Metric Space Inversions and Quasihyperbolic Geometry

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Outline

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- Show that $\dfrac{|x-y|}{|x||y|}$ determines a distance function on $\mathbb{R}^n \setminus \{0\}$.

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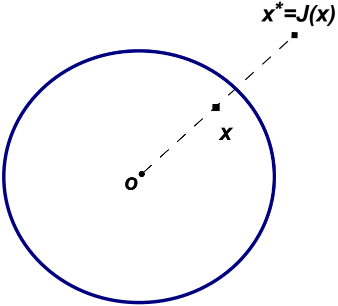
- Find an example when this fails to be a distance function.
- For which metric spaces will this quantity define a distance?

Definition of Euclidean Inversion

Inversion wrt the origin (reflection across $\mathbb{S}(0;1)$):



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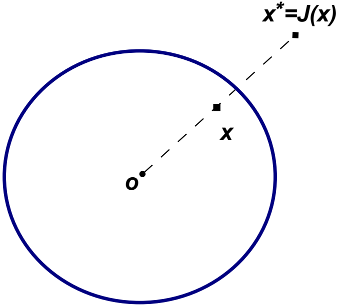
self-homeo $x \mapsto x^* = J(x) := \frac{x}{|x|^2}$ of $\mathbb{R}^n \setminus \{0\}$



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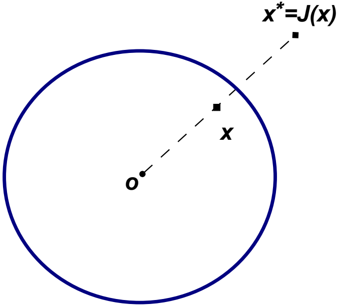

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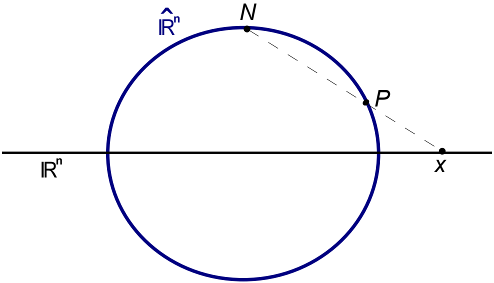
- Can pullback Euclidean distance to get

$$\|x - y\| := |J(x) - J(y)| = |x^* - y^*| = \left| \frac{x}{|x|^2} - \frac{y}{|y|^2} \right| = \frac{|x - y|}{|x||y|}$$

a new distance on $\mathbb{R}^n \setminus \{0\}$

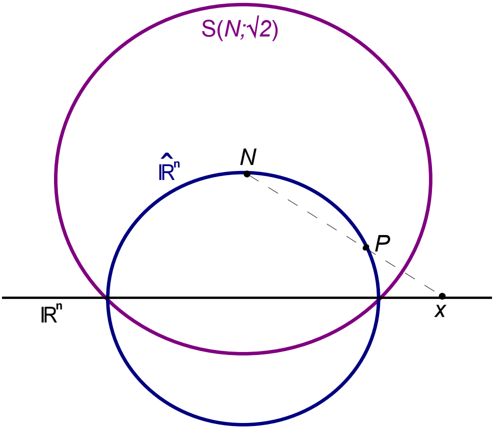
Euclidean Sphericalization

Stereographic Projection



Euclidean Sphericalization

Stereographic Projection is Inversion across $\mathbb{S}(N;\sqrt{2}) \subset \mathbb{R}^{n+1}$



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$\hat{X} \supset Z \xrightarrow{f} \hat{Y}$ is ϑ-quasimöbius if $[0, \infty) \xrightarrow{\vartheta} [0, \infty)$ homeo and

$$|x, y, z, w| \leq t \implies |f(x), f(y), f(z), f(w)| \leq \vartheta(t)$$

where absolute cross ratio of distinct $x, y, z, w \in Z$ is

$$|x, y, z, w| = \frac{|x - y||z - w|}{|x - z||y - w|}$$

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Get d_o a distance function on X_o because $\forall x, y \in X_o$

$$\frac{1}{4} i_o(x,y) \le d_o(x,y) \le i_o(x,y) = \frac{|x-y|}{|x||y|} \le \frac{1}{|x|} + \frac{1}{|y|}.$$

