Metric Space Inversions

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Virginia Polytechnic University

1. Introduction
   - Euclidean Inversions
   - Notation
2. Metric Space Inversions
   - Definitions
   - Properties
3. Inversions and QuasiConvexity
   - Annular QuasiConvexity
   - Preservation of QuasiConvexity
4. Inversions and Uniformity
   - Uniform Spaces
   - Preservation of Uniform SubSpaces
5. Ptolemaic Spaces

Outline

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Collaborators

Mostly joint work with:

- Stephen M. Buckley at NUI,
- Xiangdong Xie at Virginia Tech.

If time permits, will discuss recent related work of
S. Buckley, K. Falk, D. Wraith —all at NUI.

Things to Ponder

- Show that $\dfrac{|x-y|}{|x||y|}$ determines a distance function on $\mathbb{R}^n \setminus \{0\}$.
- Find an example when this fails to be a distance function.
- For which metric spaces will this quantity define a distance?

Start Talk

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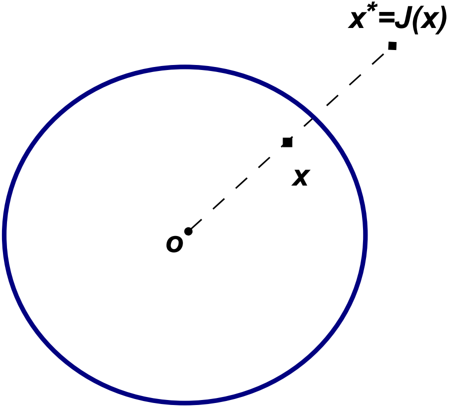
Finish Talk Summary

Definition of Euclidean Inversion

Inversion wrt the origin (reflection across $\mathbb{S}^{n-1}(0;1)$):



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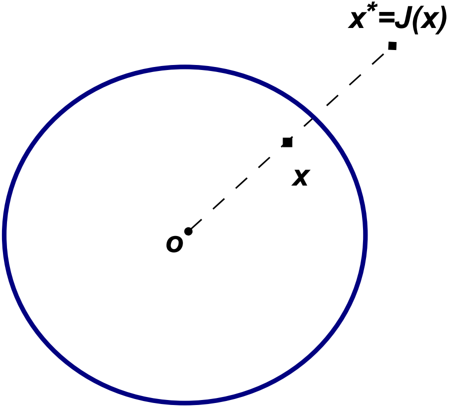
self-homeo $x \mapsto x^* = J(x) := \dfrac{x}{|x|^2}$ of $\mathbb{R}^n \setminus \{0\}$



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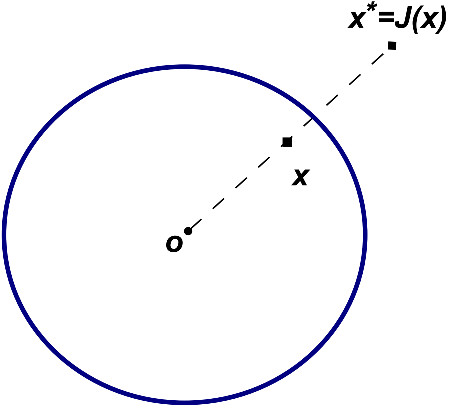

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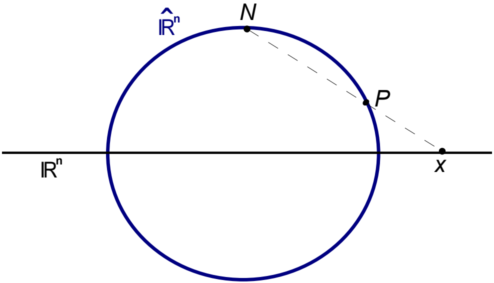
- J preserves uniform subspaces of $\mathbb{R}^n \setminus \{0\}$
- Can pullback Euclidean distance to get

$$\|x - y\| := |J(x) - J(y)| = \left| \frac{x}{|x|^2} - \frac{y}{|y|^2} \right| = \frac{|x - y|}{|x||y|}$$

a new distance on $\mathbb{R}^n \setminus \{0\}$

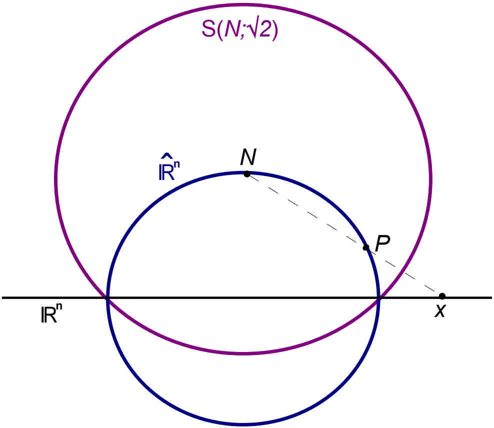
Euclidean Sphericalization

Stereographic Projection



Euclidean Sphericalization

Stereographic Projection is Inversion across $\mathbb{S}^n(N;\sqrt{2}) \subset \mathbb{R}^{n+1}$



