QuasiHyperbolic Type Metrics and Universal Convexity

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Honoring Roger W. Barnard

1 Introduction

2 Examples of Metrics

3 Theorems

Outline

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2 Examples of Metrics

3 Theorems

Conformal Deformations

$\Omega \subset \hat{\mathbb{R}}^n := \mathbb{R}^n \cup \{\infty\}$ ($n \geq 2$) is a *quasihyperbolic domain*, meaning that $\Omega^c := \hat{\mathbb{R}}^n \setminus \Omega$ contains at least 2 points.

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$$\Omega_\rho := (\Omega, d_\rho)$$

where d_ρ is length distance given via

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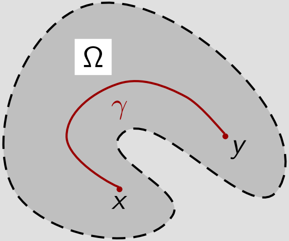

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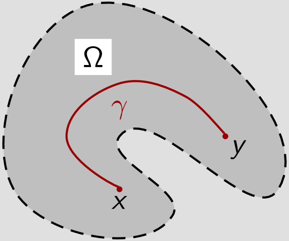

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Call γ a *ρ-geodesic* if $d_\rho(x, y) = \ell_\rho(\gamma)$; sometimes write $[x, y]_\rho$, but these need not be unique!

Geodesic Convexity

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Question

When is a ball in Ω convex in Ω_ρ?

Recall that an open ball in $\hat{\mathbb{R}}^n$ is an open Euclidean ball or an open half-space (both in $\mathbb{R}^n$) or the complement of a closed Euclidean ball.

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If B is a ball (in $\mathbb{R}^n$) and $B \subset \Omega$, then B is k-convex in Ω.

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Usually ignore trivial cases $U = \hat{\mathbb{R}}^n$, $U = \mathbb{R}^n$, etc.

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Martin: Balls (in $\mathbb{R}^n$) are universally quasi-hyperbolically convex.

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Brown (1982)

Disks (in $\hat{\mathbb{C}}$) are the only universally hyperbolically convex objects.

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Most domains in space do not support such a metric, but do have for balls in $\hat{\mathbb{R}}^n$. E.g., for Euclidean ball $\mathrm{B}(a;r)$,

$$\lambda\, ds = \lambda(x)|dx| = \frac{2r\,|dx|}{r^2-|x-a|^2}.$$

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Can define Möb invariant analog of $|dx|/|x|$ in regions $\hat{\mathbb{R}}^n_{ab} := \hat{\mathbb{R}}^n \setminus \{a, b\}$ by

$$\tau_{ab}(x)|dx| := \frac{|a-b|}{|x-a||x-b|}|dx|\,,$$

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with standard interpretation if one of a or b is the point at infinity. In fact, $\tau_{ab} ds$ is the unique metric on $\hat{\mathbb{R}}^n_{ab}$ with the property that for any Möbius tfm T with $T(\hat{\mathbb{R}}^n_{ab}) = \mathbb{R}^n_*$,

$$\tau_{ab}(x)|dx| = T^*\left[\frac{|dy|}{|y|}\right] = \frac{|T'(x)||dx|}{|T(x)|}\,.$$

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These are Möbius invariant metrics that are bilipschitz equivalent to the quasihyperbolic metric (in finite domains).

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Proof via pictured implications:

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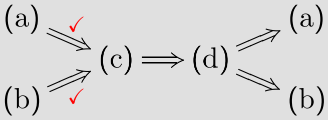

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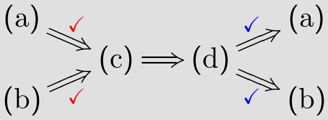

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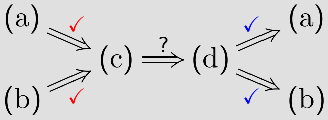

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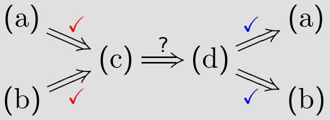

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(d) $\Longrightarrow$ (a)

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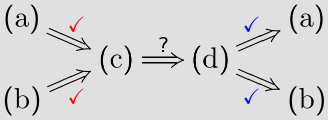

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Corollary

For $\rho = \varphi, \mu, \tau$, the only non-trivial universally ρ-convex objects are balls.

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Corollary

Spp $\mathcal{M} = \{\rho_\Omega ds\}_{\Omega \in \mathcal{O}}$ is Möb invariant class of metrics with $\mathbb{R}^n_* \in \mathcal{O}$.

If U is ρ-UC and non-trivial, then U is an open ball.

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Example

There are large Möbius invariant classes $\mathcal{M}$ of conformal metrics with the property that
$\forall\, \Omega \in \mathcal{O}\,, \forall\, r > 0\,,\ \exists$ balls in Ω with radius $\leq r$ which are not ρ-convex.

