now investigate the embedding of arbitrary smooth manifolds as regular submanifolds of $\mathbb{R}^k$. We shall first show by a straightforward argument that any smooth manifold may be embedded in some $\mathbb{R}^N$ for some sufficiently large N. We will then explain how to cut down on N and approach the optimal $N = 2\dim M$ which Whitney showed (we shall reach $2\dim M+1$ and possibly at the end of the course, show $N = 2\dim M$.)

Theorem 3.14 (Compact Whitney embedding in $\mathbb{R}^N$)**.** *Any* compact *manifold may be embedded in* $\mathbb{R}^N$ *for sufficiently large* N.

Proof. Let $\{(U_i \supset V_i, \varphi_i)\}_{i=1}^k$ be a *finite* regular covering, which exists by compactness. Choose a partition of unity $\{f_1, \ldots, f_k\}$ as in Theorem 1.19 and define the following "zoom-in" maps $M \longrightarrow \mathbb{R}^{\dim M}$:

$$\tilde{\varphi}_i(x) = \begin{cases} f_i(x)\varphi_i(x) & x \in U_i, \\ 0 & x \notin U_i. \end{cases}$$

Then define a map $\Phi : M \longrightarrow \mathbb{R}^{k(\dim M+1)}$ which zooms simultaneously into all neighbourhoods, with extra information to guarantee injectivity:

$$\Phi(x) = (\tilde{\varphi}_1(x), \ldots, \tilde{\varphi}_k(x), f_1(x), \ldots, f_k(x)).$$

Note that $\Phi(x) = \Phi(x')$ implies that for some i, $f_i(x) = f_i(x') \neq 0$ and hence $x, x' \in U_i$. This then implies that $\varphi_i(x) = \varphi_i(x')$, implying $x = x'$. Hence Φ is injective.

We now check that $D\Phi$ is injective, which will show that it is an injective immersion. At any point x the differential sends $v \in T_xM$ to the following vector in $\mathbb{R}^{\dim M} \times \cdots \times \mathbb{R}^{\dim M} \times \mathbb{R} \times \cdots \times \mathbb{R}$.

$$(Df_1(v)\varphi_1(x) + f_1(x)D\varphi_1(v), \ldots, Df_k(v)\varphi_k(x) + f_k(x)D\varphi_1(v), Df_1(v), \ldots, Df_k(v)$$

But this vector cannot be zero. Hence we see that Φ is an immersion.

But an injective immersion from a compact space must be an embedding: view Φ as a bijection onto its image. We must show that Φ^{-1} is continuous, i.e. that Φ takes closed sets to closed sets. If $K \subset M$ is closed, it is also compact and hence $\Phi(K)$ must be compact, hence closed (since the target is Hausdorff). ☐

Theorem 3.15 (Compact Whitney embedding in $\mathbb{R}^{2n+1}$)**.** *Any* compact *n-manifold may be embedded in* $\mathbb{R}^{2n+1}$.

Proof. Begin with an embedding $\Phi : M \longrightarrow \mathbb{R}^N$ and assume $N > 2n+1$. We then show that by projecting onto a hyperplane it is possible to obtain an embedding to $\mathbb{R}^{N-1}$.

A vector $v \in S^{N-1} \subset \mathbb{R}^N$ defines a hyperplane (the orthogonal complement) and let $P_v : \mathbb{R}^N \longrightarrow \mathbb{R}^{N-1}$ be the orthogonal projection to this hyperplane. We show that the set of v for which $\Phi_v = P_v \circ \Phi$ fails to be an embedding is a set of measure zero, hence that it is possible to choose v for which Φ_v is an embedding.

Φ_v fails to be an embedding exactly when Φ_v is not injective or $D\Phi_v$ is not injective at some point. Let us consider the two failures separately:

If v is in the image of the map $\beta_1 : (M \times M)\backslash\Delta_M \longrightarrow S^{N-1}$ given by

$$\beta_1(p_1, p_2) = \frac{\Phi(p_2) - \Phi(p_1)}{||\Phi(p_2) - \Phi(p_1)||},$$

then Φ_v will fail to be injective. Note however that β_1 maps a $2n$-dimensional manifold to a $N-1$-manifold, and if $N > 2n+1$ then baby Sard's theorem implies the image has measure zero.

The immersion condition is a local one, which we may analyze in a chart (U, φ). Φ_v will fail to be an immersion in U precisely when v coincides with a vector in the normalized image of $D(\Phi \circ \varphi^{-1})$ where

$$\Phi \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N.$$

Hence we have a map (letting $N(w) = ||w||$)

$$\frac{D(\Phi \circ \varphi^{-1})}{N \circ D(\Phi \circ \varphi^{-1})} : U \times S^{n-1} \longrightarrow S^{N-1}.$$

The image has measure zero as long as $2n - 1 < N - 1$, which is certainly true since $2n < N - 1$. Taking union over countably many charts, we see that immersion fails on a set of measure zero in S^{N-1}.

Hence we see that Φ_v fails to be an embedding for a set of $v \in S^{N-1}$ of measure zero. Hence we may reduce N all the way to $N = 2n + 1$. □

Corollary 3.16. *We see from the proof that if we do not require injectivity but only that the manifold be* immersed *in $\mathbb{R}^N$, then we can take $N = 2n$ instead of $2n + 1$.*

We now use Whitney embedding to prove genericity of transversality for all target manifolds, not just $\mathbb{R}^n$. We do this by embedding the manifold M into $\mathbb{R}^N$, translating it, and projecting back onto M.

If $Y \subset \mathbb{R}^N$ is an embedded submanifold, the normal space at $y \in Y$ is defined by $N_yY = \{v \in \mathbb{R}^N \ : \ v \perp T_yY\}$. The collection of all normal spaces of all points in Y is called the normal bundle:

$$NY = \{(y, v) \in Y \times \mathbb{R}^N \ : \ v \in N_yY\}.$$

This is an embedded submanifold of $\mathbb{R}^N \times \mathbb{R}^N$ of dimension N, and it has a projection map $\pi : (y, v) \mapsto y$ such that (NY, π, Y) is a vector bundle. We may take advantage of the embedding in $\mathbb{R}^N$ to define a smooth map $E : NY \longrightarrow \mathbb{R}^N$ via

$$E(x, v) = x + v.$$

Definition 24. A tubular neighbourhood of the embedded submanifold $Y \subset \mathbb{R}^N$ is a neighbourhood U of Y in $\mathbb{R}^N$ that is the diffeomorphic image under E of an open subset $V \subset NY$ of the form

$$V = \{(y, v) \in NY \ : \ |v| < \delta(y)\},$$

for some positive continuous function $\delta : M \longrightarrow \mathbb{R}$.