Metric Space Notation & Definitions

(X, d) metric; $o \in X$ base point; $X_o := X \setminus \{o\}$; $|x - y| = d(x, y)$;

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$\hat{X} \supset Z \xrightarrow{f} \hat{Y}$ is ϑ-quasimöbius if $[0,\infty) \xrightarrow{\vartheta} [0,\infty)$ homeo and

$$|x,y,z,w| \leq t \implies |f(x), f(y), f(z), f(w)| \leq \vartheta(t)$$

where absolute cross ratio of distinct $x,y,z,w \in Z$ is

$$|x,y,z,w| = \frac{|x-y||z-w|}{|x-z||y-w|}$$

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Get distance function because for all $x, y \in X_o$

$$\frac{1}{4} i_o(x,y) \le d_o(x,y) \le i_o(x,y) = \frac{|x-y|}{|x||y|} \le \frac{1}{|x|} + \frac{1}{|y|}.$$

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Know that for all pts $x, y \in X_o$

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Definition

The inversion of (X, d) wrt o is

$$(\mathrm{Inv}_o(X), d_o) := (\hat{X}_o, d_o) = (\hat{X} \setminus \{o\}, d_o)\,.$$

Elementary Properties

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- $(\hat{X}, d) \xrightarrow{\text{id}} (\hat{Y}, d_o)$ is $16t$-quasimöbius (careful :-)
- d_o-topology on X_o agrees with original subspace topology

Metric Space Sphericalization (Bonk-Kleiner)

This $\boxed{(\mathrm{Sph}_o(X), \hat{d}_o) := (\hat{X}, \hat{d}_o)}$ where $\boxed{s_o(x,y) := \dfrac{|x-y|}{(1+|x|)(1+|y|)}}$

and

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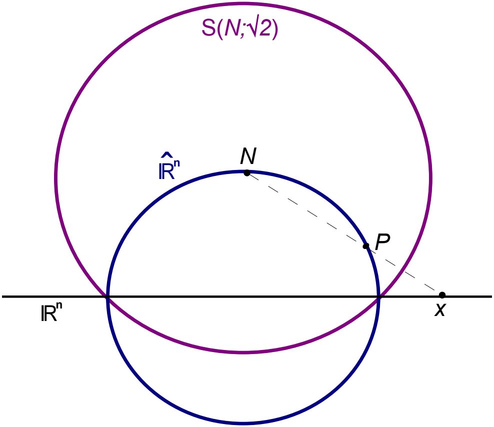
For all $x, y \in \mathrm{Sph}_o(X)$ get

$$\frac{1}{4} s_o(x,y) \le \hat{d}_o(x,y) \le s_o(x,y) \le \frac{1}{1+|x|} + \frac{1}{1+|y|}.$$

Metric Space Sphericalization is Inversion

Recall that stereographic proj $\hat{\mathbb{R}}^n \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ is inversion.



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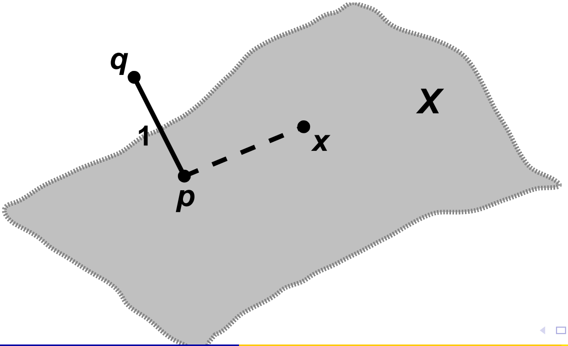
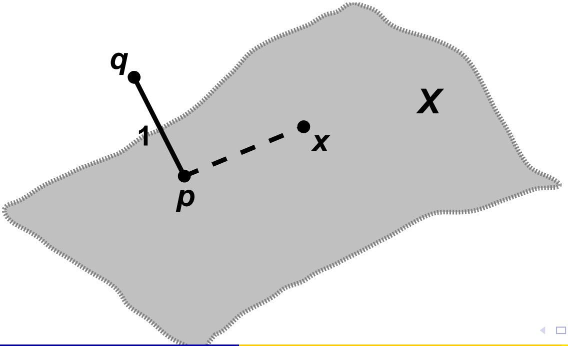


Metric Space Sphericalization is Inversion

Phenomenon also true in metric space setting.

Get $(\mathrm{Sph}_p(X), \hat{d}_p) \equiv (\mathrm{Inv}_q(X \sqcup \{q\}), d'_q)$ (isometric) where

$$d'(y,x) := d'(x,y) := \begin{cases} 0 & \text{if } x = q = y \\ d(x,y) & \text{if } x \neq q \neq y \\ d(x,p) + 1 & \text{if } x \neq q = y\,. \end{cases}$$



Mapping Properties

Theorem

Natural identity maps associated with following processes are bilipschitz.