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What if $\mathbb{R}^n_* \in \mathcal{O}$?

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Call A *circle convex* provided for all circles C in $\hat{\mathbb{R}}^n$, $C \cap A$ is connected.

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Skip Proof

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Let $p \in B$. Get unique circle C thru p, a, b. Evidently, $C \cap \partial B = \{a, b\}$, so must get $C \cap A = C \cap \bar{B}$ which is the closed subarc of C that contains p and has endpts a, b.

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Let $p \in B$. Get unique circle C thru p, a, b. Evidently, $C \cap \partial B = \{a, b\}$, so must get $C \cap A = C \cap \bar{B}$ which is the closed subarc of C that contains p and has endpts a, b. See that $p \in A$.

Circle Convex Sets are Balls

Spp $A \subsetneq \hat{\mathbb{R}}^n$ is circle convex. Assume A bounded. Let $\bar{B}$ be the closed circumball containing A. We show that $A = \bar{B}$. Know that $A \subset \bar{B}$.
Since $\bar{B}$ is circumball, $A \cap \partial B = \partial A \cap \partial B \supset \{a, b\}$ with $a \neq b$.
Since A is circle convex, if C is any circle thru a, b then $C \cap A$ connected.

Basic Circle Geometry: the intersection of two distinct circles contains exactly 0 or 1 or 2 points.

Let $p \in B$. Get unique circle C thru p, a, b. Evidently, $C \cap \partial B = \{a, b\}$, so must get $C \cap A = C \cap \bar{B}$ which is the closed subarc of C that contains p and has endpts a, b. See that $p \in A$. Thus, $B \subset A \subset \bar{B}$, so $A = \bar{B}$.

The End

$\left(\ddot{\smile}\right)$

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Proof that Balls are Universally Ferrand Convex

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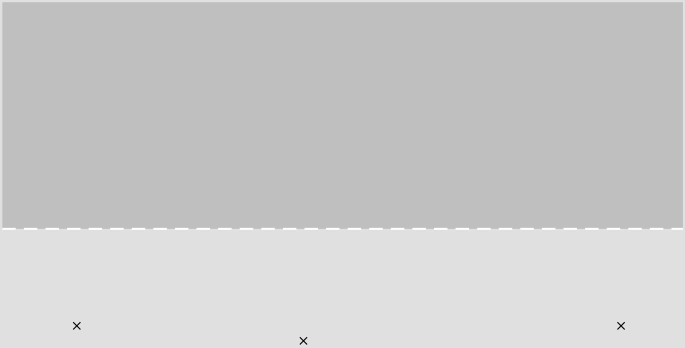

Proof that Balls are Universally Ferrand Convex

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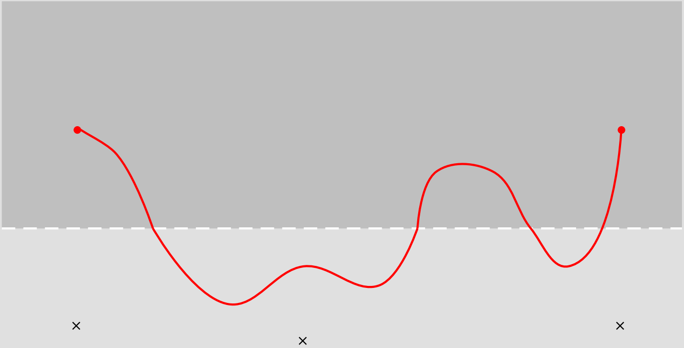

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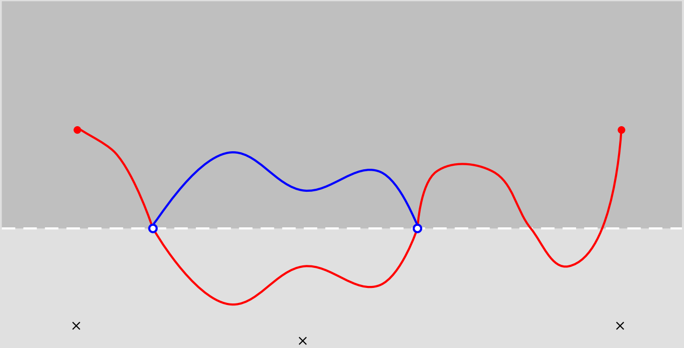
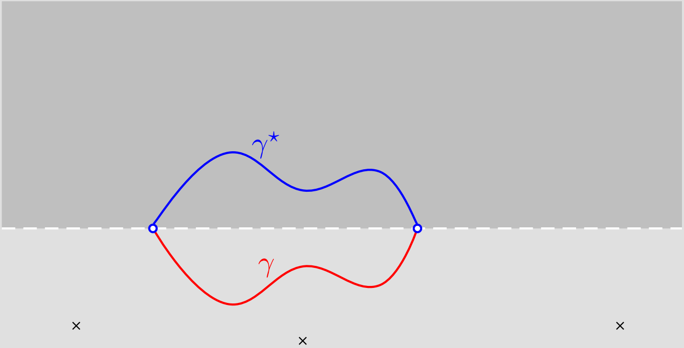


