Euclidean QuasiConvexity

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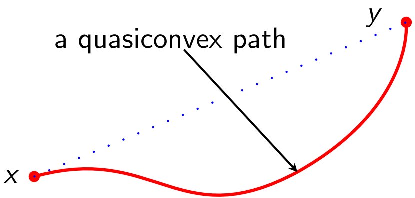
AMS Special Session
Complex Dynamics and Complex Function Theory

Outline

Definition of QuasiConvexity

A metric space is quasiconvex iff it is bilipschitz equivalent to some length space; each pair of points can be joined by a rectifiable path whose length is comparable to the distance between its endpoints.



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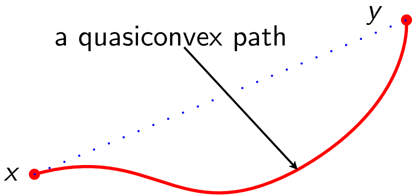
Definition

A metric space is c-*quasiconvex* if each pair of points x, y can be joined by a rectfiable path γ satisfying

$$\ell(\gamma) \leq c\,|x - y|\,.$$



Examples of QuasiConvex Spaces

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- Sobolev extension domains in Euclidean space
- a John disk is a quasidisk if and only if it is quasiconvex

Main Questions & Outline of Talk

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- Next, exhibit sufficient conditions for quasiconvexity of domains in $\mathbb{R}^n$.
- Last, present some especially relevant examples.

Outline

Notation

Call $C \subset \mathbb{R}^2$ a *Jordan curve* if it is a Jordan loop or a Jordan line:

a *Jordan loop* is homeomorphic image of a round circle,

so always compact;

a *Jordan line* is image of injective path $\mathbb{R} \xrightarrow{\lambda} \mathbb{R}^2$ with

$\lambda(t) \to \infty$ (in $\hat{\mathbb{R}}^2$) as $t \to \pm\infty$.

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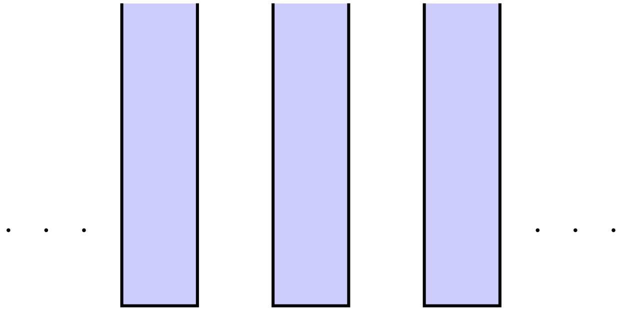
A *Jordan curve domain* is an open connected plane region each of whose boundary components is either a single point or a Jordan curve.

A Simply Connected Jordan Curve Domain



A simply connected Jordan curve domain

Figure: Infinitely many unbounded boundary components

Necessary Conditions for QuasiConvexity

Theorem

Suppose $D \subsetneq \mathbb{R}^2$ is a c-quasiconvex domain. Then:

(i) *D is a Jordan curve domain,*

▸ Proof of (i)

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▸ Proof

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Can weaken hypothesis *finitely many boundary components* if instead require that all boundary points be joinable by quasiconvex paths.

QuasiConvexity Characterization

Corollary

A finitely connected $D \subsetneq \mathbb{R}^2$ is c-quasiconvex iff

(i) *D is a Jordan curve domain, and*

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For necessity, can take any $b > c$; for sufficiency, $c = b$ works.

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Recall that there exist simply connected Jordan curve domains with infinitely many boundary components.

See lotsa pix

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General Sufficient Conditions

Fact

Suppose $A \subset \mathbb{R}^n$ is closed and each projection onto a coordinate $(n-1)$-plane has $(n-1)$-measure zero. Then A^c is quasiconvex.

See Proof

General Sufficient Conditions

Theorem

Suppose $A \subset \mathbb{R}^n$ is closed and each projection onto a coordinate $(n-1)$-plane has $(n-1)$-measure zero or is nowhere dense. Then A^c is quasiconvex.

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Thus A^c is quasiconvex if

- $\dim_{\mathcal{H}} A < n-1$, or $\mathcal{H}^{n-1}(A) = 0$, or
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So, there exists quasiconvex $D \subset \mathbb{R}^n$ with $\mathcal{H}^n(\partial D) > 0$.

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Statements

Theorem

There exists a compact totally disconnected set $A \subset \mathbb{R}^n$ *with Hausdorff dimension* $\dim_{\mathcal{H}} A = n - 1$ *and* A^c *non-quasiconvex.*

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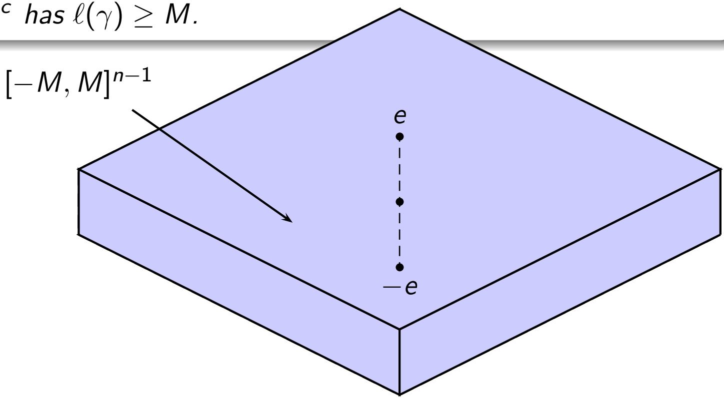
Key Tool

Proposition

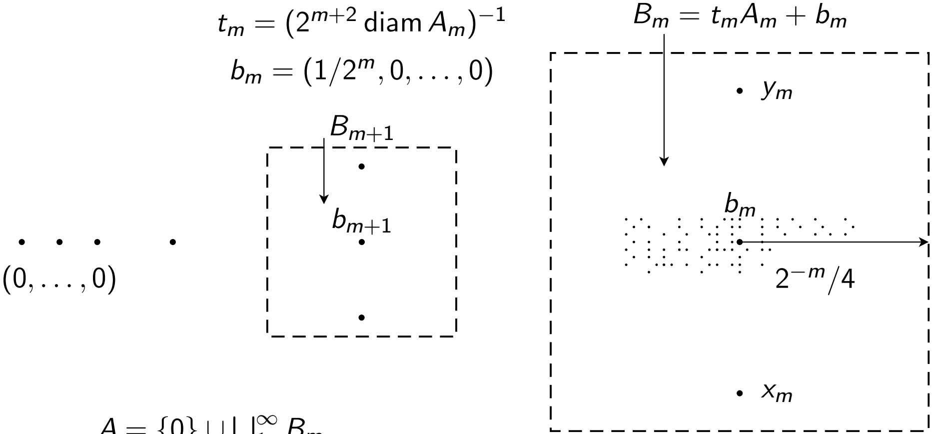
$\forall M > 0, \exists$ compact totally disconn $A \subset [-M, M]^{n-1} \times [-1/2, 1/2]$ with $\pm e \in A^c$, $e = (0, \ldots, 0, 1)$, $\dim_{\mathcal{H}} A \leq n-1$, and st each rectifiable path γ joining $\pm e$ in A^c has $\ell(\gamma) \geq M$.