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Know that $\forall x, y \in X_o$

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Definition

The inversion of (X,d) with respect to the base point o is

$$(\mathrm{Inv}_o(X), d_o) := (\hat{X}_o, d_o) = (\hat{X} \setminus \{o\}, d_o) \,.$$

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- $\mathrm{Inv}_o(X)$ is bdd iff o is an isolated pt in X in which case

$$\frac{\operatorname{diam} X_o}{\operatorname{dist}(o, X_o) + \operatorname{diam} X_o}\,\frac{1}{8\operatorname{dist}(o, X_o)} \le \operatorname{diam}_o \mathrm{Inv}_o(X) \le \frac{2}{\operatorname{dist}(o, X_o)}$$

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- d_o-topology on X_o agrees with original subspace topology

Metric Space Sphericalization (Bonk-Kleiner)

This $\boxed{(\mathrm{Sph}_o(X), \hat{d}_o) := (\hat{X}, \hat{d}_o)}$ where $\boxed{s_o(x,y) := \dfrac{|x-y|}{(1+|x|)(1+|y|)}}$

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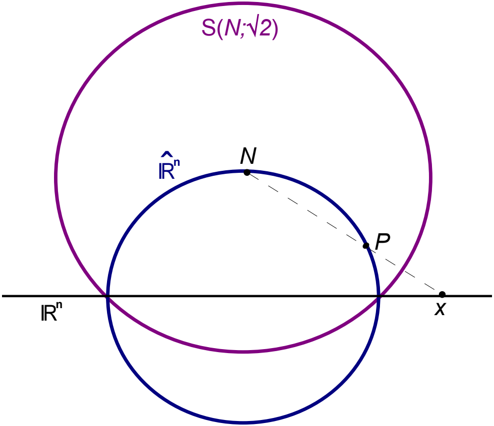
For all $x, y \in \mathrm{Sph}_o(X)$ get

$$\frac{1}{4} s_o(x,y) \le \hat{d}_o(x,y) \le s_o(x,y) \le \frac{1}{1+|x|} + \frac{1}{1+|y|} \, ;$$

$$\hat{\mathrm{d}}\mathrm{iam}_o \, \mathrm{Sph}_o(X) \le 1 \quad \text{and} \quad \hat{\mathrm{d}}\mathrm{iam}_o \, \mathrm{Sph}_o(X) \ge \begin{cases} \hat{d}_o(o, \hat{o}) \ge \frac{1}{4} & X \text{ unbdd}\,, \\ \dfrac{1}{4} \dfrac{\mathrm{diam}\, X}{2 + \mathrm{diam}\, X} & X \text{ bdd}\,. \end{cases}$$

Metric Space Sphericalization <u>is</u> Inversion

Recall that in Euclidean setting stereographic projection $\hat{\mathbb{R}}^n \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ can be viewed as inversion about $\mathbb{S}(N;\sqrt{2}) \subset \mathbb{R}^{n+1}$.



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$$d^{p,q}(x,y) := d^{p,q}(y,x) := \begin{cases} 0 & \text{if } x = q = y \\ d(x,y) & \text{if } x \neq q \neq y \\ d(x,p)+1 & \text{if } x \neq q = y \,. \end{cases}$$

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Then

- $(X^q, d^{p,q})$ is a metric space
- $(\mathrm{Inv}_q(X^q), (d^{p,q})_q) \equiv (\mathrm{Sph}_p(X), \hat{d}_p)$ (isometric)

Mapping Properties

First an an elementary observation.

Lemma

If $X \xrightarrow{f} Y$ *is* K*-bilipschitz, then the induced maps* $\mathrm{Inv}_o(X) \to \mathrm{Inv}_{f(o)}(Y)$ *and* $\mathrm{Sph}_o(X) \to \mathrm{Sph}_{f(o)}(Y)$ *are* $4K^3$*-bilipschitz.*

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Theorem

The natural identity maps associated with following processes are bilipschitz:

- *inversion followed by inversion:* $X \xrightarrow{\mathrm{id}} \mathrm{Inv}_{o'}(\mathrm{Inv}_o X)$,
- *sphericalization followed by inversion:* $X \xrightarrow{\mathrm{id}} \mathrm{Inv}_{\hat{o}}(\mathrm{Sph}_o X)$,
- *inversion followed by sphericalization:* $X \xrightarrow{\mathrm{id}} \mathrm{Sph}_p(\mathrm{Inv}_o X)$.