Proof that Balls are Universally Ferrand Convex

WTS

$\ell_\varphi(\gamma) > \ell_\varphi(\gamma^\star)$



Proof that Balls are Universally Ferrand Convex

WTS

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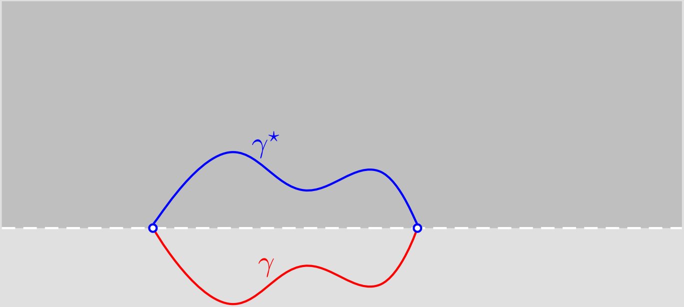

But, $\ell_\varphi(\gamma) = \int_\gamma \varphi\, ds$ and $\ell_\varphi(\gamma^\star) = \ell_{\varphi^\star}(\gamma) = \int_\gamma \varphi^\star\, ds$.

Proof that Balls are Universally Ferrand Convex

WTS

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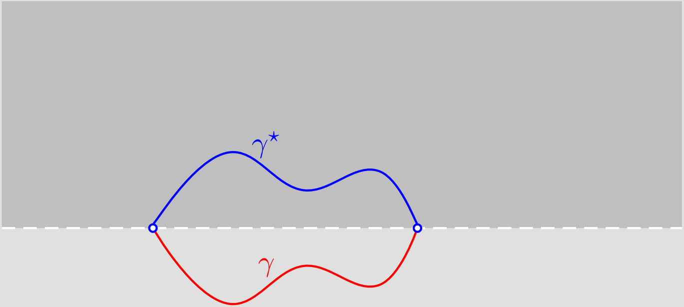
$$\int_\gamma \varphi\, ds > \int_\gamma \varphi^\star\, ds$$



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Proof that Balls are Universally Ferrand Convex

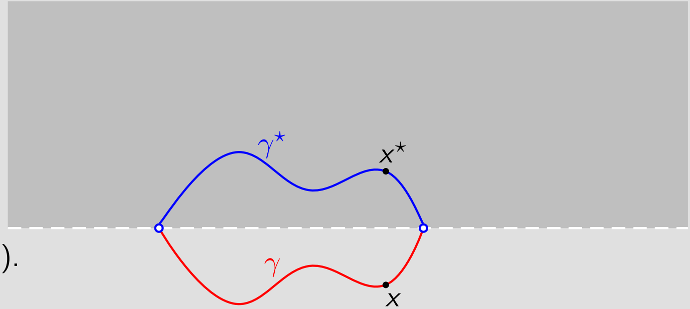
WTS

$\ell_\varphi(\gamma) > \ell_\varphi(\gamma^\star)$, or,

$\displaystyle\int_\gamma \varphi\, ds > \int_\gamma \varphi^\star\, ds$,

or, $\forall\, x \in \gamma$,

$\varphi(x) > \varphi^\star(x) = \varphi(x^\star)$.



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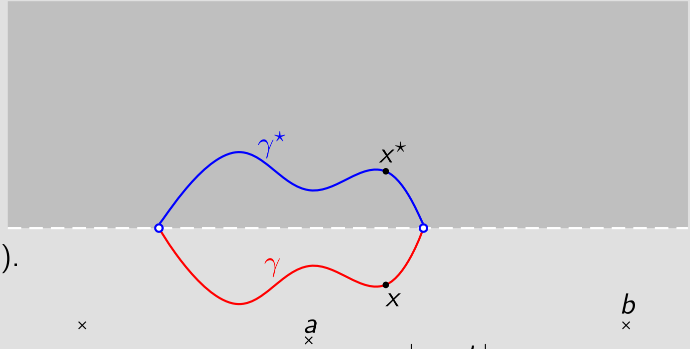
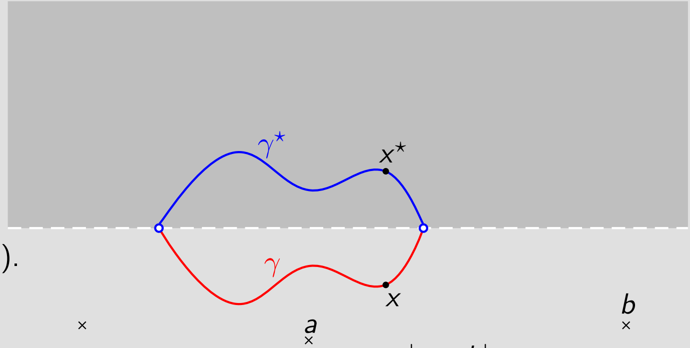


Fix $x \in \gamma$. Pick $a, b \in \partial\Omega$ st $\varphi(x^\star) = \tau_{ab}(x^\star) = \dfrac{|a-b|}{|x^\star - a||x^\star - b|}$.