- *inversion followed by inversion:* $X \xrightarrow{\text{id}} \mathrm{Inv}_{o'}(\mathrm{Inv}_o X)$
- *sphericalization followed by inversion:* $X \xrightarrow{\text{id}} \mathrm{Inv}_{\hat{o}}(\mathrm{Sph}_o X)$
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Caution! E.g. X unbdd in first, second but bdd in third.

Inversion Followed by Inversion

Spp X, $Y = \mathrm{Inv}_o(X)$ both unbdd. Let $Z = \mathrm{Inv}_{o'}(Y)$. Get

$$(X, o, \infty) \overset{\mathrm{Inv}_o}{\rightsquigarrow} (Y, \infty, o') \overset{\mathrm{Inv}'_o}{\rightsquigarrow} (Z, o'', \infty) .$$

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Here is analogue of Euclidean inversions having order two.

Proposition

Suppose X unbdd and o non-isolated. Then $(X, d) \xrightarrow{\text{id}} (\mathsf{Inv}_{o'} \mathsf{Inv}_o(X), (d_o)_{o'})$ is 16-bilipschitz.

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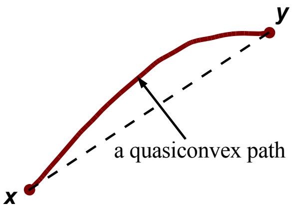
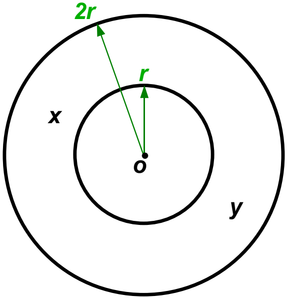
QuasiConvex Spaces

Definition

A path γ with endpoints x, y is c-quasiconvex if $\ell(\gamma) \leq c|x - y|$.
Call X c-quasiconvex if all pts joinable by c-quasiconvex paths.



What is Annular QuasiConvexity?



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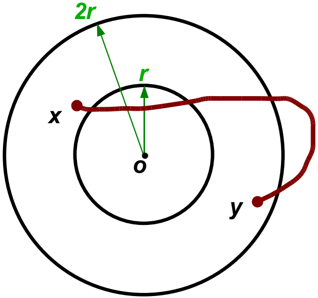

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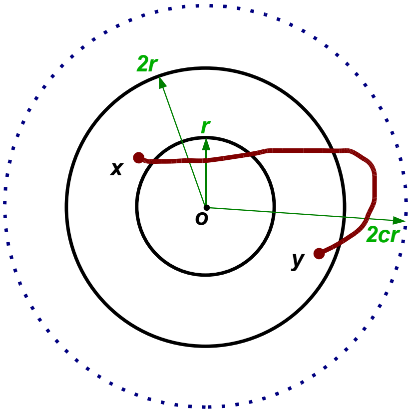

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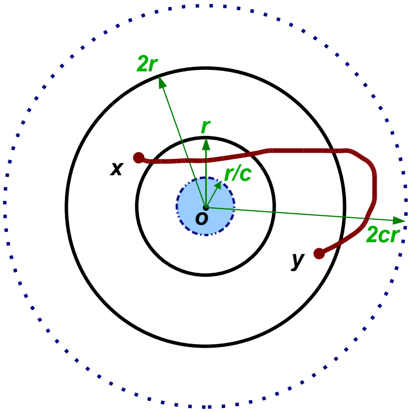
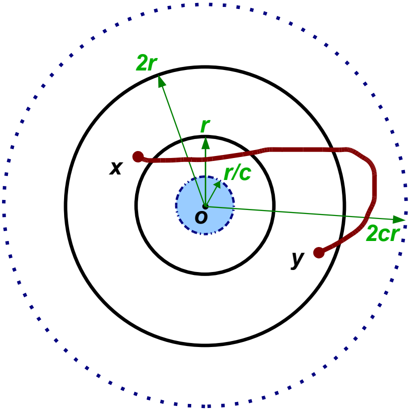


Annular QuasiConvex Spaces: Definition & Examples

Definition

X is c-annular quasiconvex if pts in $A(o; r, 2r)$ joinable by c-quasiconvex paths in $A(o; r/c, 2cr)$ (alternatively, $\gamma \cap B(o; r/c) = \emptyset$)



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- Doubling metric measure spaces supporting a Poincaré inequality
- Trees are not annular quasiconvex

Using Annular QuasiConvexity

Proposition

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- $\forall\, x, y \in A(o; r, R)$: $\quad \dfrac{|x-y|}{4R^2} \le d_o(x,y) \le \dfrac{|x-y|}{r^2}$ (quasidilation)