If $U \subset \mathbb{R}^N$ is such a tubular neighbourhood of Y, then there does exist a positive continuous function $\epsilon : Y \longrightarrow \mathbb{R}$ such that $U_\epsilon = \{x \in \mathbb{R}^N \ : \ \exists y \in Y \text{ with } |x - y| < \epsilon(y)\}$ is contained in U. This is simply

$$\epsilon(y) = \sup\{r \ : \ B(y, r) \subset U\}.$$

Theorem 3.17 (Tubular neighbourhood theorem)**.** *Every embedded submanifold of $\mathbb{R}^N$ has a tubular neighbourhood.*

Corollary 3.18. *Let X be a manifold with boundary and $f : X \longrightarrow Y$ be a smooth map to a manifold Y. Then there is an open ball $S = B(0, 1) \subset \mathbb{R}^N$ and a smooth map $F : X \times S \longrightarrow Y$ such that $F(x, 0) = f(x)$ and for fixed x, the map $f_x : s \mapsto F(x, s)$ is a submersion $S \longrightarrow Y$. In particular, F and ∂F are submersions.*

Proof. Embed Y in $\mathbb{R}^N$, and let $S = B(0, 1) \subset \mathbb{R}^N$. Then use the tubular neighbourhood to define

$$F(y, s) = (\pi \circ E^{-1})(f(y) + \epsilon(y)s),$$

□

The transversality theorem then guarantees that given any smooth $g : Z \longrightarrow Y$, for almost all $s \in S$ the maps $f_s, \partial f_s$ are transverse to g. We improve this slightly to show that f_s may be chosen to be *homotopic* to f.

Corollary 3.19 (Transverse deformation of maps)**.** *Given any smooth maps $f : X \longrightarrow Y$, $g : Z \longrightarrow Y$, where only X has boundary, there exists a smooth map $f' : X \longrightarrow Y$ homotopic to f with $f', \partial f'$ both transverse to g.*

Proof. Let S, F be as in the previous corollary. Away from a set of measure zero in S, the functions $f_s, \partial f_s$ are transverse to g, by the transversality theorem. But these f_s are all homotopic to f via the homotopy $X \times [0, 1] \longrightarrow Y$ given by

$$(x, t) \mapsto F(x, ts).$$

□

The last theorem we shall prove concerning transversality is a very useful extension result which is essential for intersection theory:

Theorem 3.20 (Transverse deformation of homotopies)**.** *Let X be a manifold with boundary and $f : X \longrightarrow Y$ a smooth map to a manifold Y. Suppose that ∂f is transverse to the closed map $g : Z \longrightarrow Y$. Then there exists a map $f' : X \longrightarrow Y$, homotopic to f and with $\partial f' = \partial f$, such that f' is transverse to g.*

Proof. First observe that since ∂f is transverse to g on ∂X, f is also transverse to g there, and furthermore since g is closed, f is transverse to g in a neighbourhood U of ∂X. (if $x \in \partial X$ but x not in $f^{-1}(g(Z))$ then since the latter set is closed, we obtain a neighbourhood of x for which f is transverse to g. If $x \in \partial X$ and $x \in f^{-1}(g(Z))$, then transversality at x implies transversality near x.)

Now choose a smooth function $\gamma : X \longrightarrow [0, 1]$ which is 1 outside U but 0 on a neighbourhood of ∂X. (why does γ exist? exercise.) Then set $\tau = \gamma^2$, so that $d\tau(x) = 0$ wherever $\tau(x) = 0$. Recall the map $F : X \times S \longrightarrow Y$ we used in proving the transversality homotopy theorem 3.19 and modify it via

$$F'(x, s) = F(x, \tau(x)s).$$

Then F' and $\partial F'$ are transverse to g, and we can pick s so that $f' : x \mapsto F'(x, s)$ and $\partial f'$ are transverse to g. Finally, if x is in the neighbourhood of ∂X for which $\tau = 0$, then $f'(x) = F(x, 0) = f(x)$. □

Corollary 3.21. *if $f : X \longrightarrow Y$ and $f' : X \longrightarrow Y$ are homotopic smooth maps of manifolds, each transverse to the closed map $g : Z \longrightarrow Y$, then the fiber products $W = X_f \times_g Z$ and $W' = X_{f'} \times_g Z$ are cobordant.*

Proof. if $F : X \times [0, 1] \longrightarrow Y$ is the homotopy between $\{f, f'\}$, then by the previous theorem, we may find a (homotopic) homotopy $F' : X \times [0, 1] \longrightarrow Y$ which is transverse to g. Hence the fiber product $U = (X \times [0, 1])_{F'} \times_g Z$ is the cobordism with boundary $W \sqcup W'$. □

3.4 Intersection theory

The previous corollary allows us to make the following definition:

Definition 25. Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ be smooth maps with X compact, g closed, and $\dim X + \dim Z = \dim Y$. Then we define the (mod 2) intersection number of f and g to be

$$I_2(f, g) = \sharp(X_{f'} \times_g Z) \pmod 2,$$

where $f' : X \longrightarrow Y$ is any smooth map smoothly homotopic to f but transverse to g, and where we assume the fiber product to consist of a finite number of points (this is always guaranteed, e.g. if g is proper, or if g is a closed embedding).

Example 3.22. *If C_1, C_2 are two distinct great circles on S^2 then they have two transverse intersection points, so $I_2(C_1, C_2) = 0$ in $\mathbb{Z}_2$. Of course we can shrink one of the circles to get a homotopic one which does not intersect the other at all. This corresponds to the standard cobordism from two points to the empty set.*

Example 3.23. *If (e_1, e_2, e_3) is a basis for $\mathbb{R}^3$ we can consider the following two embeddings of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ into $\mathbb{R}P^2$: $\iota_1 : \theta \mapsto \langle\cos(\theta/2)e_1 + \sin(\theta/2)e_2\rangle$ and $\iota_2 : \theta \mapsto \langle\cos(\theta/2)e_2 + \sin(\theta/2)e_3\rangle$. These two embedded submanifolds intersect transversally in a single point $\langle e_2\rangle$, and hence $I_2(\iota_1, \iota_2) = 1$ in $\mathbb{Z}_2$. As a result, there is no way to deform ι_i so that they intersect transversally in zero points. In particular, $\mathbb{R}P^2$ has a noncontractible loop.*

Example 3.24. *Given a smooth map $f : X \longrightarrow Y$ for X compact and $\dim Y = 2 \dim X$, we may consider the self-intersection $I_2(f, f)$. In the previous examples we may check $I_2(C_1, C_1) = 0$ and $I_2(\iota_1, \iota_1) = 1$. Any embedded S^1 in an oriented surface has no self-intersection. If the surface is nonorientable, the self-intersection may be nonzero.*

Example 3.25. *Let $p \in S^1$. Then the identity map $\mathrm{Id} : S^1 \longrightarrow S^1$ is transverse to the inclusion $\iota : p \longrightarrow S^1$ with one point of intersection. Hence the identity map is not (smoothly) homotopic to a constant map, which would be transverse to ι with zero intersection. Using smooth approximation, get that Id is not continuously homotopic to a constant map, and also that S^1 is not contractible.*

Example 3.26. *By the previous argument, any compact manifold is not contractible.*

Example 3.27. *Consider $SO(3) \cong \mathbb{R}P^3$ and let $\ell \subset \mathbb{R}P^3$ be a line, diffeomorphic to S^1. This line corresponds to a path of rotations about an axis by $\theta \in [0, \pi]$ radians. Let $\mathcal{P} \subset \mathbb{R}P^3$ be a plane intersecting ℓ in one point. Since this is a transverse intersection in a single point, ℓ cannot be deformed to a point (which would have zero intersection with $\mathcal{P}$. This shows that the path of rotations is not homotopic to a constant path.*

If $\iota : \theta \mapsto \iota(\theta)$ is the embedding of S^1, then traversing the path twice via $\iota' : \theta \mapsto \iota(2\theta)$, we obtain a map ι' which is transverse to $\mathcal{P}$ but with two intersection points. Hence it is possible that ι' may be deformed so as not to intersect $\mathcal{P}$. Can it be done?