Proof that Balls are Universally Ferrand Convex



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$$\varphi(x) \geq \tau_{ab}(x) = \tau_{a^\star b^\star}(x^\star) = \frac{|a^\star - b^\star|}{|x^\star - a^\star||x^\star - b^\star|}$$

Proof that Balls are Universally Ferrand Convex

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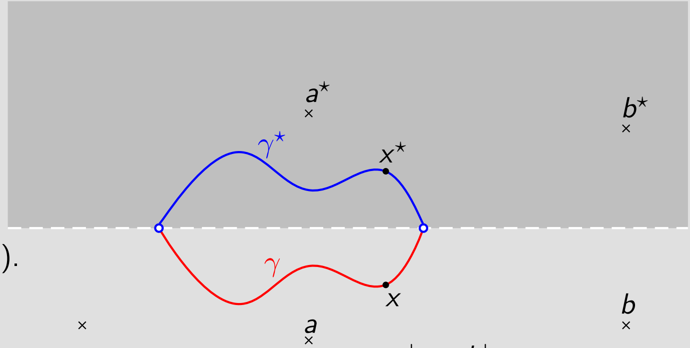

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$$= \frac{|a-b|}{|x^\star - a||x^\star - b|} \frac{|x^\star - a|}{|x^\star - a^\star|} \frac{|x^\star - b|}{|x^\star - b^\star|}$$

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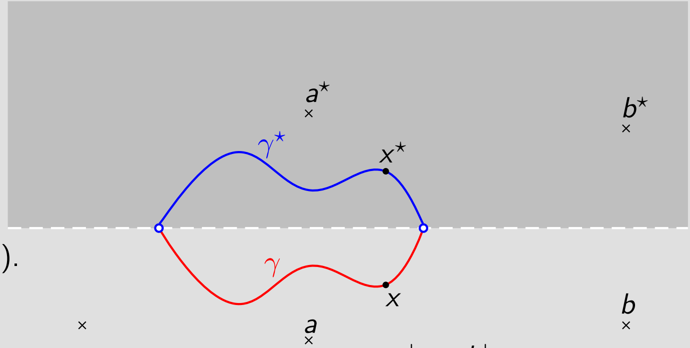

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$$\begin{aligned}\varphi(x) \geq \tau_{ab}(x) = \tau_{a^\star b^\star}(x^\star) &= \frac{|a^\star - b^\star|}{|x^\star - a^\star||x^\star - b^\star|} \\ &= \frac{|a-b|}{|x^\star - a||x^\star - b|}\,\frac{|x^\star - a|}{|x^\star - a^\star|}\,\frac{|x^\star - b|}{|x^\star - b^\star|} > \tau_{ab}(x^\star)\end{aligned}$$

Proof that Balls are Universally Ferrand Convex

Back

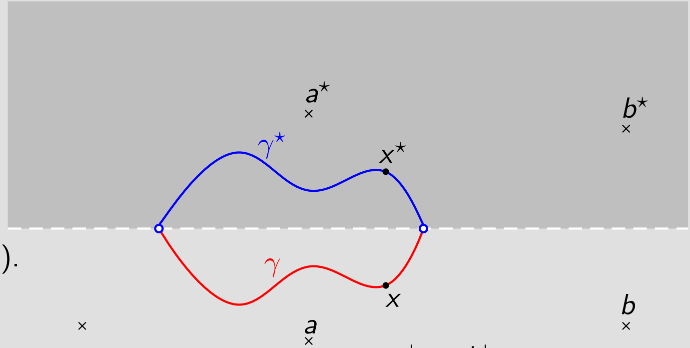
WTS

$\ell_\varphi(\gamma) > \ell_\varphi(\gamma^\star)$, or,

$\displaystyle\int_\gamma \varphi\, ds > \int_\gamma \varphi^\star\, ds$,

or, $\forall\, x \in \gamma$,

$\varphi(x) > \varphi^\star(x) = \varphi(x^\star)$.



Fix $x \in \gamma$. Pick $a, b \in \partial\Omega$ st $\varphi(x^\star) = \tau_{ab}(x^\star) = \dfrac{|a-b|}{|x^\star - a||x^\star - b|}$. Then

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