Proposition $\implies$ Theorem

$$t_m = (2^{m+2}\operatorname{diam} A_m)^{-1}$$

$$b_m = (1/2^m, 0, \ldots, 0)$$



$$A = \{0\} \cup \bigcup_1^\infty B_m$$

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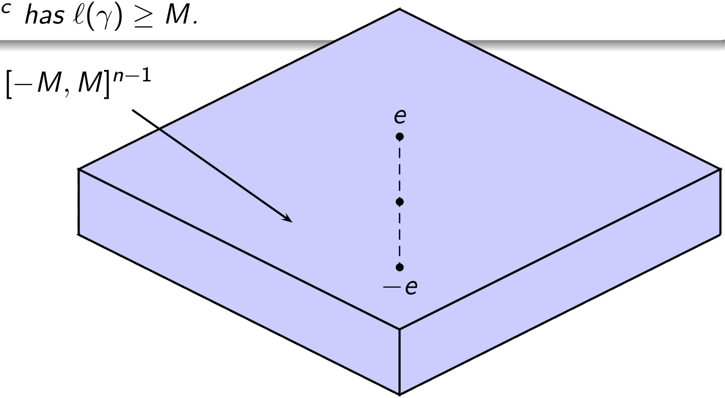

Idea for Proof of Proposition

Use Cantor type construction:
get $A := \cap_i E_i$ where $E_1 \supset E_2 \supset \dots$, $E_i = \cup_j B_{ij}$ compact, with B_{ij} nested closed rectangular boxes satisfying

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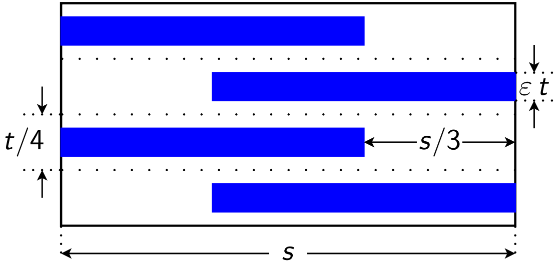
Construction in $\mathbb{R}^2$

Start with thin flat $[0, s] \times [0, t]$ rectangle. Divide into four horizontal corridors ($[0, s] \times [0, t/4]$, etc.). Place $(2s/3) \times (\varepsilon\, t)$ barriers into vertical middles of each of these corridors. Alternate horizontal placement of barriers.



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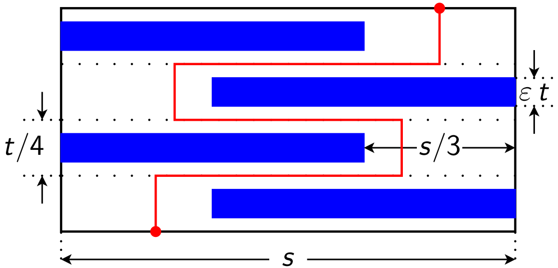
a 'penetrating path' traversing a plane maze



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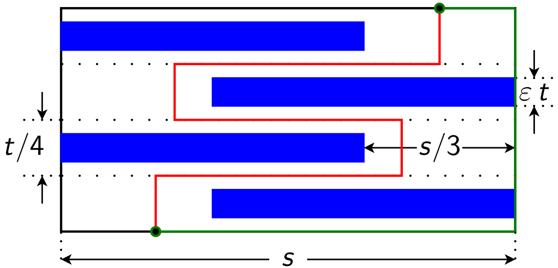
an 'avoiding path' traversing the boundary
a 'penetrating path' traversing a plane maze