Example 3.28. *Consider $\mathbb{R}P^4$ and two transverse hyperplanes P_1, P_2 each an embedded copy of $\mathbb{R}P^3$. These then intersect in $P_1 \cap P_2 = \mathbb{R}P^2$, and since $\mathbb{R}P^2$ is not null-cobordant, we cannot deform the planes to remove all intersection.*

Intersection theory also allows us to define the degree of a map modulo 2. The degree measures how many generic preimages there are of a local diffeomorphism.

Definition 26. Let $f : M \longrightarrow N$ be a smooth map of manifolds of the same dimension, and suppose M is compact and N connected. Let $p \in N$ be any point. Then we define $\deg_2(f) = I_2(f, p)$.

4 Differential forms

Differential forms are an essential tool in differential geometry: first, the k-forms are intended to give a notion of k-dimensional volume (this is why they are multilinear and skew-symmetric, like the determinant) and in a way compatible with the boundary map (this leads to the exterior derivative and Stokes' theorem). Second, they behave well functorially, better than vector fields or other tensors. In a way, you may think of them as generalized functions (in fact, viewing TM as a supermanifold, differential forms *are* its functions).

4.1 Associated vector bundles

Recall that the tangent bundle (TM, π_M, M) is a bundle of vector spaces

$$TM = \sqcup_p T_p M$$

which has local trivializations $\Phi : \pi_M^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ which preserve the projections to U, and which are linear maps for fixed p.

By the smoothness and linearity of these local trivializations, we note that, for charts (U_i, φ_i), we have

$$g_{ij} = (\Phi_j \circ \Phi_i^{-1}) = T(\varphi_j \circ \varphi_i^{-1}) : U_i \cap U_j \longrightarrow GL(\mathbb{R}^k)$$

is a collection of smooth matrix-valued functions, called the "transition functions" of the bundle. These obviously satisfy the gluing conditions or "cocycle condition" $g_{ij} g_{jk} g_{ki} = 1_{k\times k}$ on $U_i \cap U_j \cap U_k$.

We will now use the tangent bundle to create other bundles, and for this we will use functors from the category of vector spaces $\mathbf{Vect}_{\mathbb{R}}$ to itself obtained from linear algebra.

Example 4.1 (Cotangent bundle)**.** *Consider the duality functor $V \mapsto V^* = \mathrm{Hom}(V, \mathbb{R})$ which is contravariant, i.e. if $A : V \longrightarrow W$ then $A^* : W^* \longrightarrow V^*$. Also, it is a smooth functor in the sense that the map $A \mapsto A^*$ is smooth map of vector spaces (in this case it is the identity map, essentially).*

The idea is to apply this functor to the bundle fibrewise, to apply it to the trivializations fiberwise, and to use the smoothness of the functor to obtain the manifold structure on the result.

Therefore we can form the set

$$T^*M = \sqcup_p (T_p M)^*,$$

which also has a projection map p_M to M. And, for each trivialization $\Phi : \pi_M^{-1}(U) \longrightarrow U \times \mathbb{R}^k$, we obtain bijections $F = (\Phi^*)^{-1} : {p_M}^{-1}(U) \longrightarrow U \times \mathbb{R}^k$. We use these bijections as charts for T^*M, and we check the smoothness by computing the transition functions:

$$(P_j \circ P_i^{-1}) = (g_{ij}^*)^{-1}.$$

Therefore we see that the transition functions for T^*M are the inverse duals of the transition functions for TM. Since this is still smooth, we obtain a smooth vector bundle. It is called the **cotangent bundle***.*

Example 4.2. *There is a well-known functor* $\mathbf{Vect}_{\mathbb{R}} \times \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ *given by* $(V, W) \mapsto V \oplus W$*. This is a smooth functor and we may apply it to our vector bundles to obtain new ones, such as* $TM \oplus T^*M$*. The transition functions for this particular example would be*

$$\begin{pmatrix} g_{ij} & 0 \\ 0 & (g_{ij}^*)^{-1} \end{pmatrix}$$

Example 4.3 (Bundle of multivectors and differential forms)**.** *Recall that for any finite-dimensional vector space* V*, we can form the exterior algebra*

$$\wedge^\bullet V = \mathbb{R} \oplus V \oplus \wedge^2 V \oplus \cdots \oplus \wedge^n V,$$

for $n = \dim V$*. The product is usually denoted* $(a, b) \mapsto a \wedge b$*, and it satisfies* $a \wedge b = (-1)^{|a||b|} b \wedge a$*. With this product, the algebra is generated by the degree 1 elements in* V*. So,* $\wedge^\bullet V$ *is a "finite dimensional* $\mathbb{Z}$*-graded algebra generated in degree 1".*

If $(v_1, \ldots v_n)$ *is a basis for* V*, then* $v_{i_1} \wedge \cdots \wedge v_{i_k}$ *for* $i_1 < \cdots < i_k$ *form a basis for* $\wedge^k V$*. This space then has dimension* $\binom{n}{k}$*, hence the algebra* $\wedge^\bullet V$ *has dimension* 2^n*.*

Note in particular that $\wedge^n V$ *has dimension 1, is also called the determinant line* $\det V$*, and a choice of nonzero element in* $\det V$ *is called an "orientation" on the vector space* V*.*

Recall that if $f : V \longrightarrow W$ *is a linear map, then* $\wedge^k f : \wedge^k V \longrightarrow \wedge^k W$ *is defined on monomials via*

$$\wedge^k f(a_1 \wedge \cdots \wedge a_k) = f(a_1) \wedge \cdots \wedge f(a_k).$$

In particular, if $A : V \longrightarrow V$ *is a linear map, then for* $n = \dim V$*, the top exterior power* $\wedge^n A : \wedge^n V \longrightarrow \wedge^n V$ *is a linear map of a 1-dimensional space onto itself, and is hence given by a number, called* $\det A$*, the determinant of* A*.*

We may now apply this functor to the tangent and cotangent bundles: we obtain new bundles $\wedge^\bullet TM$ *and* $\wedge^\bullet T^*M$*, called the* **bundle of multivectors** *and the* **bundle of differential forms***. Each of these bundles is a sum of degree* k *sub-bundles, called the* k*-multivectors and* k*-forms, respectively. We will be concerned primarily with sections of the bundle of* k*-forms, i.e.*

$$\Omega^k(M) = \Gamma(M, \wedge^k T^*M).$$

4.2 Coordinate representations

We are familiar with vector fields, which are sections of TM, and we know that a vector field is written locally in coordinates $(x^1, \ldots, x^n)$ as

$$X = \sum_i v^i \tfrac{\partial}{\partial x^i},$$

with coefficients v^i smooth functions.

There is an easy way to produce examples of 1-forms in $\Omega^1(M)$, using smooth functions f. We note that the action $X \mapsto X(f)$ defines a dual vector at each point of M, since $(X(f))_p$ depends only on the vector X_p and not the behaviour of X away from p. Recall that $X(f) = \pi_2 \circ Tf \circ X$.

