Quasiconvex Plane Domains

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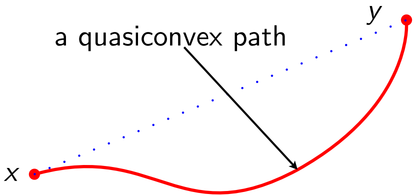
Analysis and Potential Theory on Metric Spaces

UNIVERSITY OF Cincinnati

Outline

Definition of QuasiConvexity

A metric space is quasiconvex iff it is bilipschitz equivalent to some length space; each pair of points can be joined by a rectifiable path whose length is comparable to the distance between its endpoints.



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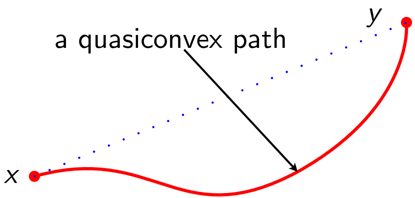
A metric space is c-*quasiconvex* if each pair of points x, y can be joined by a rectfiable path γ satisfying

$$\ell(\gamma) \leq c\,|x - y|\,.$$



Examples of QuasiConvex Spaces

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- doubling metric measure spaces supporting $(1, p)$-Poincaré inequality
- Sobolev extension domains in Euclidean space
- a John disk is a quasidisk if and only if it is quasiconvex

Basic Example

Given $0 < \theta \leq \pi/2$, $C_\theta = \{z \in \mathbb{C} : |\operatorname{Arg}(z)| \leq \theta\}$ is closed convex sector and the concave sector $D_\theta = \mathbb{R}^2 \setminus C_\theta$ is $\csc\theta$-quasiconvex.

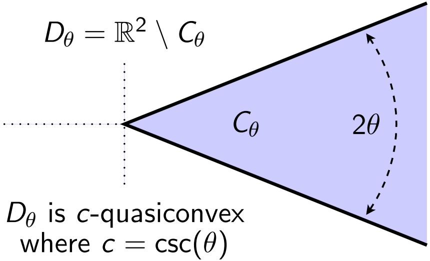

Figure: A concave sector is quasiconvex.

Extremal Examples

$\theta = \pi/n$, $\zeta_k = e^{2ki\theta}$ $(1 \le k \le n)$, $C_k = \zeta_k C_\theta + \zeta_k$ (closed convex sectors obtained by rotating C_θ and then translating) $\implies$

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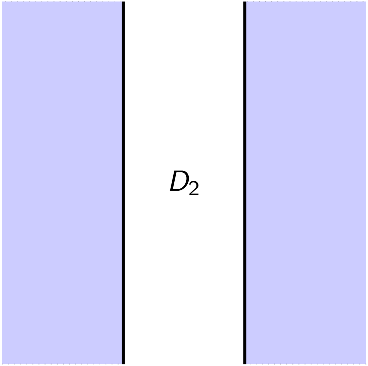
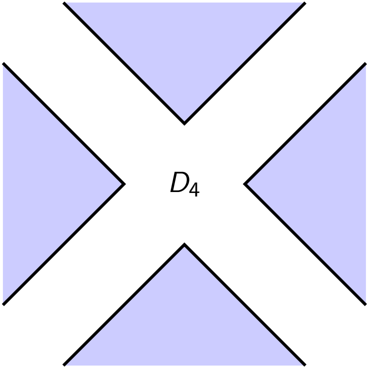


A convex domain



A $\sqrt{2}$-quasiconvex domain

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Fact

Suppose $A \subset \mathbb{R}^n$ is closed and each projection onto a coordinate $(n-1)$-plane has $(n-1)$-measure zero. Then A^c is quasiconvex.

See Proof

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Theorem

Suppose $A \subset \mathbb{R}^n$ is closed and each projection onto a coordinate $(n-1)$-plane is nowhere dense. Then A^c is quasiconvex.

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Thus A^c is quasiconvex if

- $\dim_{\mathcal{H}} A < n-1$, or A itself has $(n-1)$-measure zero, or
- A is n-fold product of a positive measure nowhere dense subset of $\mathbb{R}$.