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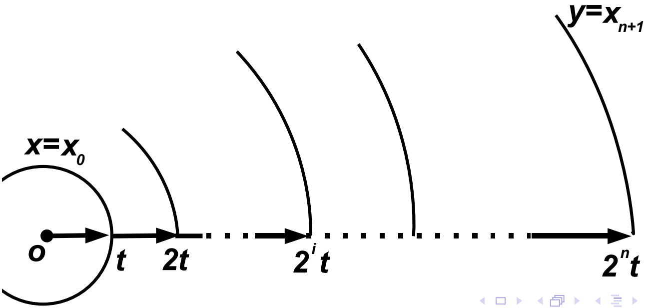
- $\ell_o(\gamma) = \displaystyle\int_\gamma \frac{|dx|}{|x|^2}$ *(arclength diff'l is $|dx|_o = |dx|/|x|^2$)*
- *for all rect $\gamma \subset A(o; r, R)$:* $\dfrac{\ell(\gamma)}{R^2} \le \ell_o(\gamma) \le \dfrac{\ell(\gamma)}{r^2}$

Ideas in Proof that $\mathrm{Inv}_o(X)$ is Quasiconvex

Spp $t = |x| \le |y|$; even $2^n t < |y| \le 2^{n+1} t$.



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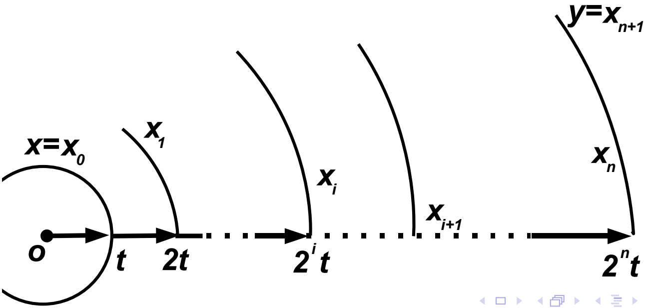

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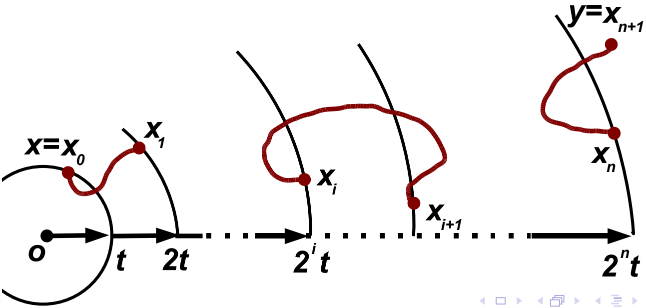

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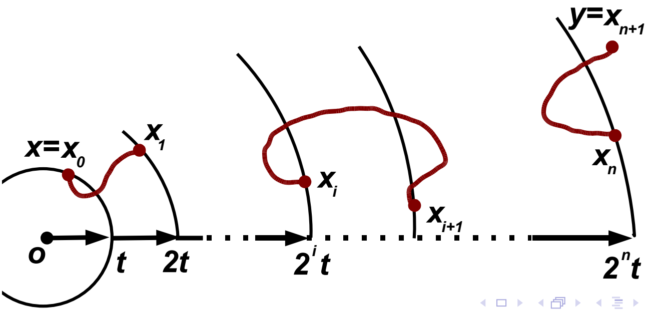

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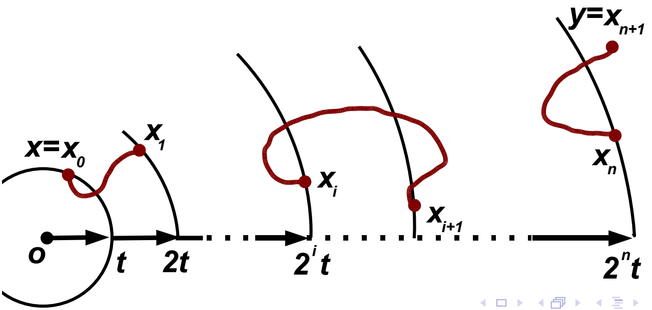

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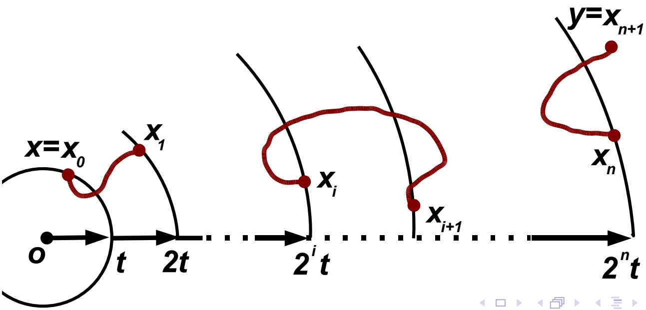

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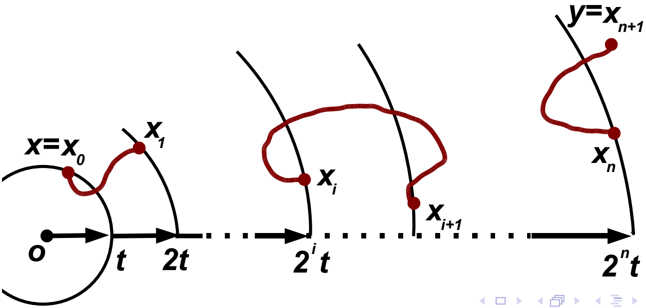

Main Result

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For any metric space X, these are quantitively equivalent:

(a) X *is quasiconvex and annular quasiconvex*

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Above theme employed repeatedly.