Definition 27. The exterior derivative of a function f, denoted df, is the section of T^*M given by the fiber projection $\pi_2 \circ Tf$.

In a coordinate chart, we can apply d to the coordinates x^i; we obtain dx^i, which satisfy $dx^i(\frac{\partial}{\partial x^j}) = \delta^i_j$. Therefore $(dx^1, \ldots, dx^n)$ is the dual basis to $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$. Therefore, a section of T^*M has local expression

$$\xi = \sum_i \xi_i dx^i,$$

for ξ_i smooth functions, given by $\xi_i = \xi(\frac{\partial}{\partial x^i})$. In particular, the exterior derivative of a function df can be written

$$df = \sum_i \tfrac{\partial f}{\partial x^i} dx^i.$$

A general differential form $\rho \in \Omega^k(M)$ can be written

$$\rho = \sum_{i_1 < \cdots < i_k} \rho_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

4.3 Pullback of forms

Given a smooth map $f : M \longrightarrow N$, we obtain bundle maps $f_* : TM \longrightarrow TN$ and hence $f^* := \wedge^k (f_*)^* : \wedge^k T^*N \longrightarrow \wedge^k T^*M$. Hence we have the diagram

$$\begin{array}{ccc} \wedge^k T^*M & \xleftarrow{f^*} & \wedge^k T^*N \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow[f]{} & N \end{array}$$

The interesting thing is that if $\rho \in \Omega^k(N)$ is a differential form on N, then it is a section of π_N. Composing with f, f^*, we obtain a section $f^*\rho := f^* \circ \rho \circ f$ of π_M. Hence we obtain a natural map

$$\Omega^k(N) \xrightarrow{f^*} \Omega^k(M).$$

Such a natural map does not exist (in either direction) for multivector fields, for instance.

Suppose that $\rho \in \Omega^k(N)$ is given in a coordinate chart by $\rho = \sum \rho_{i_1 \cdots i_k} dy^{i_1} \wedge \cdots \wedge dy^{i_k}$. Now choose a coordinate chart for M with coordinates $x^1, \ldots x^m$. What is the local expression for $f^*\rho$? We need only compute $f^* dy_i$. We use a notation where f^k denotes the k^{th} component of f in the coordinates $(y^1, \ldots y^n)$, i.e. $f^k = y^k \circ f$.

$$f^* dy_i(\tfrac{\partial}{\partial x^j}) = dy_i(f_* \tfrac{\partial}{\partial x^j}) \tag{18}$$

$$= dy_i(\sum_k \tfrac{\partial f^k}{\partial x^j} \tfrac{\partial}{\partial y_k}) \tag{19}$$

$$= \tfrac{\partial f^i}{\partial x^j}. \tag{20}$$

Hence we conclude that

$$f^* dy_i = \sum_j \tfrac{\partial f^i}{\partial x^j} dx^j.$$

Finally we compute

$$f^*\rho = \sum_{i_1 < \cdots < i_k} f^* \rho_{i_1 \cdots i_k} f^*(dy^{i_1}) \wedge \cdots \wedge f^*(dy^{i_k}) \tag{21}$$

$$= \sum_{i_1 < \cdots < i_k} (\rho_{i_1 \cdots i_k} \circ f) \sum_{j_1} \cdots \sum_{j_k} \tfrac{\partial f^{i_1}}{\partial x^{j_1}} \cdots \tfrac{\partial f^{i_k}}{\partial x^{j_k}} dx^{j_1} \wedge \cdots dx^{j_k}. \tag{22}$$

4.4 The exterior derivative

Differential forms are equipped with a natural differential operator, which extends the exterior derivative of functions to all forms: $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$. The exterior derivative is uniquely specified by the following requirements: first, it satisfies $d(df) = 0$ for all functions f. Second, it is a graded derivation of the algebra of exterior differential forms of degree 1, i.e.

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta.$$

This allows us to compute its action on any 1-form $d(\xi_i dx^i) = d\xi_i \wedge dx^i$, and hence, in coordinates, we have

$$d(\rho dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_k \tfrac{\partial \rho}{\partial x^k} dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Extending by linearity, this gives a local definition of d on all forms. Does it actually satisfy the requirements? this is a simple calculation: let $\tau_p = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ and $\tau_q = dx^{j_1} \wedge \cdots \wedge dx^{j_q}$. Then

$$d((f\tau_p) \wedge (g\tau_q)) = d(fg\tau_p \wedge \tau_q) = (gdf + fdg) \wedge \tau_p \wedge \tau_q = d(f\tau_p) \wedge g\tau_q + (-1)^p f\tau_p \wedge d(g\tau_q),$$

as required.

Therefore we have defined d, and since the definition is coordinate-independent, we can be satisfied that d is well-defined.

Definition 28. d is the unique degree $+1$ graded derivation of $\Omega^\bullet(M)$ such that $df(X) = X(f)$ and $d(df) = 0$ for all functions f.

Example 4.4. *Consider $M = \mathbb{R}^3$. For $f \in \Omega^0(M)$, we have*

$$df = \tfrac{\partial f}{\partial x^1}dx^1 + \tfrac{\partial f}{\partial x^2}dx^2 + \tfrac{\partial f}{\partial x^3}dx^3.$$

Similarly, for $A = A_1dx^1 + A_2dx^2 + A_3dx^3$, we have

$$dA = (\tfrac{\partial A_2}{\partial x^1} - \tfrac{\partial A_1}{\partial x^2})dx^1 \wedge dx^2 + (\tfrac{\partial A_3}{\partial x^1} - \tfrac{\partial A_1}{\partial x^3})dx^1 \wedge dx^3 + (\tfrac{\partial A_3}{\partial x^2} - \tfrac{\partial A_2}{\partial x^3})dx^2 \wedge dx^3$$

Finally, for $B = B_{12}dx^1 \wedge dx^2 + B_{13}dx^1 \wedge dx^3 + B_{23}dx^2 \wedge dx^3$, we have

$$dB = (\tfrac{\partial B_{12}}{\partial x^3} - \tfrac{\partial B_{13}}{\partial x^2} + \tfrac{\partial B_{23}}{\partial x^1})dx^1 \wedge dx^2 \wedge dx^3.$$

Definition 29. The form $\rho \in \Omega^\bullet(M)$ is called *closed* when $d\rho = 0$ and *exact* when $\rho = d\tau$ for some τ.

Example 4.5. *A function $f \in \Omega^0(M)$ is closed if and only if it is constant on each connected component of M: This is because, in coordinates, we have*

$$df = \tfrac{\partial f}{\partial x^1}dx^1 + \cdots + \tfrac{\partial f}{\partial x^n}dx^n,$$

and if this vanishes, then all partial derivatives of f must vanish, and hence f must be constant.