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Outline

Notation

Call $C \subset \mathbb{R}^2$ a *Jordan curve* if it is a Jordan loop or a Jordan line:

a *Jordan loop* is homeomorphic image of a round circle,

so always compact;

a *Jordan line* is image of injective path $\mathbb{R} \xrightarrow{\lambda} \mathbb{R}^2$ with

$\lambda(t) \to \infty$ (in $\hat{\mathbb{R}}^2$) as $t \to \pm\infty$.

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All topology with respect to $\mathbb{R}^2$.

A *Jordan curve domain* is an open connected plane region each of whose boundary components is either a single point or a Jordan curve.

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- There are simply connected Jordan curve domains having infinitely many boundary components.

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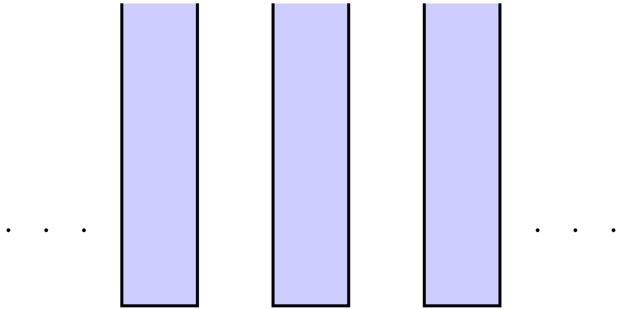

A simply connected Jordan curve domain

Figure: Infinitely many unbounded boundary components

Necessary Conditions for QuasiConvexity

Theorem

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Can weaken hypothesis *finitely many boundary components* if instead require that all boundary points be joinable by quasiconvex paths.

QuasiConvexity Characterization

Corollary

A finitely connected $D \subsetneq \mathbb{R}^2$ is c-quasiconvex iff

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Construct $A := \cap_i E_i$ where $E_1 \supset E_2 \supset \ldots$ closed, $E_i = \cup_j F_{ij}$, with F_{ij} nested compact sets satisfying