Outline

1 Introduction
- Euclidean Inversions
- Notation

2 Metric Space Inversions
- Definitions
- Properties

3 Inversions and QuasiConvexity
- Annular QuasiConvexity
- Preservation of QuasiConvexity

4 Inversions and Uniformity
- Uniform Spaces
- Preservation of Uniform SubSpaces

5 Ptolemaic Spaces

What is a Uniform Space?

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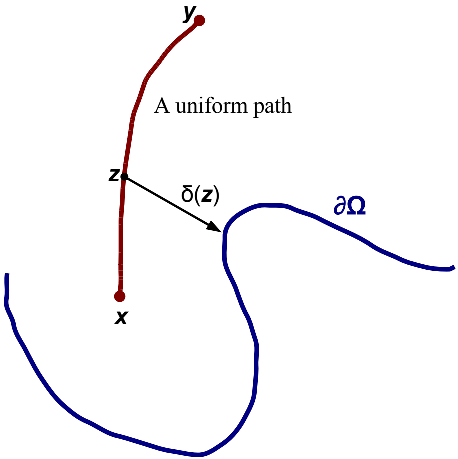
One in which points can be joined by uniform paths.

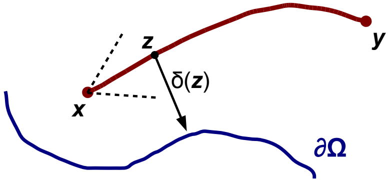


Uniform Spaces: Definition & Examples

Definition

(Ω, d) is c-uniform space if all pts joinable by c-uniform paths.
γ joining x, y in Ω is such if c-quasiconvex and c-double cone, which means $\ell(\gamma) \leq c|x-y|$, and

$$\forall z \in \gamma : \quad \min\{\ell(\gamma[x,z]), \ell(\gamma[y,z])\} \leq c\delta(z) .$$



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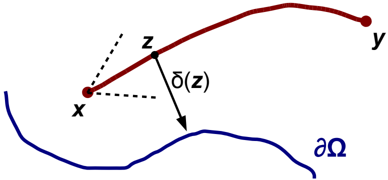

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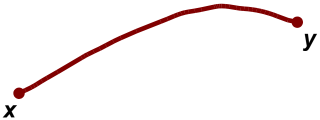
- Not infinite cylinders nor infinite slabs

Gromov Hyperbolic Spaces

Every (unbounded proper geodesic) Gromov hyperbolic space can be conformally dampened to a bounded uniform space.

If (Ω, h) is Gromov δ-hyperbolic, then for all $\varepsilon \in (0, \varepsilon_0]$ ($\varepsilon_0 = \varepsilon_0(\delta)$) (Ω, d_ε) is 20-uniform. Here d_ε is the length distance given by

$$d_\varepsilon(x, y) := \inf_\gamma \int_\gamma e^{-\varepsilon h(z,o)} \, |dz| \, .$$



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X complete; $\Omega \subset X_o$ open locally compact with $\partial\Omega \neq \emptyset \neq \partial_o\Omega$
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Ideas in Proof of Ω Uniform $\implies I_o(\Omega)$ Uniform

Skip Proof

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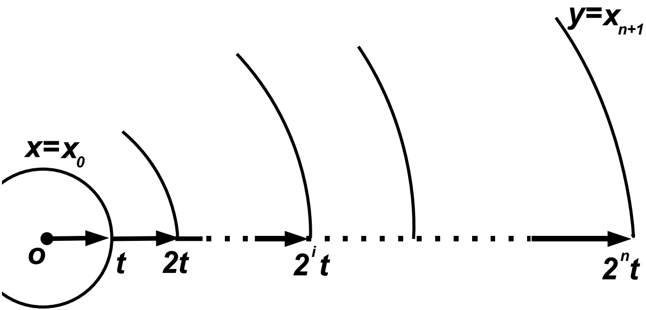

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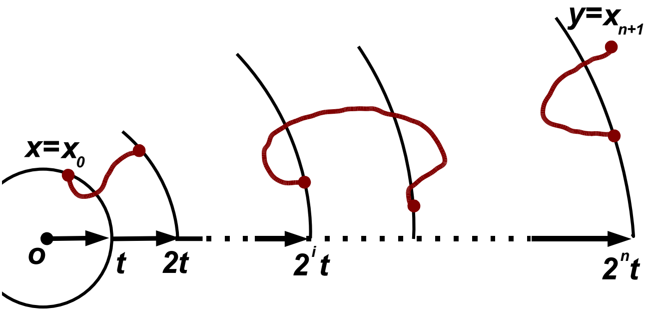