Theorem 4.6. *The exterior derivative of an exact form is zero, i.e. $d \circ d = 0$. Usually written $d^2 = 0$.*

Proof. The graded commutator $[d_1, d_2] = d_1 \circ d_2 - (-1)^{|d_1||d_2|}d_2 \circ d_1$ of derivations of degree $|d_1|, |d_2|$ is always (why?) a derivation of degree $|d_1| + |d_2|$. Hence we see $[d, d] = d \circ d - (-1)^{1 \cdot 1}d \circ d = 2d^2$ is a derivation of degree 2 (and so is d^2). Hence to show it vanishes we must test on functions and exact 1-forms, which locally generate forms since every form is of the form $f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

But $d(df) = 0$ by definition and this certainly implies $d^2(df) = 0$, showing that $d^2 = 0$. □

The fact that $d^2 = 0$ is dual to the fact that $\partial(\partial M) = \emptyset$ for a manifold with boundary M. We will see later that Stokes' theorem explains this duality. Because of the fact $d^2 = 0$, we have a very special algebraic structure: we have a sequence of vector spaces $\Omega^k(M)$, and maps $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ which are such that any successive composition is zero. This means that the image of d is contained in the kernel of the next d in the sequence. This arrangement of vector spaces and operators is called a *cochain complex* of vector spaces [2]. We often simply refer to this as a "complex" and omit the term "cochain". The reason for the "co" is that the differential increases the degree k, which is opposite to the usual boundary map on manifolds, which decreases k. We will see chain complexes when we study homology.

A complex of vector spaces is usually drawn as a linear sequence of symbols and arrows as follows: if $f : U \longrightarrow V$ is a linear map and $g : V \longrightarrow W$ is a linear map such that $g \circ f = 0$, then we write

$$U \xrightarrow{f} V \xrightarrow{g} W$$

In general, this simply means that $\mathrm{im} f \subset \ker g$, and to measure the difference between them we look at the quotient $\ker g/\mathrm{im} f$, which is called the **cohomology** of the complex at the position V (or homology, if d decreases degree). If we are lucky, and the complex has no cohomology at V, meaning that $\ker g$ is precisely equal to $\mathrm{im} f$, then we say that the complex is **exact** at V. If the complex is exact everywhere, we call it an exact sequence (and it has no cohomology!) In our case, we have a longer cochain complex:

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \longrightarrow 0$$

There is a bit of terminology to learn: we have seen that if $d\rho = 0$ then ρ is called *closed*. But these are also called **cocycles** and denoted $Z^k(M)$. Similarly the exact forms $d\alpha$ are also called **coboundaries**, and are denoted $B^k(M)$. Hence the cohomology groups may be written $H^k_{dR}(M) = Z^k_{dR}(M)/B^k_{dR}(M)$.

Definition 30. The de Rham complex is the complex $(\Omega^\bullet(M), d)$, and its cohomology at $\Omega^k(M)$ is called $H^k_{dR}(M)$, the de Rham cohomology.

[2] since this complex appears for $\Omega^\bullet(U)$ for any open set $U \subset M$, this is actually a cochain complex of *sheaves* of vector spaces, but this won't concern us right away.

Exercise: Check that the graded vector space $H^\bullet_{dR}(M) = \bigoplus_{k\in\mathbb{Z}} H^k(M)$ inherits a product from the wedge product of forms, making it into a $\mathbb{Z}$-graded ring. This is called the de Rham cohomology ring of M, and the product is called the *cup product*.

It is clear from the definition of d that it commutes with pullback via diffeomorphisms, in the sense $f^* \circ d = d \circ f^*$. But this is only a special case of a more fundamental property of d:

Theorem 4.7. *Exterior differentiation commutes with pullback: for $f : M \longrightarrow N$ a smooth map, $f^* \circ d_N = d_M \circ f^*$.*

Proof. We need only check this on functions g and exact 1-forms dg: let X be a vector field on M and $g \in C^\infty(N, \mathbb{R})$.

$$f^*(dg)(X) = dg(f_* X) = \pi_2 g_* f_* X = \pi_2 (g \circ f)_* X = d(f^* g)(X),$$

giving $f^* dg = df^* g$, as required. For exact 1-forms we have $f^* d(dg) = 0$ and $d(f^* dg) = d(df^* g) = 0$ by the result for functions. □

This theorem may be interpreted as follows: The differential forms give us a $\mathbb{Z}$-graded ring, $\Omega^\bullet(M)$, which is equipped with a differential $d : \Omega^k \longrightarrow \Omega^{k+1}$. This sequence of vector spaces and maps which compose to zero is called a *cochain complex*. Beyond it being a cochain complex, it is equipped with a wedge product.

Cochain complexes $(C^\bullet, d_C)$ may be considered as objects of a new category, whose morphisms consist of a sum of linear maps $\psi_k : C^k \longrightarrow D^k$ commuting with the differentials, i.e. $d_D \circ \psi_k = \psi_{k+1} \circ d_C$. The previous theorem shows that pullback f^* defines a morphism of cochain complexes $\Omega^\bullet(N) \longrightarrow \Omega^\bullet(M)$; indeed it even preserves the wedge product, hence it is a morphism of differential graded algebras.

Corollary 4.8. *We may interpret the previous result as showing that $\Omega^\bullet$ is a functor from manifolds to differential graded algebras (or, if we forget the wedge product, to the category of cochain complexes). As a result, we see that the de Rham cohomology $H^\bullet_{dR}$ may be viewed as a functor, from smooth manifolds to $\mathbb{Z}$-graded commutative rings.*

Example 4.9. *S^1 is connected, and hence $H^0_{dR}(S^1) = \mathbb{R}$. So it remains to compute $H^1_{dR}(S^1)$.*

Let $\frac{\partial}{\partial\theta}$ be the rotational vector field on S^1 of unit Euclidean norm, and let $d\theta$ be its dual 1-form, i.e. $d\theta(\frac{\partial}{\partial\theta}) = 1$. Note that θ is not a well-defined function on S^1, so the notation $d\theta$ may be misleading at first.

Of course, $d(d\theta) = 0$, since $\Omega^2(S^1) = 0$. We might ask, is there a function $f(\theta)$ such that $df = d\theta$? This would mean $\frac{\partial f}{\partial\theta} = 1$, and hence $f = \theta + c_2$. But since f is a function on S^1, we must have $f(\theta + 2\pi) = f(\theta)$, which is a contradiction. Hence $d\theta$ is not exact, and $[d\theta] \neq 0$ in $H^1_{dR}(S^1)$.

Any other 1-form will be closed, and can be represented as $gd\theta$ for $g \in C^\infty(S^1, \mathbb{R})$. Let $\overline{g} = \frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi} g(\theta)d\theta$ be the average value of g, and consider $g_0 = g - \overline{g}$. Then define

$$f(\theta) = \int_{t=0}^{t=\theta} g_0(t)dt.$$

Clearly we have $\frac{\partial f}{\partial\theta} = g_0(\theta)$, and furthermore f is a well-defined function on S^1, since $f(\theta + 2\pi) = f(\theta)$. Hence we have that $g_0 = df$, and hence $g = \overline{g} + df$, showing that $[gd\theta] = \overline{g}[d\theta]$.

Hence $H^1_{dR}(S^1) = \mathbb{R}$, and as a ring, $H^0_{dR} + H^1_{dR}$ is simply $\mathbb{R}[x]/(x^2)$.

Note that technically we have proven that $H^1_{dR}(S^1) \cong \mathbb{R}$, but we will see from the definition of integration later that this isomorphism is canonical.

The de Rham cohomology is an important invariant of a smooth manifold (in fact it doesn't even depend on the smooth structure, only the topological structure). To compute it, there are many tools available. There are three particularly important tools: first, there is Poincaré's lemma, telling us the cohomology of $\mathbb{R}^n$. Second, there is integration, which allows us to prove that certain cohomology classes are non-trivial. Third, there is the Mayer-Vietoris sequence, which allows us to compute the cohomology of a union of open sets, given knowledge about the cohomology of each set in the union.