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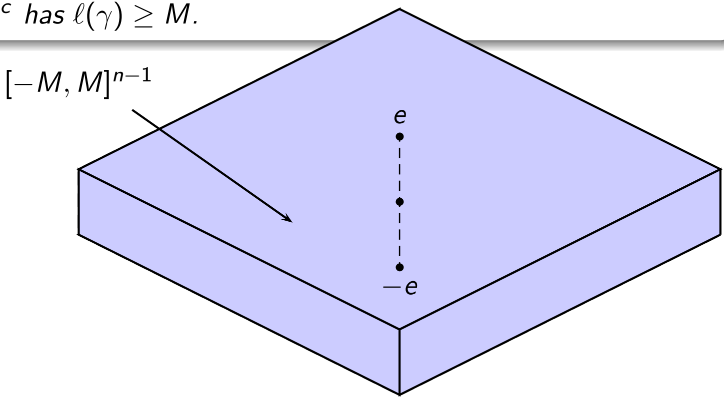
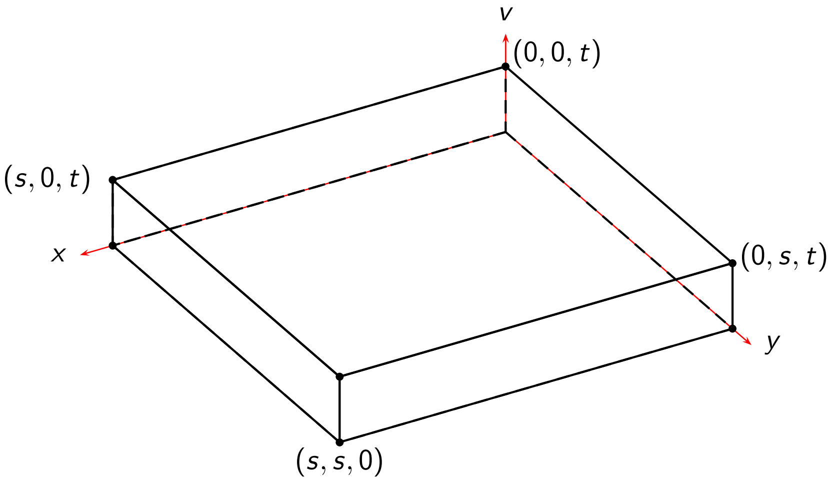
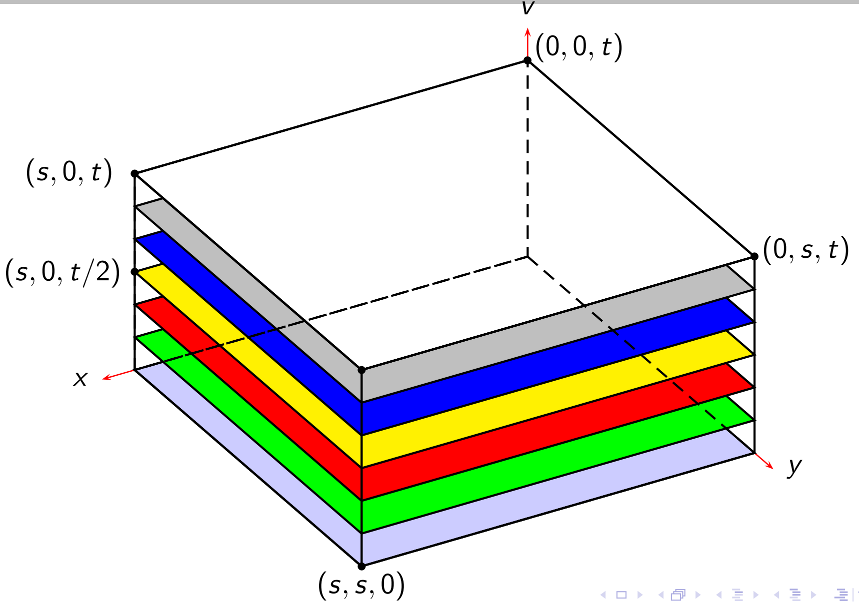


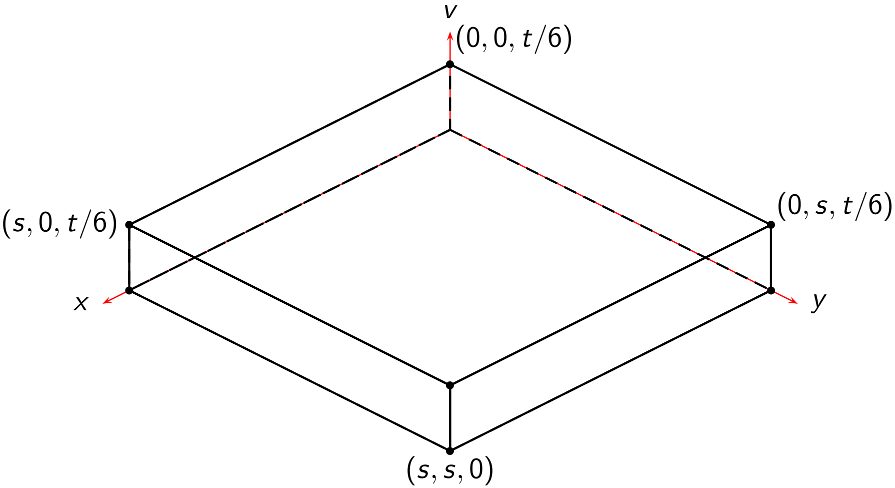
B – a Thin Flat $s \times s \times t$ Rectangular Box in $\mathbb{R}^3$



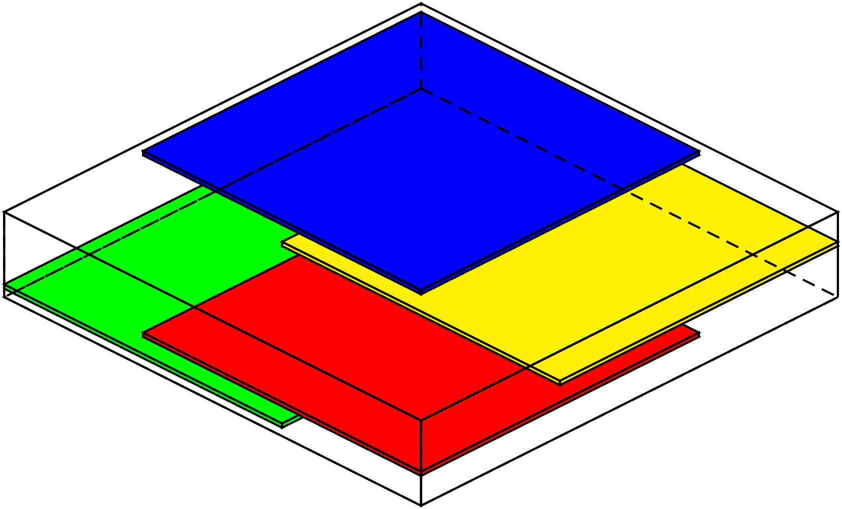
The Rectangular Box B Divided into 6 Layers



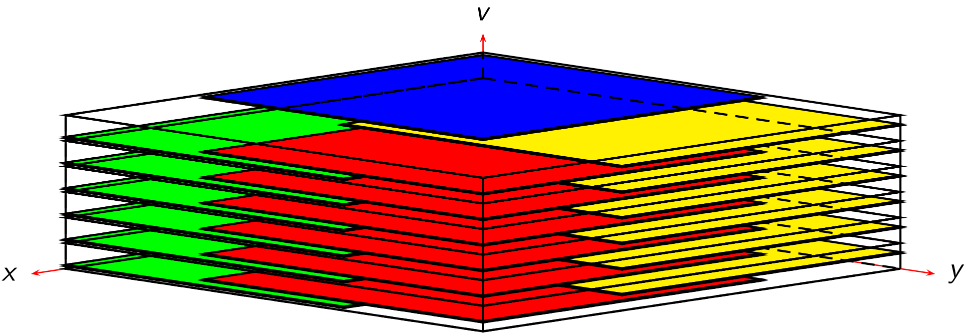
C – a Thin Flat $s \times s \times (t/6)$ Layer of B