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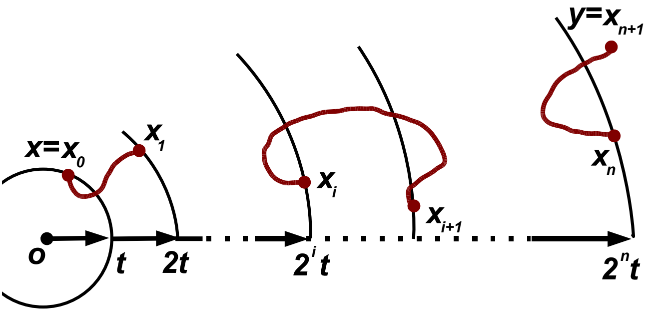

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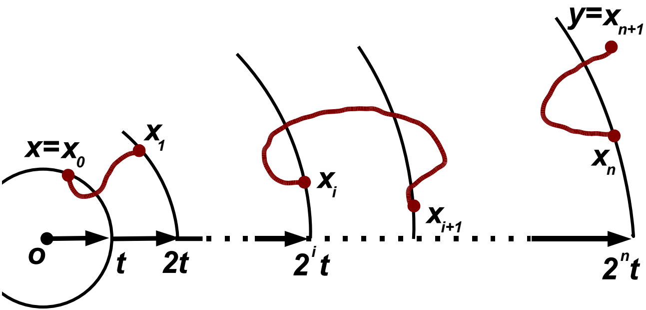

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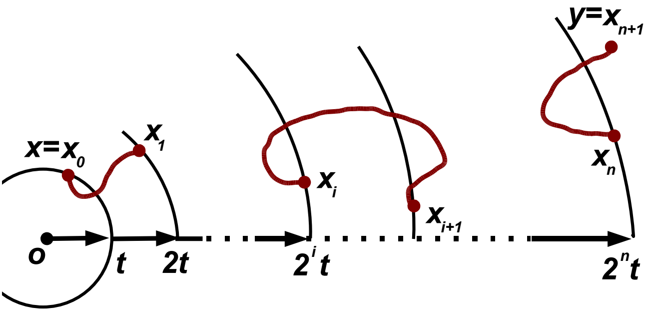

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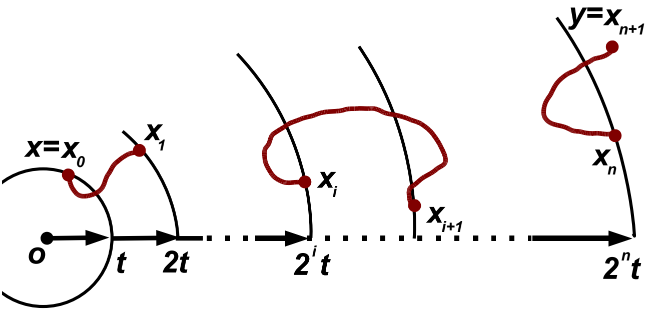

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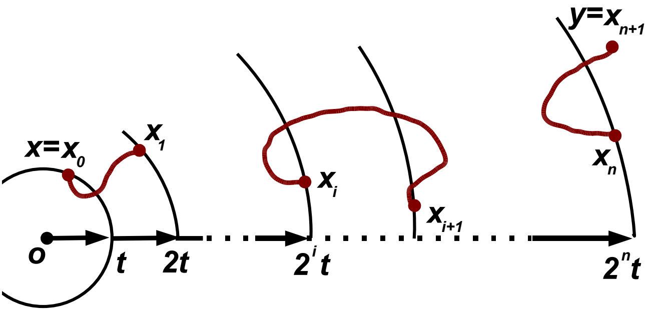

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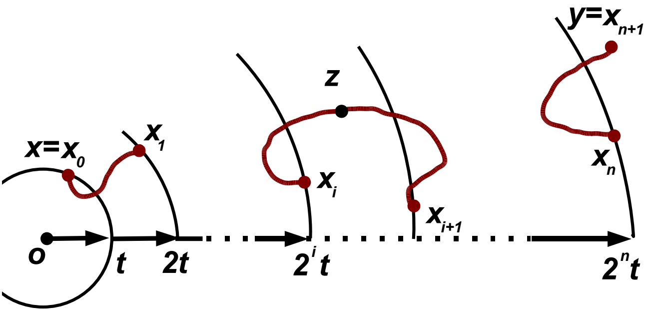

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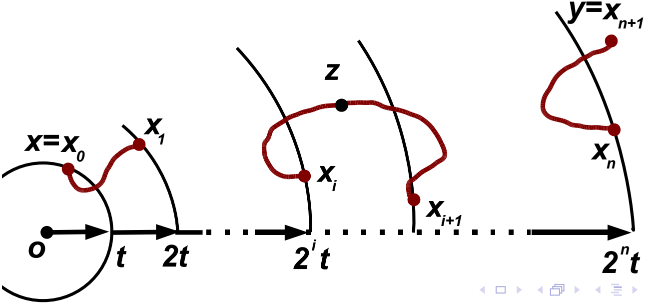