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Caution! E.g. X unbdd in first, second but bdd in third.

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Proposition

X unbdd; $Y = \mathrm{Inv}_o(X)$; $d' = (d_o)_{o'}$ dist on $\mathrm{Inv}_{o'}(Y) = \mathrm{Inv}_{o'}(\mathrm{Inv}_o X)$

- *If o non-isolated, then $(X, d) \xrightarrow{\text{id}} (\mathrm{Inv}_{o'}(Y), d')$ is 16-bilipschitz.*
- *If o isolated, then $(X_o, d) \xrightarrow{\text{id}} (\mathrm{Inv}_{o'}(Y), d')$ is 16-bilipschitz.*

Sphericalization Followed by Inversion

add this!?

Inversion Followed by Sphericalization

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QuasiConvex Spaces

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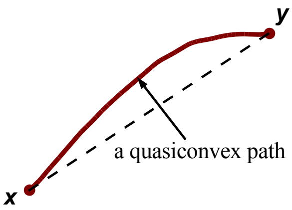
A path γ with endpoints x, y is c-quasiconvex if $\ell(\gamma) \le c|x-y|$.
Call X c-quasiconvex if all pts joinable by c-quasiconvex paths.



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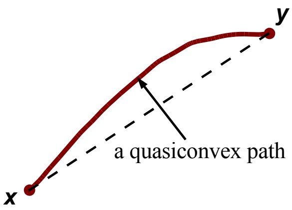

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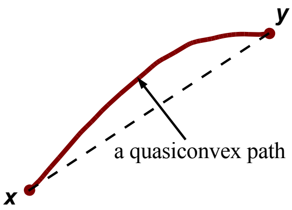
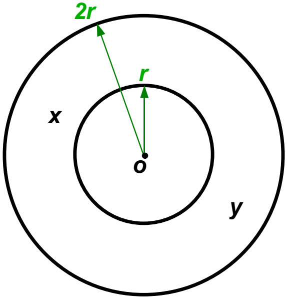
- (X,d) is quasiconvex iff $(X,d) \xrightarrow{\text{id}} (X,\ell)$ is bilipschitz.
- If (X,d) locally quasiconvex, so are (X_o, d_o) and $(X, \hat{d}_o)$.



What is Annular QuasiConvexity?



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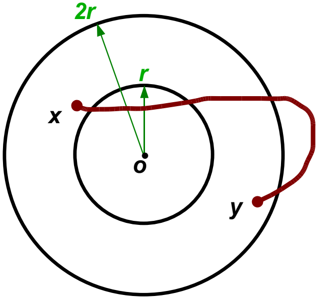

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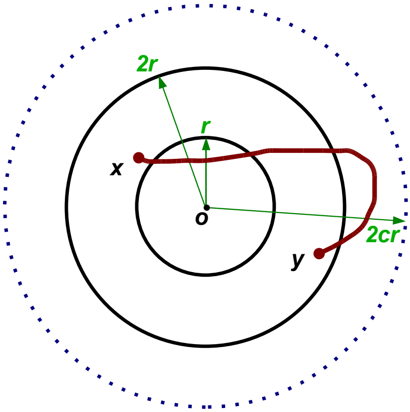