A SubMaze in C with $(2s/3) \times (2s/3) \times (\varepsilon\, t)$ Barriers



A Box Maze in B Built with 6 Stacked SubMazes



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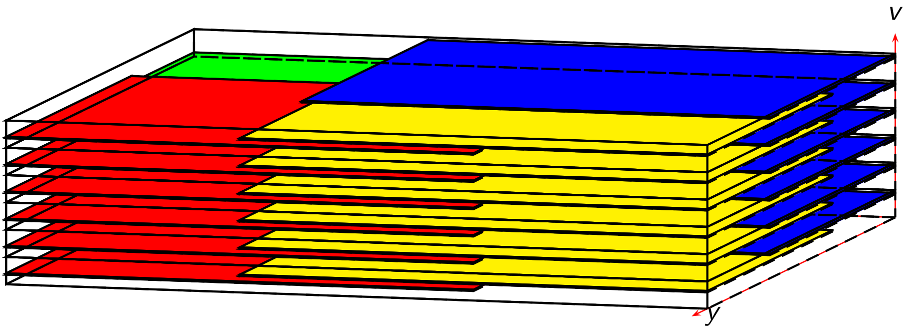

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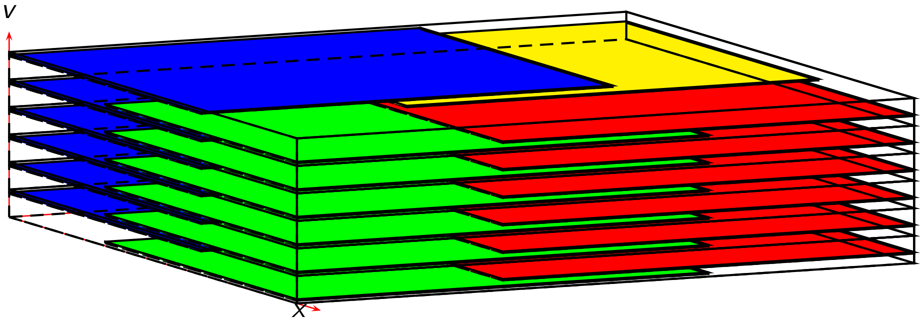

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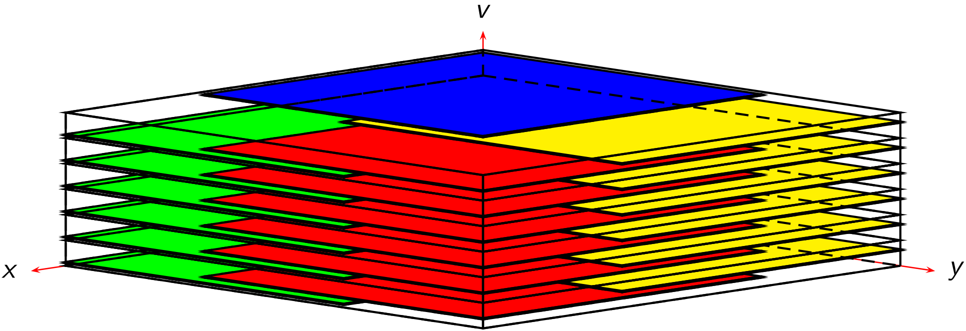
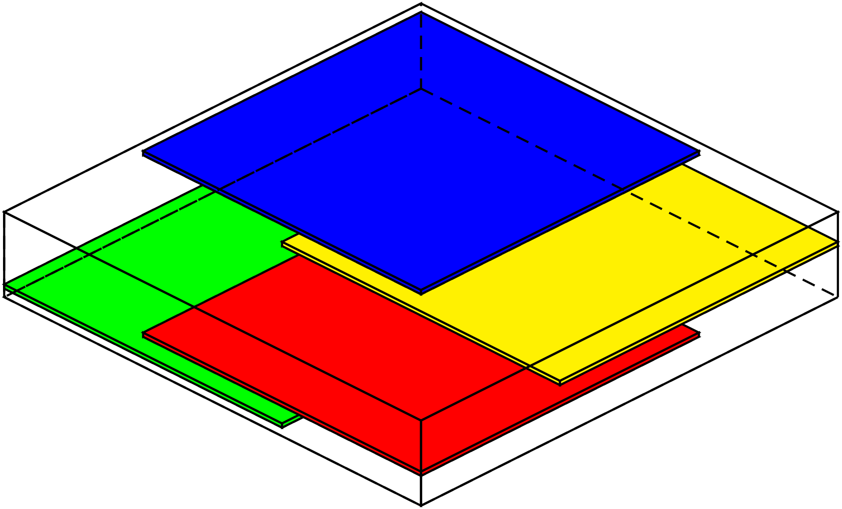


A SubMaze in C with $(2s/3) \times (2s/3) \times (\varepsilon\, t)$ Barriers



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The End

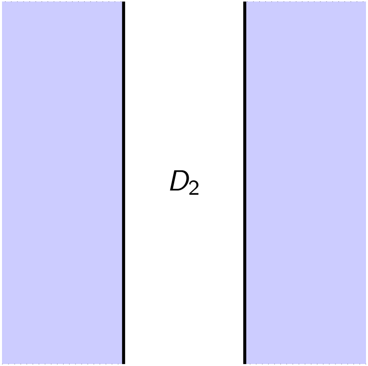
$$\left(\ddot{\smile}\right)$$

5 Appendix
- Extremal Examples
- Proof of Theorem A(i)
- Proof of Theorem A(ii)
- Proof of Theorem B
- Proof of Theorem

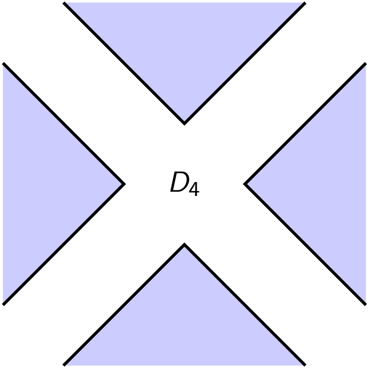
Outline

Complements of Sectors

$\theta = \pi/n$, $\zeta_k = e^{2ki\theta}$ $(1 \le k \le n)$, $C_k = \zeta_k C_\theta + \zeta_k$ (closed convex sectors obtained by rotating C_θ and then translating) $\implies$ $D_n = \mathbb{R}^2 \setminus \cup_{k=1}^n C_k$ is simply conn $\csc\theta$-quasicvx domain with n unbdd bdry cmpnts Go Back