$$\lim_{i \to \infty} \sup_j \operatorname{diam} F_{ij} = 0 \,.$$

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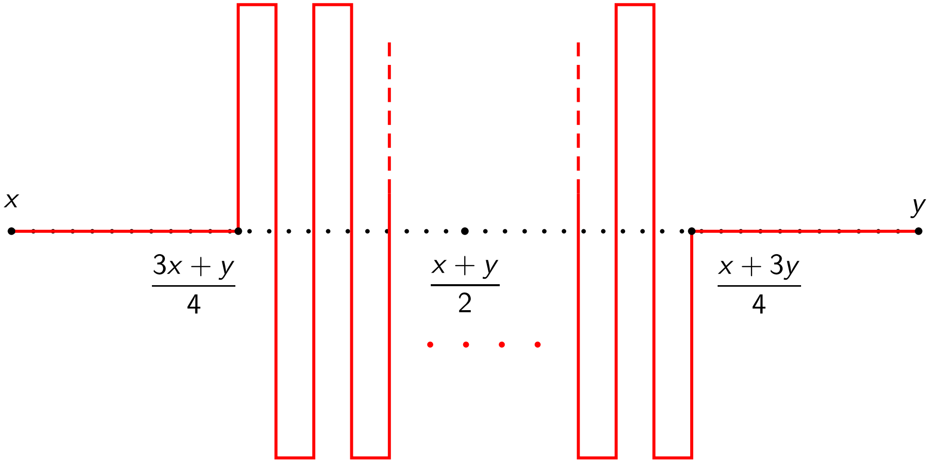
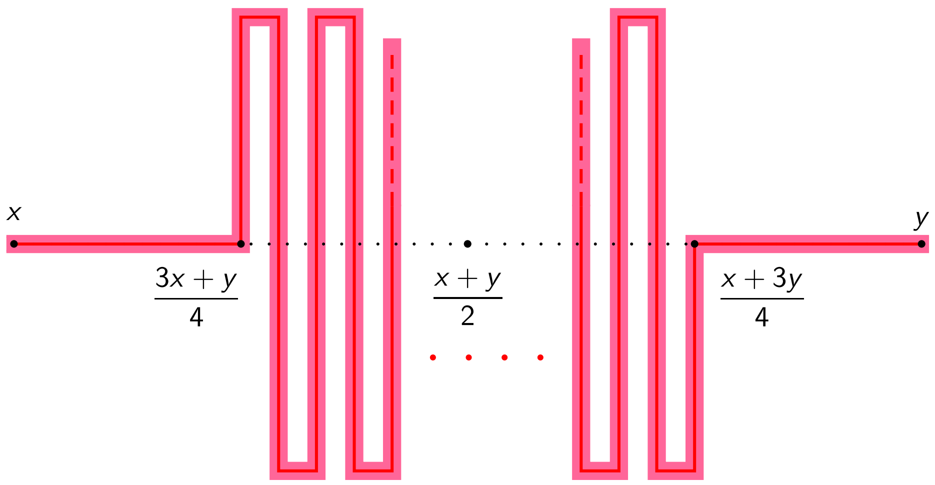
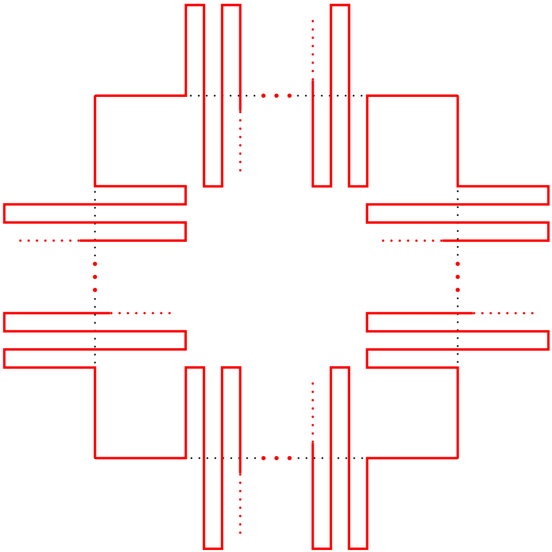
$[x, y] \rightsquigarrow$ Long Broken Line Segment