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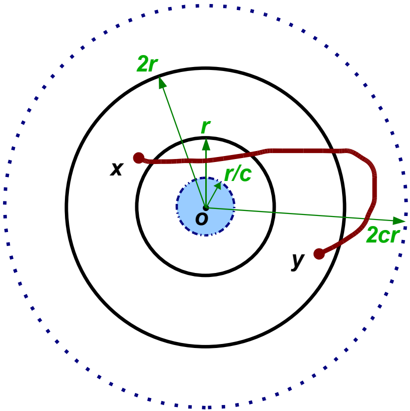
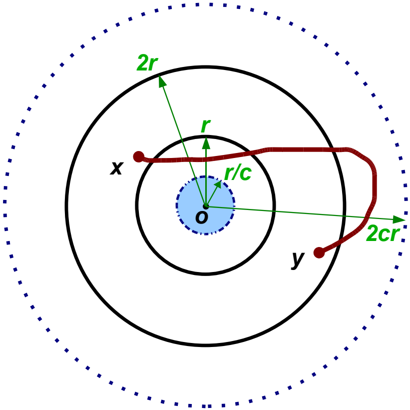


Annular QuasiConvex Spaces: Definition & Examples

Definition

X is c-annular quasiconvex if pts in $A(o; r, 2r)$ joinable by c-quasiconvex paths in $A(o; r/c, 2cr)$ (alternatively, $\gamma \cap B(o; r/c) = \emptyset$)



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- Doubling metric measure spaces supporting a Poincaré inequality (recent result by Rikka Korte)

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- Doubling metric measure spaces supporting a Poincaré inequality (recent result by Rikka Korte)
- Trees are not annular quasiconvex

Using Annular QuasiConvexity

Proposition

Suppose X is connected and c-annular quasiconvex at o. Then X is $9c$-quasiconvex and $\mathrm{Inv}_o(X)$ is $72c^3$-quasiconvex.

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Proof that $\mathrm{Inv}_o(X)$ is quasiconvex.

Spp $t = |x| \leq |y|$; even $2^n t < |y| \leq 2^{n+1} t$.

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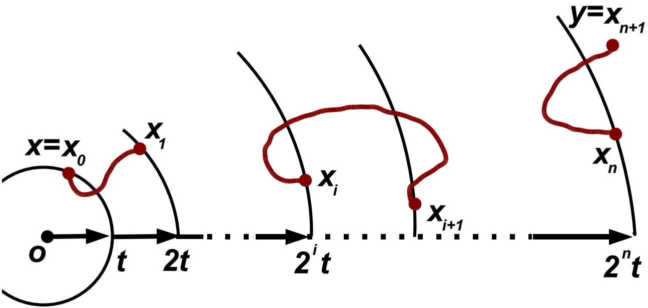

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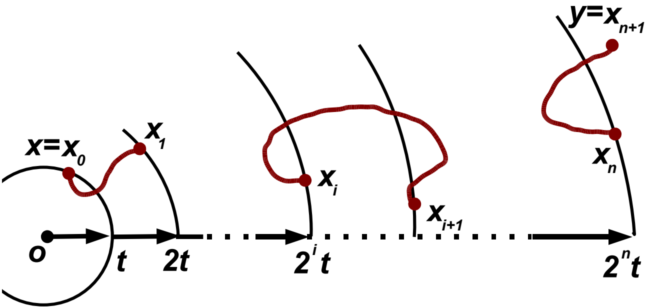

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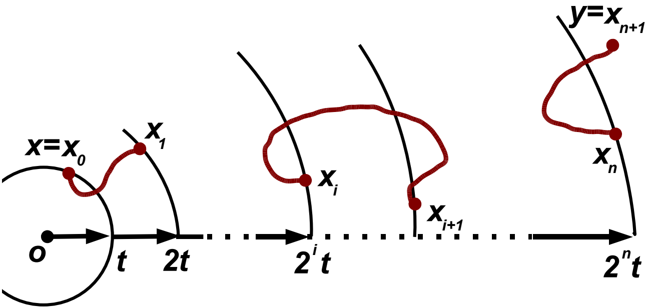

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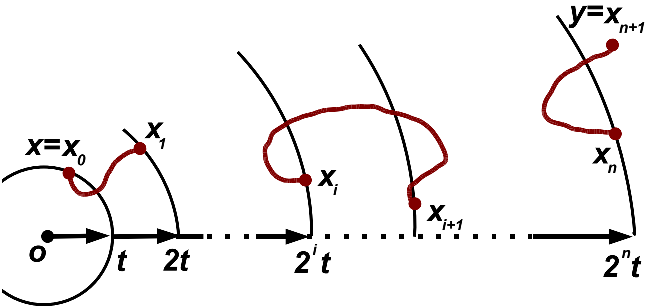
Therefore, $\ell_o(\gamma) \lesssim \sum_0^n 1/(2^i t) \simeq 1/t \lesssim |x-y|/|x||y| \lesssim d_o(x,y)$. □



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Theorem

For any metric space X, these are quantitively equivalent:

- X *is quasiconvex and annular quasiconvex*
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The quasiconvexity constants depend only on each other.