A convex domain



A $\sqrt{2}$-quasiconvex domain

Complements of Sectors

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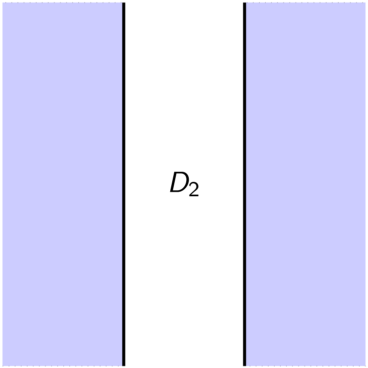

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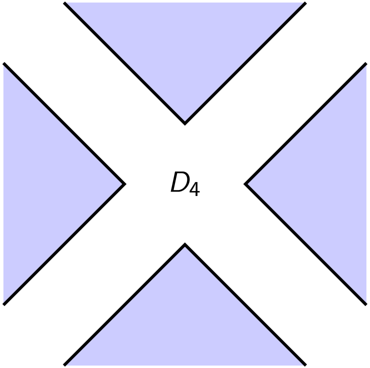

A $\sqrt{2}$-quasiconvex domain

Lots of Unbdd Boundary Components

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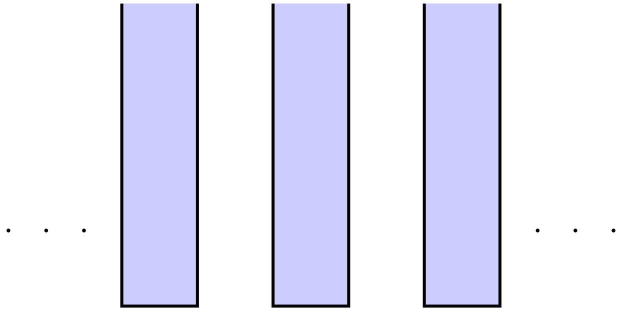
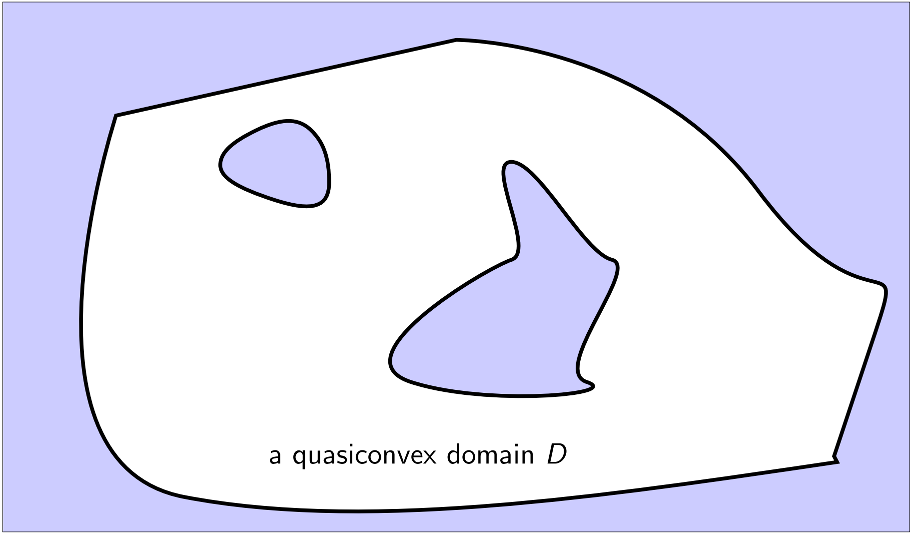


A simply connected Jordan curve domain

Figure: Infinitely many unbounded boundary components

QuasiConvex Plane Domains are Jordan Curve Domains



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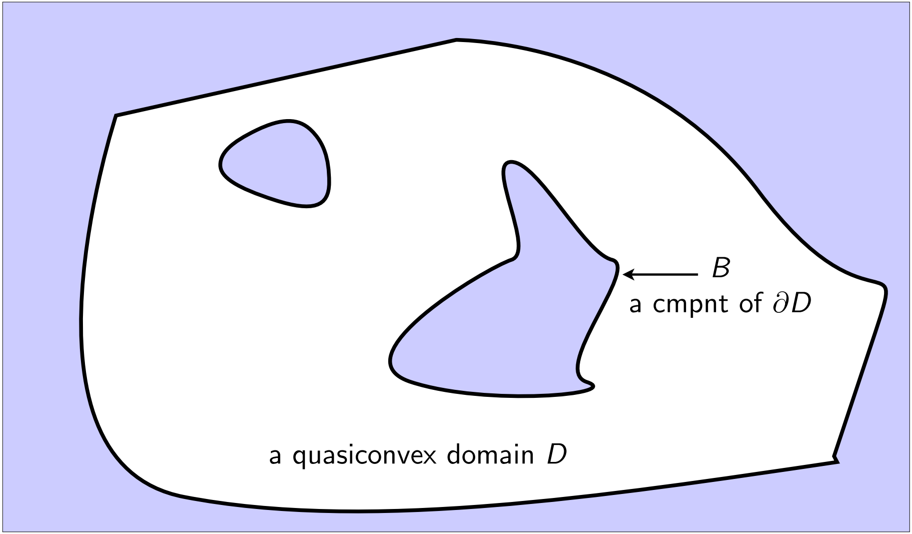