Lemma 4.10. *Consider the embeddings $J_i : M \longrightarrow M \times [0,1]$ given by $x \mapsto (x, i)$ for $i = 0, 1$. The induced morphisms of de Rham complexes J_0^* and J_1^* are chain homotopic morphisms, meaning that there is a linear map $K : \Omega^k(M \times [0,1]) \longrightarrow \Omega^{k-1}(M)$ such that*

$$J_1^* - J_0^* = dK + Kd$$

This shows that on closed forms, J_i^* may differ, but only by an exact form.

Proof. Let t be the coordinate on $[0,1]$. Define $Kf = 0$ for $f \in \Omega^0(M \times [0,1])$, and $K\alpha = 0$ if $\alpha = f\rho$ for $\rho \in \Omega^k(M)$. But for $\beta = f dt \wedge \rho$ we define

$$K\beta = (\int_0^1 f dt)\rho.$$

Then we verify that

$$dKf + Kdf = 0 + \int_0^1 \tfrac{\partial f}{\partial t} dt = (J_1^* - J_0^*)f,$$

$$dK\alpha + Kd\alpha = 0 + (\int_0^1 \tfrac{\partial f}{\partial t} dt)\rho = (J_1^* - J_0^*)\alpha,$$

$$dK\beta + Kd\beta = (\int_0^1 d_M f dt) \wedge \rho + (\int_0^1 f dt) \wedge d\rho + K(df \wedge dt \wedge \rho - f dt \wedge d\rho) = 0,$$

which agrees with $(J_1^* - J_0^*)\beta = 0 - 0 = 0$. Note that we have used $K(df \wedge dt \wedge \rho) = K(-dt \wedge d_M f \wedge \rho) = -(\int_0^1 d_M f) \wedge \rho$, and the notation $d_M f$ is a time-dependent 1-form whose value at time t is the exterior derivative on M of the function $f(-,t) \in \Omega^0(M)$. □

The previous theorem can be used in a clever way to prove that homotopic maps $M \longrightarrow N$ induce the same map on cohomology:

Theorem 4.11. *Let $f : M \longrightarrow N$ and $g : M \longrightarrow N$ be smooth maps which are (smoothly) homotopic. Then $f^* = g^*$ as maps $H^\bullet(N) \longrightarrow H^\bullet(M)$.*

Proof. Let $H : M \times [0,1] \longrightarrow N$ be a (smooth) homotopy between f, g, and let J_0, J_1 be the embeddings $M \longrightarrow M \times [0,1]$ from the previous result, so that $H \circ J_0 = f$ and $H \circ J_1 = g$. Recall that $J_1^* - J_0^* = dK + Kd$, so we have

$$g^* - f^* = (J_1^* - J_0^*)H^* = (dK + Kd)H^* = dKH^* + KH^*d$$

This shows that f^*, g^* differ, on closed forms, only by exact terms, and hence are equal on cohomology. □

Corollary 4.12. *If M, N are (smoothly) homotopic, then $H^\bullet_{dR}(M) \cong H^\bullet_{dR}(N)$.*

Proof. M, N are homotopic iff we have maps $f : M \longrightarrow N$, $g : N \longrightarrow M$ with $fg \sim 1$ and $gf \sim 1$. This shows that $f^*g^* = 1$ and $g^*f^* = 1$, hence f^*, g^* are inverses of each other on cohomology, and hence isomorphisms. □

Corollary 4.13 (Poincaré lemma)**.** *Since $\mathbb{R}^n$ is homotopic to the 1-point space ($\mathbb{R}^0$), we have*

$$H^k_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} \text{ for } k = 0 \\ 0 \text{ for } k > 0 \end{cases}$$

As a note, we should mention that the homotopy in the previous theorem need not be smooth, since any homotopy may be deformed (using a continuous homotopy) to a smooth homotopy, by smooth approximation. Hence we finally obtain that the de Rham cohomology is a homotopy invariant of smooth manifolds.

4.5 Integration

Since we are accustomed to the idea that a function may be integrated over a subset of $\mathbb{R}^n$, we might think that if we have a function on a manifold, we can compute its local integrals and take a sum. This, however, makes no sense, because the answer will depend on the particular coordinate system you choose in each open set: for example, if $f : U \longrightarrow \mathbb{R}$ is a smooth function on $U \subset \mathbb{R}^n$ and $\varphi : V \longrightarrow U$ is a diffeomorphism onto $V \subset \mathbb{R}^n$, then we have the usual change of variables formula for the (Lebesgue or Riemann) integral:

$$\int_U f dx^1 dx^2 \cdots dx^n = \int_V \varphi^* f \left| \det[\tfrac{\partial \varphi_i}{\partial x^j}] \right| dx^1 \cdots dx^n.$$

The extra factor of the absolute value of the Jacobian determinant shows that the integral of f is coordinate-dependant. For this reason, it makes more sense to view the left hand side not as the integral of f but rather as the integral of $\nu = f\,dx^1 \wedge \cdots \wedge dx^n$. Then, the right hand side is indeed the integral of $\varphi^*\nu$ (which includes the Jacobian determinant in its expression automatically) , as long as φ^* has positive Jacobian determinant.

Therefore, the integral of a differential n-form will be well-defined on an n-manifold M, as long as we can choose an atlas where the Jacobian determinants of the gluing maps are all positive: This is precisely the choice of an *orientation* on M, as we now show.

Definition 31. A n-manifold M is called *orientable* when $\det T^*M := \wedge^n T^*M$ is isomorphic to the trivial line bundle. An orientation is the choice of an equivalence class of nonvanishing sections v, where $v \sim v'$ iff $fv = v'$ for $f \in C^\infty(M, \mathbb{R})$. M is called *oriented* when an orientation is chosen, and if M is connected and orientable, there are two possible orientations.

$\mathbb{R}^n$ has a natural orientation by $dx^1 \wedge \cdots \wedge dx^n$; if M is orientable, we may choose charts which preserve orientation, as we now show.

Proposition 4.14. *If the n-manifold M is oriented by* $[v]$*, it is possible to choose an orientation-preserving atlas* (U_i, φ_i) *in the sense that* $\varphi_i^* dx^1 \wedge \cdots \wedge dx^n \sim v$ *for all* i*. In particular, the Jacobian determinants for this atlas are all positive.*

Proof. Choose any atlas (U_i, φ_i). For each i, either $\varphi_i^* dx^1 \wedge \cdots \wedge dx^n \sim v$, and if not, replace φ_i with $q \circ \varphi$, where $q : (x^1, \ldots, x^n) \mapsto (-x^1, \ldots, x^n)$. This completes the proof. □

Now we can define the integral on an oriented n-manifold M, by defining the integral on chart images and asking it to be compatible with these charts:

Theorem 4.15. *Let M be an oriented n-manifold. Then there is a unique linear map* $\int_M : \Omega_c^n(M) \longrightarrow \mathbb{R}$ *on compactly supported n-forms which has the following property: if h is an orientation-preserving diffeomorphism from* $V \subset \mathbb{R}^n$ *to* $U \subset M$*, and if* $\alpha \in \Omega_c^n(M)$ *has support contained in* U*, then*

$$\int_M \alpha = \int_V h^*\alpha.$$

Proof. Let $\alpha \in \Omega_c^n(M)$ and choose an orientation-preserving, locally finite atlas (U_i, φ_i) with subordinate partition of unity (θ_i). Then using the required properties (and noting that α is nonzero in only finitely many U_i), we have

$$\int_M \alpha = \sum_i \int_M \theta_i \alpha = \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* \theta_i \alpha.$$

This proves the uniqueness of the integral. To show existence, we must prove that the above expression actually satisfies the defining condition, and hence can be used as an explicit definition of the integral.