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Above theme employed repeatedly.

Outline

QuasiHyperbolization

(Ω, d) non-complete locally complete rectifiably connected metric space

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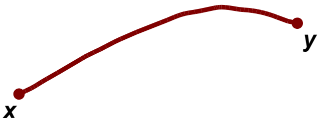

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- In Euclidean half-spaces: quasihyp dist=hyperbolic dist
- In Euclidean balls: $k/2 \le h \le 2k$

Basic Properties of QuasiHyperbolic Distance

Easy to check that for any rectifiable curve γ,

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Fact

Assume $(\Omega,\ell) \xrightarrow{\mathrm{id}} (\Omega,d)$ *is homeo. Then* $(\Omega,k) \xrightarrow{\mathrm{id}} (\Omega,d)$ *also homeo and* (Ω,k) *is complete. Thus when* (Ω,d) *is locally compact (so* (Ω,k) *too), Hopf-Rinow theorem guarantees* (Ω,k) *proper and geodesic.*

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Let $k_o, \hat{k}_o$ denote quasihyperbolic metrics in $I_o(\Omega), S_o(\Omega)$.

Theorem

Let X be complete. Suppose $\Omega \subset X_o$ is open locally c-quasiconvex subspace with $\partial\Omega \neq \emptyset \neq \partial_o\Omega$. Then we get bilipschitz identity maps

$$(\Omega, k) \xrightarrow{\mathrm{id}} (\Omega, k_o) \quad \textit{and} \quad (\Omega, k) \xrightarrow{\mathrm{id}} (\Omega, \hat{k}_o)\,.$$

The BiLipschitz Constants

For $(\Omega, k) \xrightarrow{\text{id}} (\Omega, k_o)$, get BL constant $M = 2\,c(a \vee 20\,b)$ where

$$a = \begin{cases} 1 & \text{if } \Omega \text{ is unbounded}\,, \\ \operatorname{diam}\Omega/[\operatorname{dist}(o,\partial\Omega) \vee (\operatorname{diam}\partial\Omega/2)] & \text{if } \Omega \text{ is bounded}\,. \end{cases}$$

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For $(\Omega, k) \xrightarrow{\text{id}} (\Omega, \hat{k}_o)$, Ω unbounded and $o \in \partial\Omega$, $M = 40\,c$.

Examples illustrate M may depend on quantities indicated above.

Outline

What is a Uniform Space?

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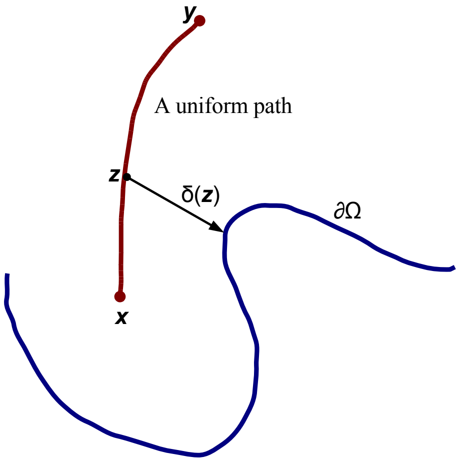
One in which points can be joined by uniform paths.



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(Ω, d) is c-uniform space if all pts joinable by c-uniform paths.
γ joining x, y in Ω is such if c-quasiconvex and c-double cone,
which means $\ell(\gamma) \leq c|x - y|$, and

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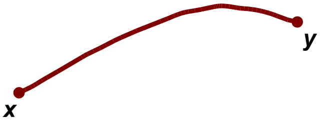

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- Not infinite cylinders nor infinite slabs

QuasiHyperbolic Geometry of Uniform Spaces

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- *If c-uniform, quasihyperbolization is proper geodesic Gromov* δ*-hyperbolic,* $\delta = \delta(c) = 10000c^8$*.*

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When X complete and $\Omega \subset X_o$ locally compact open non-complete rectifiably conn, get $I_o(\Omega)$ locally compact open non-complete (if $\partial\Omega \neq \{o\}$) rectifiably connected subspace of $\mathrm{Inv}_o(X)$.