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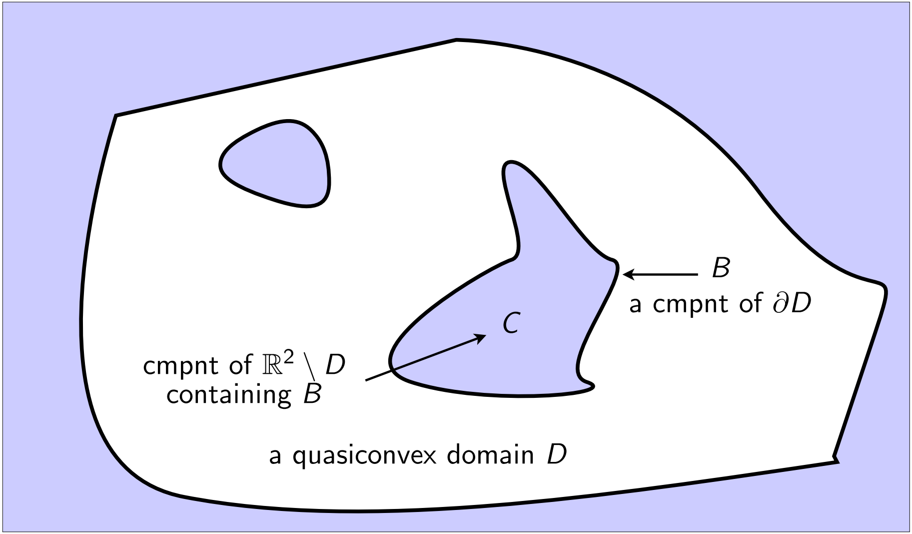
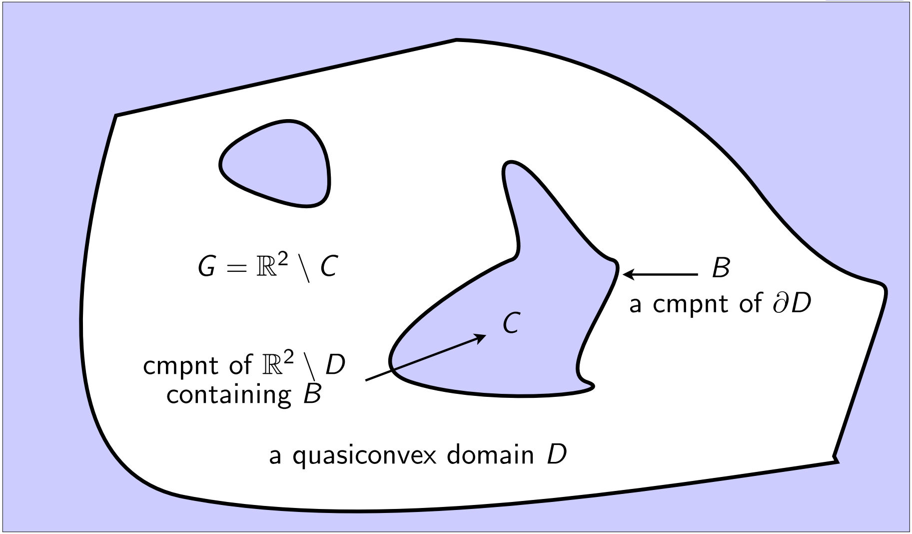


QuasiConvex Plane Domains are Jordan Curve Domains

Go Back



Qcvx Have Finitely Many Unbdd Bdry Components

Basic Example Given $0 < \theta \le \pi/2$, $C_\theta = \{z \in \mathbb{C} : |\operatorname{Arg}(z)| \le \theta\}$ is closed convex sector and the concave sector $D_\theta = \mathbb{R}^2 \setminus C_\theta$ is $\csc\theta$-quasiconvex.

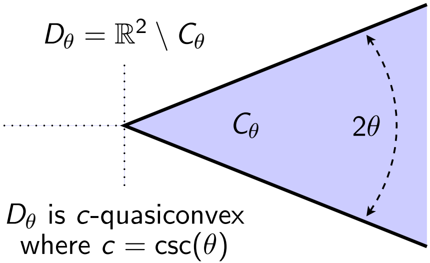


Figure: A concave sector is quasiconvex.

Qcvx Have Finitely Many Unbdd Bdry Components

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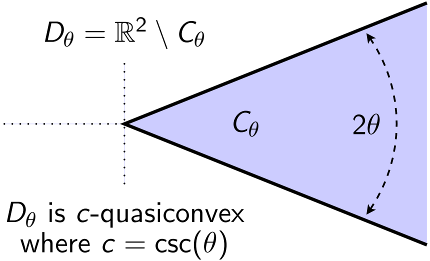

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Qcvx Have Finitely Many Unbdd Bdry Components

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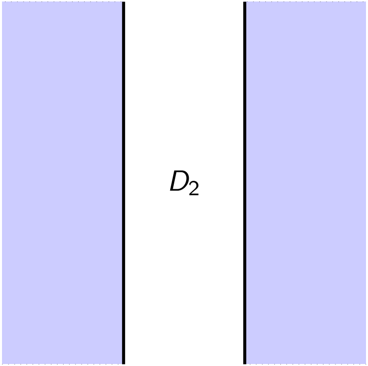
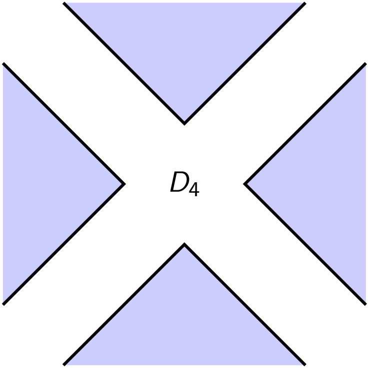
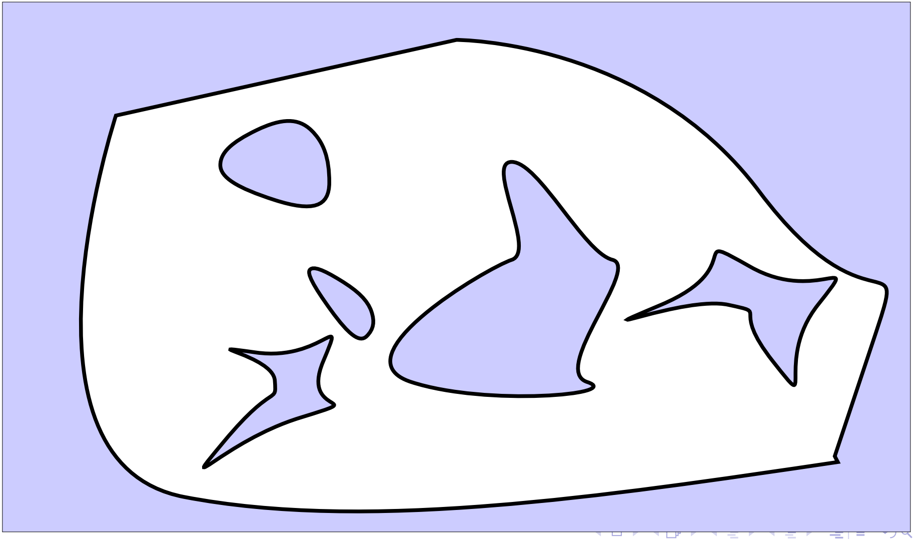