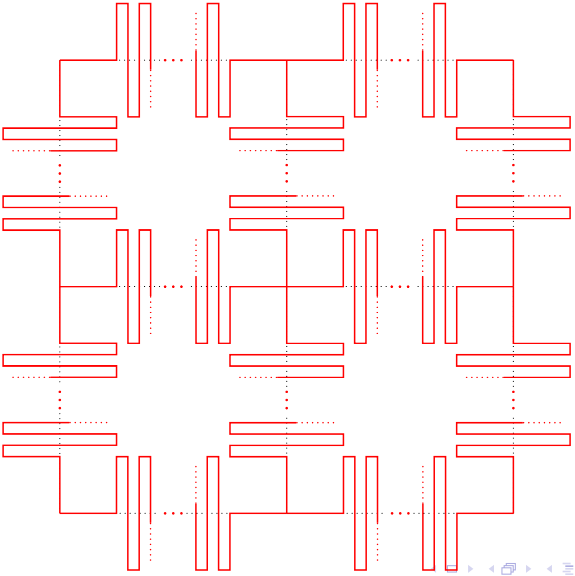
An Irrigation Canal Based on $[x, y]$



Irrigating a Square



Irrigating a Bunch of Squares



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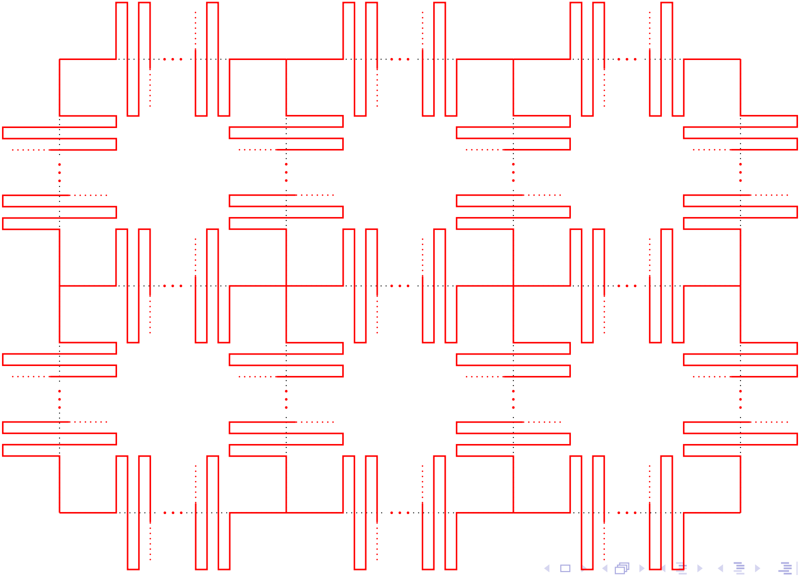

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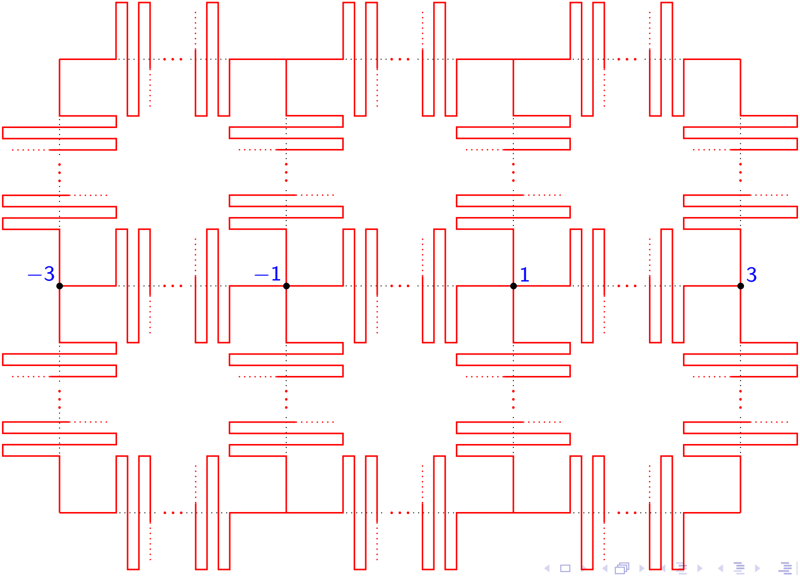