Let $h : V \longrightarrow U$ be an orientation-preserving diffeomorphism from $V \subset \mathbb{R}^n$ to $U \subset M$, and suppose α has support in U. Then $\varphi_i \circ h$ are orientation-preserving, and

$$\begin{aligned}
\int_M \alpha &= \sum_i \int_{\varphi_i(U_i) \cap \varphi_i(U)} (\varphi_i^{-1})^* \theta_i \alpha \\
&= \sum_i \int_{V \cap h^{-1}(U_i)} (\varphi_i \circ h)^* (\varphi_i^{-1})^* \theta_i \alpha \\
&= \sum_i \int_{V \cap h^{-1}(U_i)} h^*(\theta_i \alpha) \\
&= \int_V h^*\alpha,
\end{aligned}$$

as required. □

Having defined the integral, we wish to explain the duality between d and ∂: A $n-1$-form α on a n-manifold may be pulled back to the boundary ∂M and integrated. On the other hand, it can be differentiated and integrated over M. The fact that these are equal is Stokes' theorem, and is a generalization of the fundamental theorem of calculus.

First we must some simple observations concerning the behaviour of forms in a neighbourhood of the boundary.

Recall the operation of contraction with a vector field X, which maps $\rho \in \Omega^k(M)$ to $i_X\rho \in \Omega^{k-1}(M)$, defined by the condition of being a graded derivation $i_X(\alpha\wedge\beta) = i_X\alpha\wedge\beta + (-1)^{|\alpha|}\alpha\wedge i_X\beta$ such that $i_X f = 0$ and $i_X df = X(f)$ for all $f \in C^\infty(M,\mathbb{R})$.

Proposition 4.16. *Let M be a manifold with boundary. If M is orientable, then so is ∂M. Furthermore, an orientation on M induces one on ∂M.*

Proof. We need a vector field X which is tangent to M, pointing outward everywhere along the boundary (and nonvanishing on the boundary). This can be constructed by taking a locally finite covering $\{U_i\}$ of the boundary of M by charts of M, choosing vector fields X_i in U_i corresponding to the outward pointing vector field $-\frac{\partial}{\partial x^m}$ in the coordinates, and forming $X = \sum_i \theta_i X_i$ for a partition of unity $\{\theta_i\}$.

Given an orientation $[v]$ of M, we can form $[j^*(i_X v)]$, for $j : \partial M \longrightarrow M$ the inclusion of the boundary. This is then an orientation of ∂M. This depends only on $[v]$ and X being a nonvanishing outward vector field. □

We now verify a local computation leading to Stokes' theorem. If

$$\alpha = \sum_i a_i dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^m$$

is a degree $m-1$ form with compact support in $U \subset H^m$, and if U does not intersect the boundary ∂H^m, then by the fundamental theorem of calculus,

$$\int_U d\alpha = \sum_i (-1)^{i-1} \int_U \tfrac{\partial a_i}{\partial x^i} dx^1 \cdots dx^m = 0.$$

Now suppose that $V = U \cap \partial H^m \neq \emptyset$. Then

$$\begin{aligned}
\int_U d\alpha &= \sum_i (-1)^{i-1} \int_U \tfrac{\partial a_i}{\partial x^i} dx^1 \cdots dx^m \\
&= -(-1)^{m-1} \int_V a_m(x_1, \ldots, x_{m-1}, 0) dx^1 \cdots dx^{m-1} \\
&= \int_V a_m(x_1, \ldots, x_{m-1}, 0) i_{-\frac{\partial}{\partial x^m}} (dx^1 \wedge \cdots dx^m) \\
&= \int_V j^*\alpha,
\end{aligned}$$

where the last integral is with respect to the orientation induced by the outward vector field.

Theorem 4.17 (Stokes' theorem)**.** *Let M be an oriented manifold with boundary, and let the boundary be oriented with respect to an outward pointing vector field. Then for $\alpha \in \Omega_c^{m-1}(M)$ and $j : \partial M \longrightarrow M$ the inclusion of the boundary, we have*

$$\int_M d\alpha = \int_{\partial M} j^*\alpha.$$

Proof. For a locally finite atlas (U_i, φ_i), we have

$$\int_M d\alpha = \int_M d(\sum_i \theta_i\alpha) = \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* d(\theta_i\alpha)$$

By the local calculation above, if $\varphi_i(U_i) \cap \partial H^m = \emptyset$, the summand on the right hand side vanishes. On the other hand, if $\varphi_i(U_i) \cap \partial H^m \neq \emptyset$, we obtain (letting $\psi_i = \varphi_i|_{U_i\cap\partial M}$ and $j' : \partial H^m \longrightarrow \mathbb{R}^n$), using the local result,

$$\begin{aligned}
\int_{\varphi_i(U_i)} (\varphi_i^{-1})^* d(\theta_i\alpha) &= \int_{\varphi_i(U_i)\cap\partial H^m} j'^*(\varphi_i^{-1})^*(\theta_i\alpha) \\
&= \int_{\varphi_i(U_i)\cap\partial H^m} (\psi_i^{-1})^*(j^*(\theta_i\alpha)).
\end{aligned}$$

This then shows that $\int_M d\alpha = \int_{\partial M} j^*\alpha$, as desired. □

Corollary 4.18. *If $\partial M = \emptyset$, then for all $\alpha \in \Omega_c^{n-1}(M)$, we have $\int_M d\alpha = 0$.*

Corollary 4.19. *Let M be orientable and compact, and let $v \in \Omega^n(M)$ be nonvanishing. Then $\int_M v > 0$, when M is oriented by $[v]$. Hence, v cannot be exact, by the previous corollary. This tells us that the class $[v] \in H^n_{dR}(M)$ cannot be zero. In this way, integration of a closed form may often be used to show that it is nontrivial in de Rham cohomology.*

4.6 The Mayer-Vietoris sequence

Decompose a manifold M into a union of open sets $M = U \cup V$. We wish to relate the de Rham cohomology of M to that of U and V separately, and also that of $U \cap V$. These 4 manifolds are related by obvious inclusion maps as follows:

$$U \cup V \longleftarrow U \sqcup V \underset{\partial_V}{\overset{\partial_U}{\leftleftarrows}} U \cap V$$

Applying the functor $\Omega^\bullet$, we obtain morphisms of complexes in the other direction, given by simple restriction (pullback under inclusion):

$$\Omega^\bullet(U \cup V) \longrightarrow \Omega^\bullet(U) \oplus \Omega^\bullet(V) \underset{\partial_U^*}{\overset{\partial_V^*}{\rightrightarrows}} \Omega^\bullet(U \cap V)$$

Now we notice the following: if forms $\omega \in \Omega^\bullet(U)$ and $\tau \in \Omega^\bullet(V)$ come from a single global form on $U \cup V$, then they are killed by $\partial_V^* - \partial_U^*$. Hence we obtain a complex of (morphisms of cochain complexes):

$$0 \longrightarrow \Omega^\bullet(U \cup V) \longrightarrow \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{\partial_V^* - \partial_U^*} \Omega^\bullet(U \cap V) \longrightarrow 0 \tag{23}$$

It is clear that this complex is exact at the first position, since a form must vanish if it vanishes on U and V. Similarly, if forms on U, V agree on $U \cap V$, they must glue to a form on $U \cup V$. Hence the complex is exact at the middle position. We now show that the complex is exact at the last position.