Main Result

Theorem

X complete; $\Omega \subset X_o$ open locally compact with $\partial\Omega \neq \emptyset \neq \partial_o\Omega$
These are quantitively equivalent:

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If X annular quasiconvex, A, B, C depend only on each other and the quasiconvexity constants.

General Uniformity Constants

A, B, C constants for $\Omega, I_o(\Omega), S_o(\Omega)$

For Ω and $S_o(\Omega)$:

$$C = C(A, o) = c(A)[1 + \operatorname{dist}(o, \partial\Omega] \quad \text{and} \quad A = A(C) \ (\Omega \text{ unbdd}).$$

For Ω and $I_o(\Omega)$:

$$B = B(A, o) = b(A)[1 + r] \quad \text{and} \quad A = A(B, d) = a(B)d$$

where

$$r = r(o) = \begin{cases} \operatorname{dist}(o, \partial\Omega)/\operatorname{dist}(o, \Omega) & o \in X \setminus \bar{\Omega} \\ 0 & o \in \partial\Omega \end{cases}$$

$$d = \begin{cases} 1 & \Omega \text{ unbdd} \\ \operatorname{diam}\Omega/\operatorname{diam}\partial\Omega & \Omega \text{ bdd} \end{cases}$$

Ideas in Proof—fix this David!

Proof that $I_o(\Omega)$ uniform if Ω is.

Spp $t = |x| \leq |y|$; even $2^n t < |y| \leq 2^{n+1} t$.

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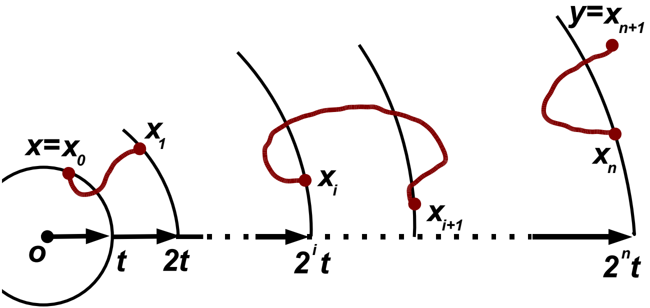

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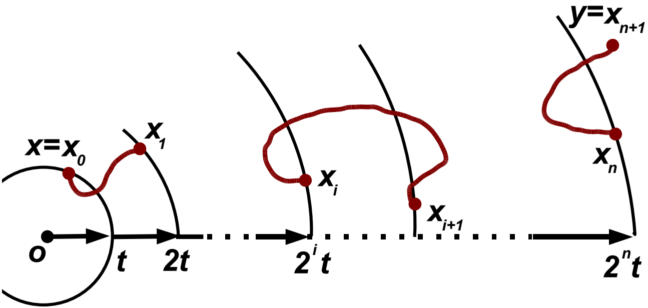

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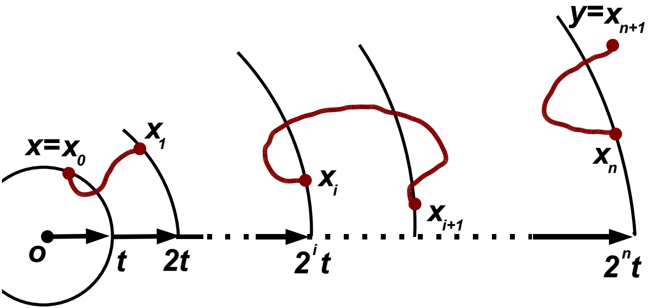

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Therefore, $\ell_o(\gamma) \lesssim \sum_0^n 1/(2^i t) \simeq 1/t \lesssim |x-y|/|x||y| \lesssim d_o(x,y)$. □