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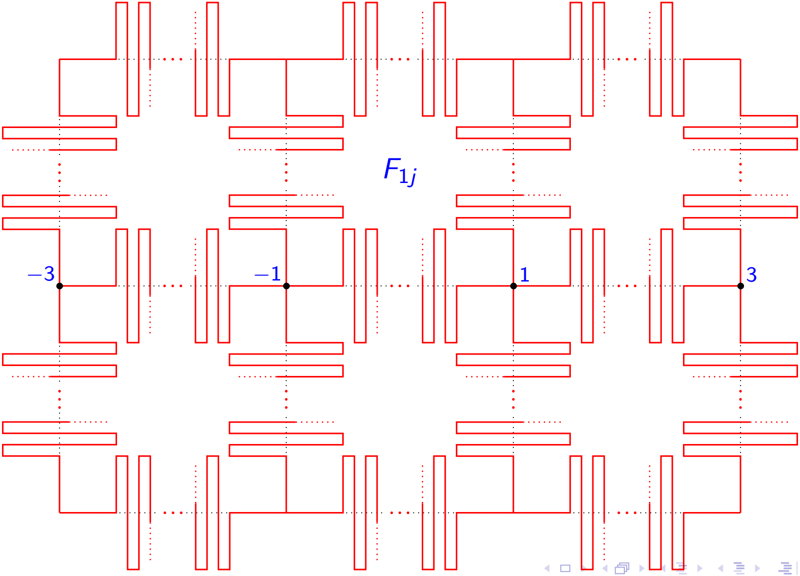
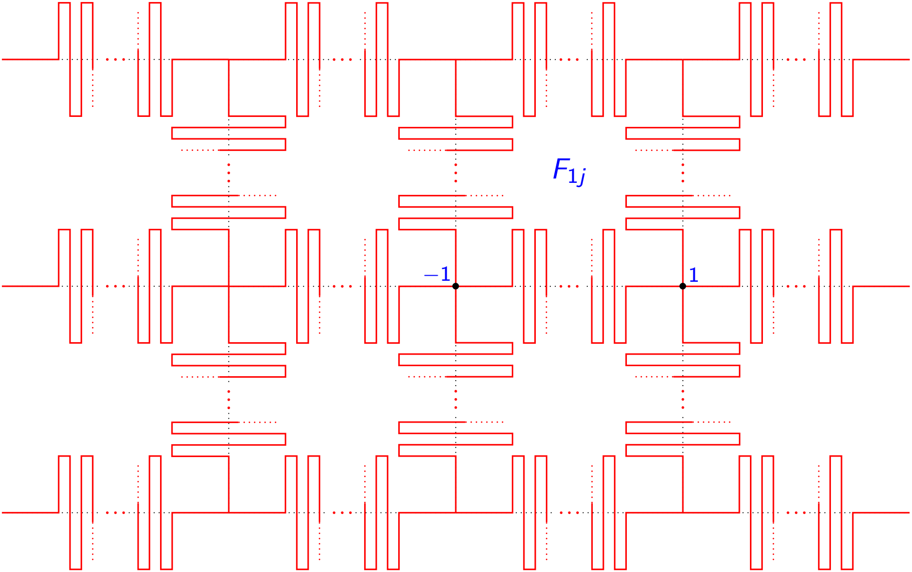
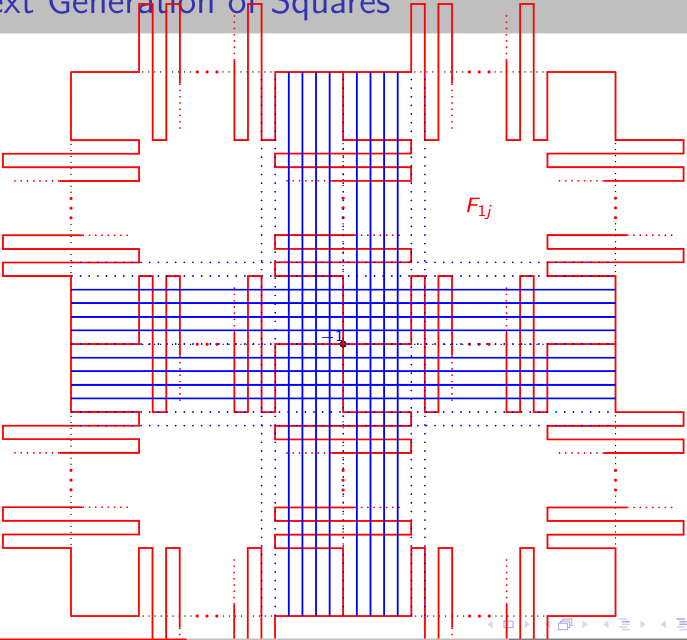