A convex domain

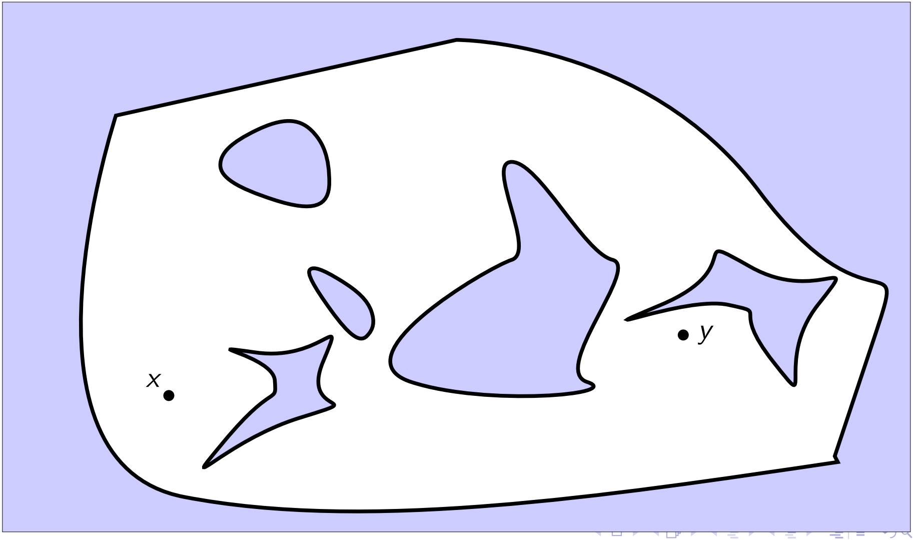


A $\sqrt{2}$-quasiconvex domain

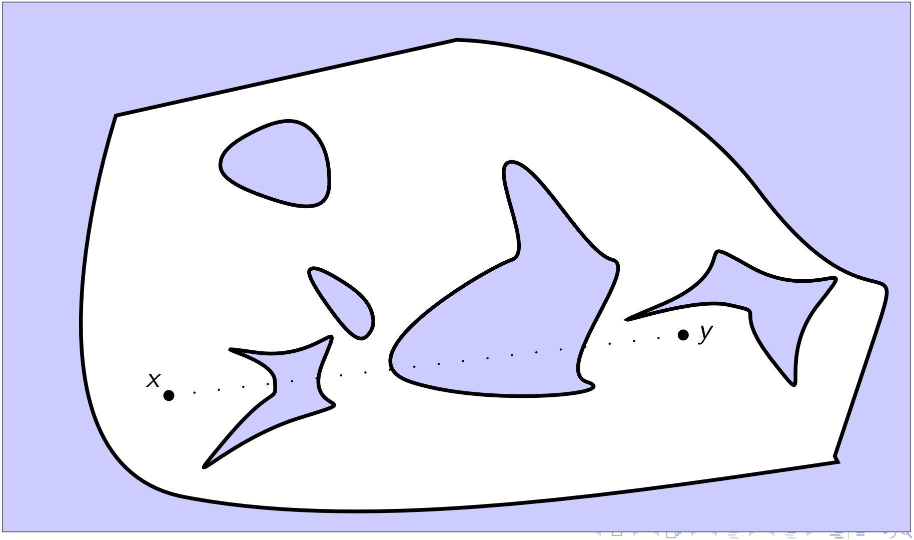
Suff Cond for Finitely Many Bdry Components



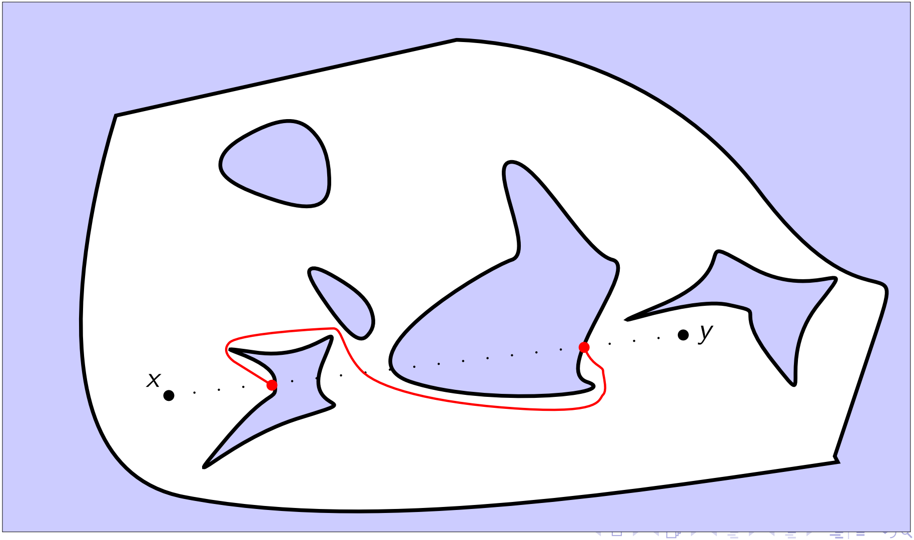
Suff Cond for Finitely Many Bdry Components



Suff Cond for Finitely Many Bdry Components



Suff Cond for Finitely Many Bdry Components



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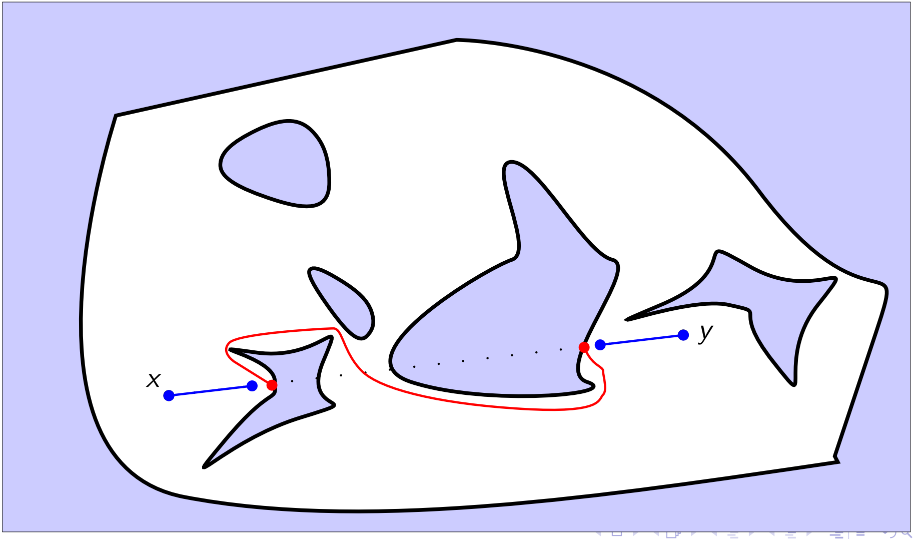