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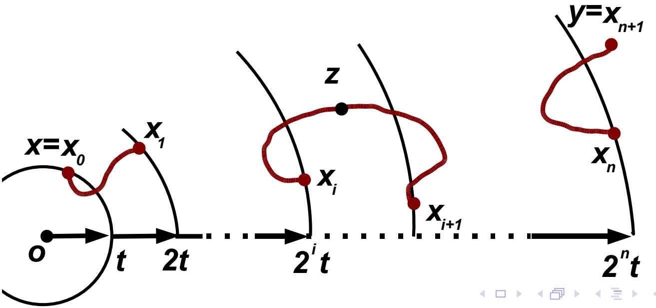

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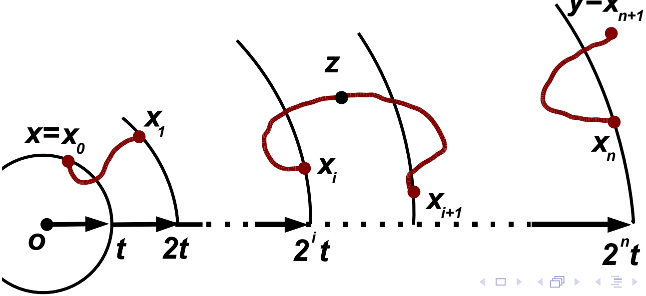
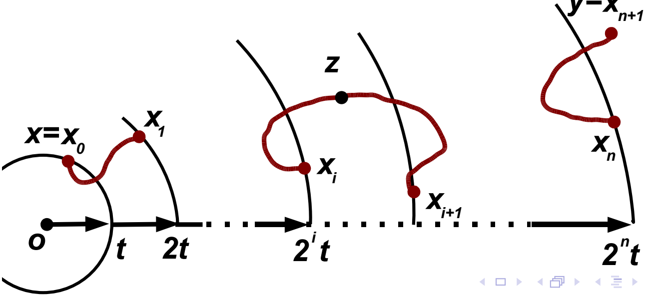


Ideas in Proof of Ω Uniform $\implies$ $I_o(\Omega)$ Uniform

Remains to show double cone condition for $z \in \gamma_i$. Have $|z| \simeq 2^i t$ and $\delta(z) \gtrsim \ell(\gamma[x,z]) \wedge \ell(\gamma[y,z]) \gtrsim 2^i t$, so $\delta_o(z) \gtrsim \frac{1}{|z|} \wedge \frac{\delta(z)}{|z|^2} \gtrsim \frac{1}{2^i t}$.

Then

$$\ell_o(\gamma[x,z]) \wedge \ell_o(\gamma[y,z]) \le \ell_o(\gamma[y,z]) \lesssim \sum_i^n \frac{1}{2^j t} \simeq \frac{1}{2^i t} \lesssim \delta_o(z)\,.$$



Outline

1 Introduction
- Euclidean Inversions
- Notation

2 Metric Space Inversions
- Definitions
- Properties

3 Inversions and QuasiConvexity
- Annular QuasiConvexity
- Preservation of QuasiConvexity

4 Inversions and Uniformity
- Uniform Spaces
- Preservation of Uniform SubSpaces

5 Ptolemaic Spaces

When is i_o a metric?

$i_o(x,y) = \dfrac{|x-y|}{|x||y|}$ always positive definite, symmetric; triangle inequality?

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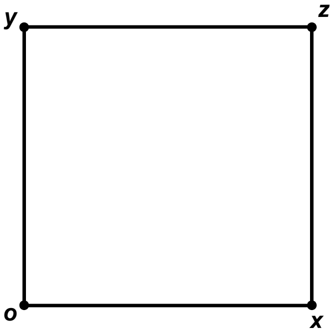
Look at $X = \mathbb{R}^2$ with ℓ^1 metric $|(x_1,y_1)-(x_2,y_2)|_1 = |x_1-y_1|+|x_2-y_2|$.

For $o=(0,0)$, $x=(1,0)$, $y=(0,1)$, $z=(1,1)$ get

$i_o(x,y)=2$, but $i_o(x,z)=\frac{1}{2}=i_o(y,z)$.



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Call X a *Ptolemaic space* in this setting; so X is Ptolemaic iff i_o is a distance function for all base points o.

CAT(0) Spaces

Ptolemaic and *CAT(0) spaces* are related.
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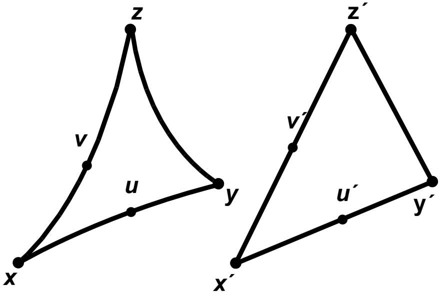
A *geodesic space* is *CAT(0)* if every *geodesic triangle* is at least as *thin* as a comparison triangle in the Euclidean plane: in below diagram

$$|u - v| \leq |u' - v'|.$$



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- For Riemannian manifolds:
  Ptolemaic $\Leftrightarrow$ CAT(0) $\Leftrightarrow$ simply conn. + sect. curv. ≤ 0
- Moreover,
  $$\forall\, o \in X : \ i_o \text{ is a length metric} \Longleftrightarrow X \text{ is Euclidean} \ .$$
  So, in manifold setting,
  inversion gives a simple characterization of Euclidean spaces.