Taking a Closer Look



The Next Generation of Squares



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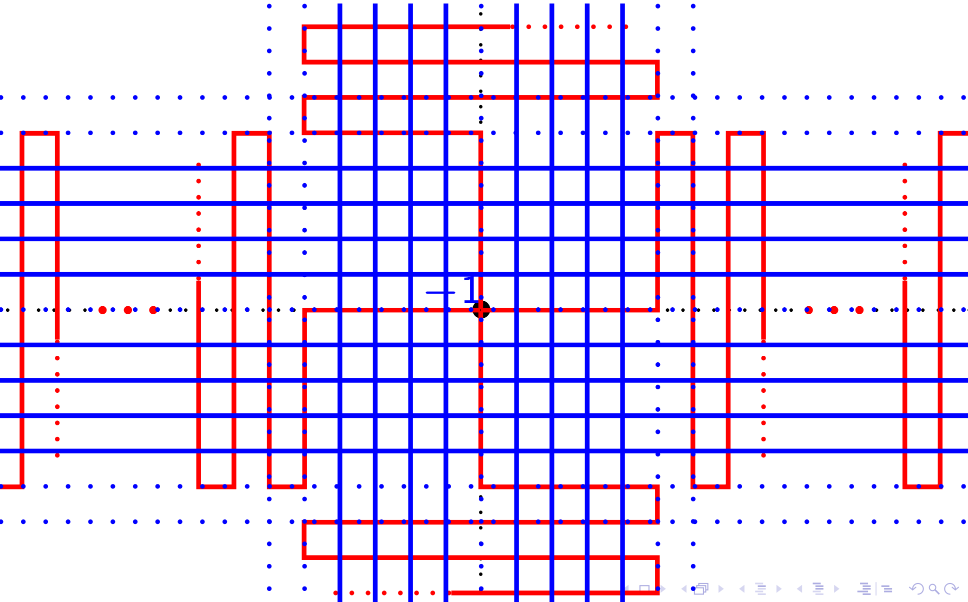

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- Although OK in many cases (sets with small dimension or nowhere dense projections).
- Are there similar examples in $\mathbb{R}^3$? In $\mathbb{R}^n$? (I guess so)
- Is there an example with Hausdorff dimension stricly less than n?

The End

$$\Big(\ddot{\smile} \Big)$$

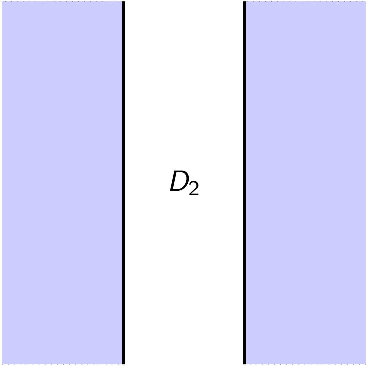
===

4 Appendix
- Extremal Examples
- Proof of Theorem

Outline

Complements of Sectors

$\theta = \pi/n$, $\zeta_k = e^{2ki\theta}$ $(1 \le k \le n)$, $C_k = \zeta_k C_\theta + \zeta_k$ (closed convex sectors obtained by rotating C_θ and then translating) $\implies$ $D_n = \mathbb{R}^2 \setminus \cup_{k=1}^n C_k$ is simply conn $\csc\theta$-quasicvx domain with n unbdd bdry cmpnts Go Back



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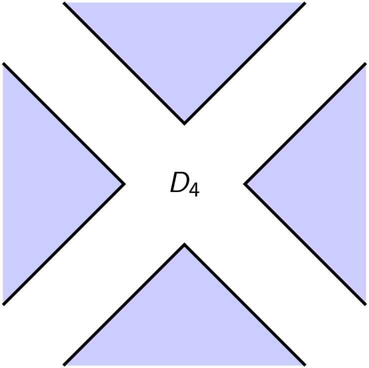

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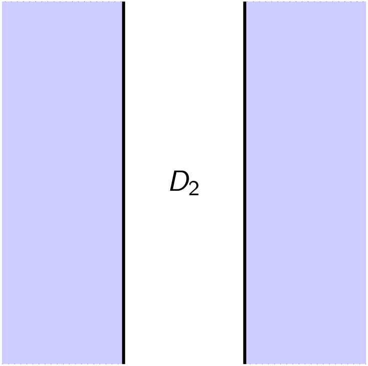
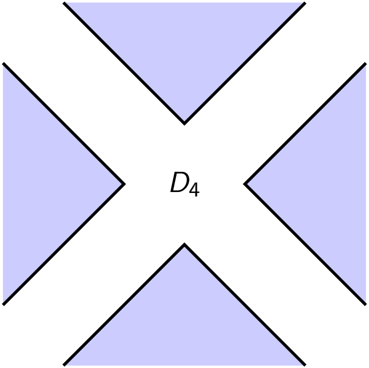