Suff Cond for Finitely Many Bdry Components

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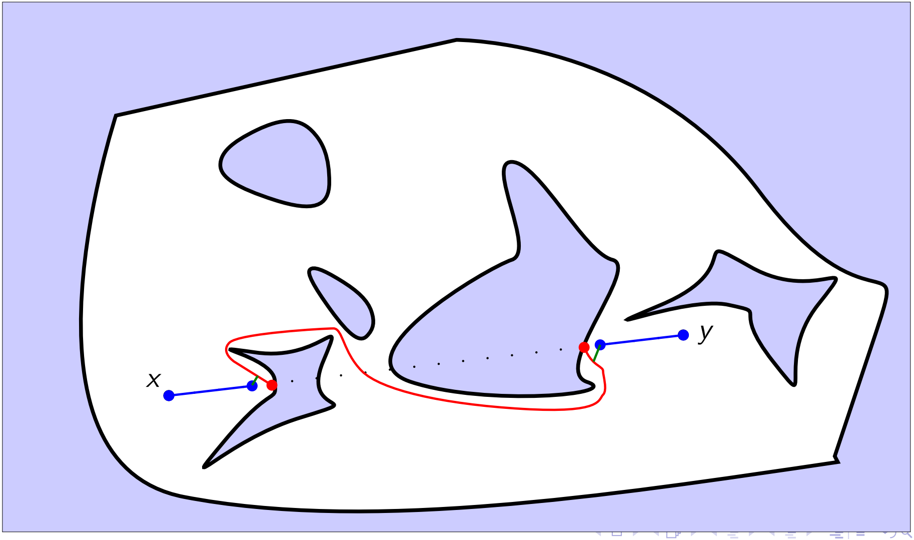
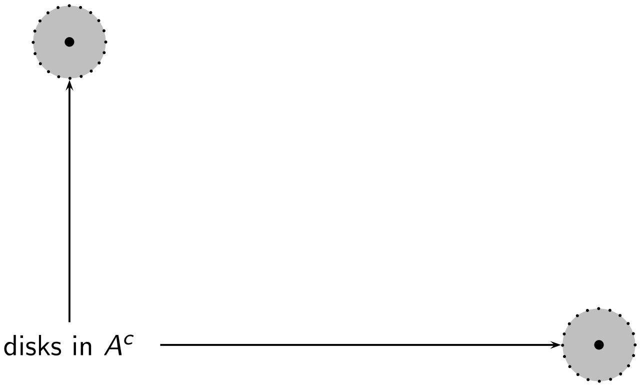


Complement of Plane Set with Zero Measure Projections



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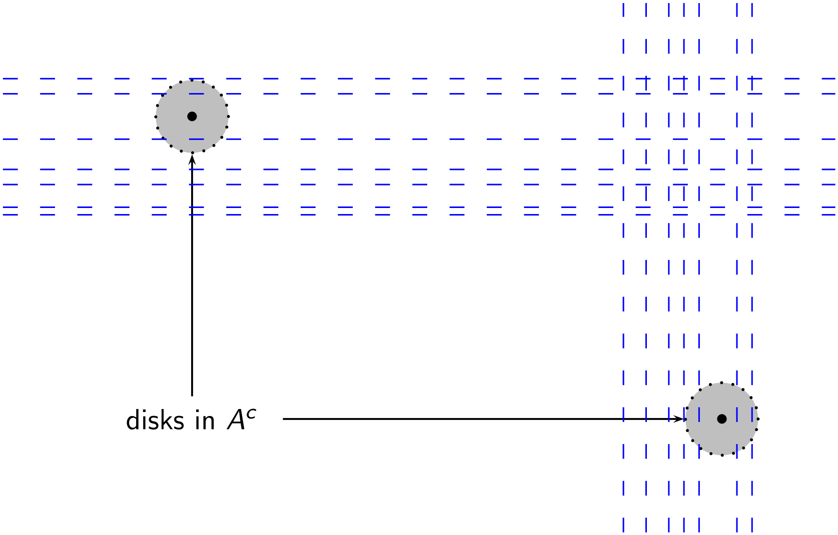

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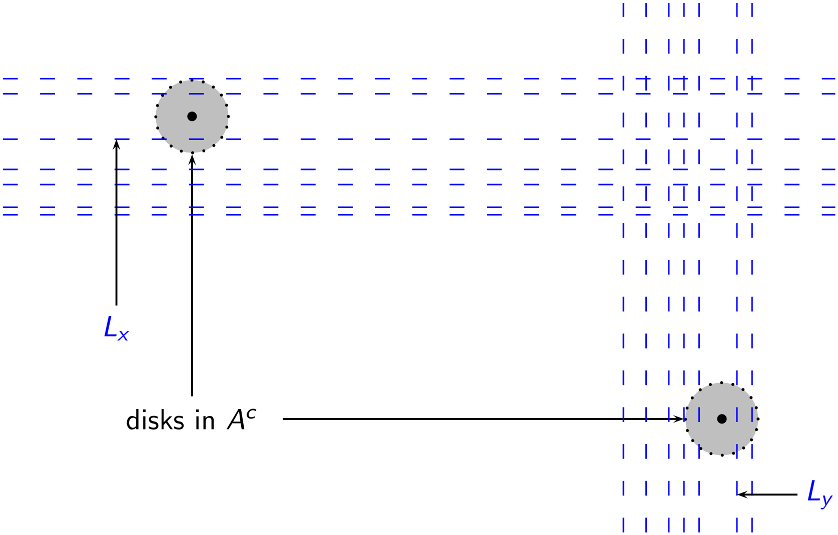

Complement of Plane Set with Zero Measure Projections

Go Back