Theorem 4.20. *The above complex (of de Rham complexes) is exact. It may be called a "short exact sequence" of cochain complexes.*

Proof. Let $\alpha \in \Omega^q(U \cap V)$. We wish to write α as a difference $\tau - \omega$ with $\tau \in \Omega^q(U)$ and $\omega \in \Omega^q(V)$. Let (ρ_U, ρ_V) be a partition of unity subordinate to (U, V). Then we have $\alpha = \rho_U\alpha - (-\rho_V\alpha)$ in $U \cap V$. Now observe that $\rho_U\alpha$ may be extended by zero in V (call the result τ), while $-\rho_V\alpha$ may be extended by zero in U (call the result ω). Then we have $\alpha = (\partial_V^* - \partial_U^*)(\tau, \omega)$, as required. ☐

It is not surprising that given an exact sequence of morphisms of complexes

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$$

we obtain maps between the cohomology groups of the complexes

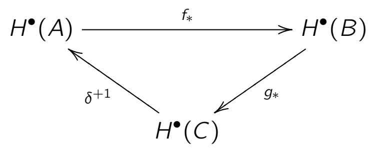
$$H^k(A^\bullet) \xrightarrow{f_*} H^k(B^\bullet) \xrightarrow{g_*} H^k(C^\bullet).$$

And it is not difficult to see that this sequence is exact at the middle term: Let $[\rho] \in H^k(B^\bullet)$, for $\rho \in B^k$ such that $d_B\rho = 0$. Suppose that $g(\rho)$ vanishes in cohomology, meaning $g(\rho) = d_C\gamma$ in C^k. Then by surjectivity of g, there exists $q \in B^{k-1}$ with $g(q) = \gamma$, and then $g(\rho - d_Bq) = 0$, so that by middle exactness, there exists $\tau \in A^k$ with $f(\tau) = \rho - d_Bq$. Then since f is a morphism of complexes, it follows that $f(d_A\tau) = d_Bf(\tau) = d_B\rho = 0$. Since $f : A^{k+1} \longrightarrow B^{k+1}$ is injective, this implies that $d_A\tau = 0$, so we have an actual cohomology class $[\tau]$ such that $f_*[\tau] = [\rho]$, as required.

The interesting thing is that the maps g_* are not necessarily surjective, nor are f_* necessarily injective. In fact, there is a natural map $\delta : H^k(C^\bullet) \longrightarrow H^{k+1}(A^\bullet)$ (called the connecting homomorphism) which extends the 3-term sequence to a full complex involving all cohomology groups of arbitrary degree:

If $[\alpha] \in H^k(C^\bullet)$, where $d_C\alpha = 0$, then there must exist $\xi \in B^k$ with $g(\xi) = \alpha$, and $g(d_B\xi) = d_C(g(\xi)) = d_C\alpha = 0$, so that there must exist $\beta \in A^{k+1}$ with $f(\beta) = d_B\xi$, and $f(d_A\beta) = d_B(f(\beta) = 0$. Hence this determines a class $[\beta] \in H^{k+1}(A^\bullet)$, and one can check that this does not depend on the choices made. We then define $\delta([\alpha]) = [\beta]$.

Exercise: with this definition of δ, we obtain a "long exact sequence" of vector spaces as follows:



Therefore, from the complex of complexes (23), we immediately obtain a long exact sequence of vector spaces, called the Mayer-Vietoris sequence:

$$\cdots \longrightarrow H^k(U \cup V) \longrightarrow H^k(U) \oplus H^k(V) \longrightarrow H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(U \cup V) \longrightarrow \cdots,$$

where the first map is simply a restriction map, the second map is the difference of the restrictions $\delta_V^* - \delta_U^*$, and the third map is the connecting homomorphism δ, which can be written explicitly as follows:

$$\delta[\alpha] = [\beta], \quad \beta = -d(\rho_V \alpha) = d(\rho_U \alpha).$$

(notice that β has support contained in $U \cap V$.)

4.7 Examples of cohomology computations

Example 4.21 (Circle)**.** *Here we present another computation of $H^\bullet_{dR}(S^1)$, by the Mayer-Vietoris sequence. Express $S_1 = U_0 \cup U_1$ as before, with $U_i \cong \mathbb{R}$, so that $H^0(U_i) = \mathbb{R}$, $H^1_{dR}(U_i) = 0$ by the Poincaré lemma. Since $U_0 \cap U_1 \cong \mathbb{R} \sqcup \mathbb{R}$, we have $H^0(U_0 \cap U_1) = \mathbb{R} \oplus \mathbb{R}$ and $H^1(U_0 \cap U_1) = 0$. Since we know that $H^2_{dR}(S^1) = 0$, the Mayer-Vietoris sequence only has 4 a priori nonzero terms:*

$$0 \longrightarrow H^0(S^1) \longrightarrow \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta_1^* - \delta_0^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta} H^1(S^1) \longrightarrow 0.$$

The middle map takes $(c_1, c_0) \mapsto c_1 - c_0$ and hence has 1-dimensional kernel. Hence $H^0(S^1) = \mathbb{R}$. Furthermore the kernel of δ must only be 1-dimensional, hence $H^1(S^1) = \mathbb{R}$ as well. Exercise: Using a partition of unity, determine an explicit representative for the class in $H^1_{dR}(S^1)$, starting with the function on $U_0 \cap U_1$ which takes values 0,1 on each respective connected component.

Example 4.22 (Spheres)**.** *To determine the cohomology of S^2, decompose into the usual coordinate charts U_0, U_1, so that $U_i \cong \mathbb{R}^2$, while $U_0 \cap U_1 \sim S^1$. The first line of the Mayer-Vietoris sequence is*

$$0 \longrightarrow H^0(S^2) \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R}.$$

The third map is nontrivial, since it is just the subtraction. Hence this first line must be exact, and $H^0(S^2) = \mathbb{R}$ (not surprising since S^2 is connected). The second line then reads (we can start it with zero since the first line was exact)

$$0 \longrightarrow H^1(S^2) \longrightarrow 0 \longrightarrow H^1(S^1) = \mathbb{R},$$

where the second zero comes from the fact that $H^1(\mathbb{R}^2) = 0$. This then shows us that $H^1(S^2) = 0$. The last term, together with the third line now give

$$0 \longrightarrow H^1(S^1) = \mathbb{R} \longrightarrow H^2(S^2) \longrightarrow 0,$$

showing that $H^2(S^2) = \mathbb{R}$.

Continuing this process, we obtain the de Rham cohomology of all spheres:

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R}, & \text{for } k = 0 \text{ or } n, \\ 0 & \text{otherwise.} \end{cases}$$