A convex domain



A $\sqrt{2}$-quasiconvex domain

Lots of Unbdd Boundary Components

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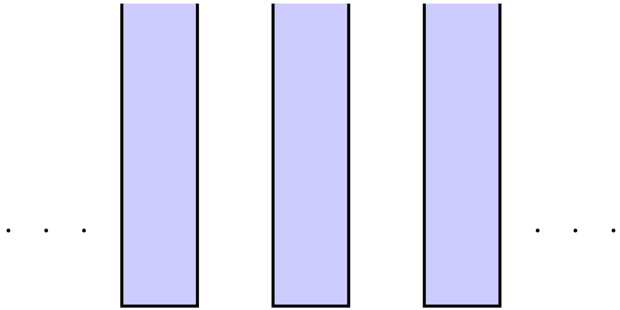
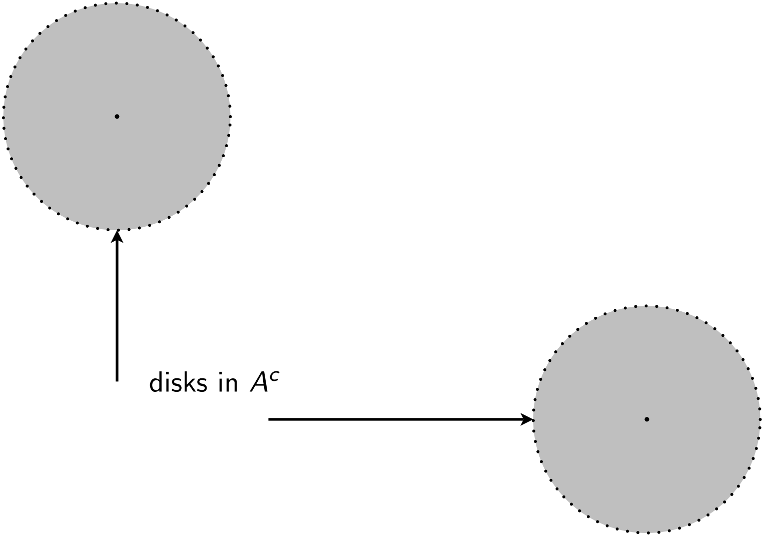


A simply connected Jordan curve domain

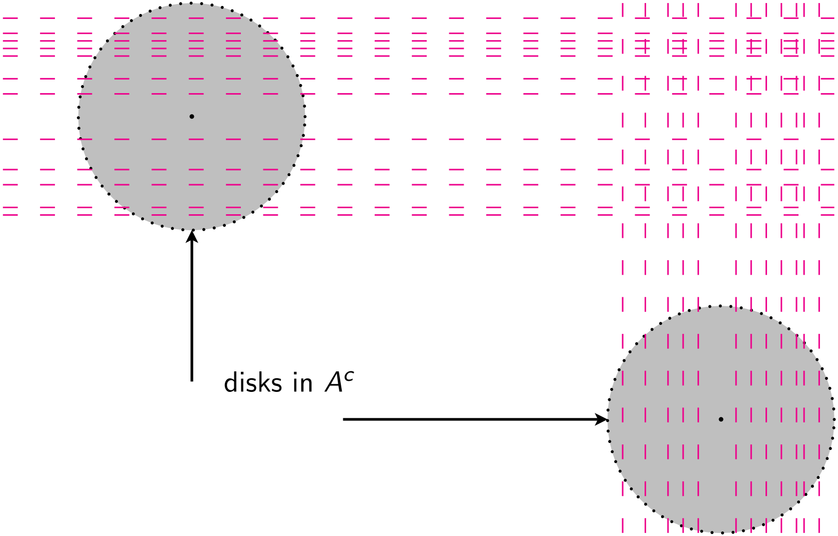
Figure: Infinitely many unbounded boundary components

Complement of Plane Set with Zero Measure Projections

Complement of Plane Set with Zero Measure Projections



Complement of Plane Set with Zero Measure Projections



Complement of Plane Set with Zero Measure Projections

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