Examples

- There is a Ptolemaic space X that cannot be isometrically imbedded into any CAT(0) space: Consider $X = \{(0,0),(0,1),(1,1),(1,2)\}$ with ℓ^∞ distance

$$|(x_1,y_1),(x_2,y_2)|_\infty = \max\{|x_1-y_1|,|x_2-y_2|\}\,.$$

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- BFW do not have an example of a geodesic Ptolemaic space that fails to be CAT(0).

Summary

- In a general metric space, we can define inversion wrt a base pt.
- This inversion preserves many nice geometric properties.
- Inversion-Sphericalization provides a handy tool to transform bdd/unbdd spaces to unbdd/bdd.

6 Appendix
- Diameter Estimates
- Quasihyperbolic Distance Estimates
- BiLipschitz Constants
- Uniformity Constants

Outline

6 Appendix

- Diameter Estimates
- Quasihyperbolic Distance Estimates
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- Uniformity Constants

Diameters of $In_o(X)$ and $\mathrm{Sph}_o(X)$

Recall that $Y = \mathrm{Inv}_o(X)$ is bdd iff o is an isolated pt in X. In this case we get

$$\frac{\operatorname{diam} X_o}{\operatorname{dist}(o, X_o) + \operatorname{diam} X_o} \frac{1}{8 \operatorname{dist}(o, X_o)} \leq \operatorname{diam}_o \mathrm{Inv}_o(X) \leq \frac{2}{\operatorname{dist}(o, X_o)} .$$

We also have

$$\frac{1}{4} s_o(x,y) \leq \hat{d}_o(x,y) \leq s_o(x,y) \leq \frac{1}{1+|x|} + \frac{1}{1+|y|} .$$

$$\hat{\operatorname{diam}}_o \mathrm{Sph}_o(X) \leq 1 \quad \text{and} \quad \hat{\operatorname{diam}}_o \mathrm{Sph}_o(X) \geq \begin{cases} \hat{d}_o(o, \hat{o}) \geq \frac{1}{4} & X \text{ unbdd} , \\ \frac{1}{4} \frac{\operatorname{diam} X}{2 + \operatorname{diam} X} & X \text{ bdd} . \end{cases}$$

Basic Properties of QuasiHyperbolic Distance

Easy to check that for any rectifiable curve γ,

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From this deduce basic (lower) estimates

$$k(x,y) \geq \log\left(1 + \frac{\ell(x,y)}{d(x)\wedge d(y)}\right) \geq j(x,y)$$
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Fact

If (Ω, d) locally compact, then (Ω, k) proper and geodesic provided $(\Omega, \ell) \xrightarrow{\text{id}} (\Omega, d)$ is homeo.

The BiLipschitz Constants

For $(\Omega, k) \xrightarrow{\mathrm{id}} (\Omega, k_o)$, get BL constant $M = 2\,c(a \vee 20\,b)$ where

$$a = \begin{cases} 1 & \text{if } \Omega \text{ is unbounded}\,, \\ \operatorname{diam}\Omega / [\operatorname{dist}(o, \partial\Omega) \vee (\operatorname{diam}\partial\Omega/2)] & \text{if } \Omega \text{ is bounded}\,. \end{cases}$$

and

$$b = \begin{cases} b' & \text{if } X \text{ is } b'\text{-quasiconvex}\,, \\ 1 & \text{if } o \in \partial\Omega\,, \\ 2\operatorname{dist}(o, \partial\Omega)/\operatorname{dist}(o, \Omega) & \text{if } o \notin \bar{\Omega}\,. \end{cases}$$

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For $(\Omega, k) \xrightarrow{\text{id}} (\Omega, \hat{k}_o)$, Ω unbounded and $o \in \partial\Omega$, $M = 40\,c$.

Examples illustrate M may depend on quantities indicated above.

General Uniformity Constants

For Ω and $S_o(\Omega)$:

$$C = C(A,o) = c(A)[1+\operatorname{dist}(o,\partial\Omega] \quad \text{and} \quad A = A(C)\ (\Omega \text{ unbdd}).$$

For Ω and $I_o(\Omega)$:

$$B = B(A,o) = b(A)[1+r] \quad \text{and} \quad A = A(B,d) = a(B)d$$

where

$$r = r(o) = \begin{cases} \operatorname{dist}(o,\partial\Omega)/\operatorname{dist}(o,\Omega) & o \in X \setminus \bar{\Omega} \\ 0 & o \in \partial\Omega \end{cases}$$

$$d = \begin{cases} 1 & \Omega \text{ unbdd} \\ \operatorname{diam}\Omega/\operatorname{diam}\partial\Omega & \Omega \text{ bdd} \end{cases}$$