QUASIEXTREMAL DISTANCE SETS

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Abstract. We present an in-depth analysis of plane continua which enjoy a quasiextremal distance property.

1. Introduction

Quasidisks in the plane can be characterized as those simply connected plane domains in which the extremal distance between two disjoint continua is comparable to the extremal distance now measured in the whole plane. For Jordan domains it suffices that this condition hold merely for continua which lie on the domain's boundary. This article is a preliminary report on our investigation of this 'boundary phenomenon'.

Motivated by Ahlfors and Beurling's study [AB50] of NED sets (null sets for extremal distance), Gehring and Martio [GM85] examined the closed sets $A \subset \bar{\mathbf{R}}^n$ whose complements $A^c = \bar{\mathbf{R}}^n \setminus A$ satisfy

$$\operatorname{mod}(E, F; \bar{\mathbf{R}}^n) \leq M \operatorname{mod}(E, F; A^c) \tag{QED}$$

for disjoint continua $E, F \subset A^c$. (Here mod denotes the conformal modulus; see Section 2 for basic definitions, notation and terminology.) In particular they showed that such sets A cannot separate space, so A^c is a domain, and they described this situation by declaring A^c to be a *quasiextremal distance*, or M-QED, domain. Herron and Koskela [HK90, 2.8] demonstrated that the above modulus inequality actually holds for disjoint continua in the closure—and hence on the boundary—of a QED domain, and introduced the class of *boundary quasiextremal distance*, or BQED, domains for which the modulus inequality (QED) is only required to hold for disjoint continua on the domain's boundary. Simple examples show that the class of QED domains is a proper subclass of the BQED domains, however, under appropriate topologic and/or geometric conditions a BQED domain is QED; see [HK91, 7.11].

We call a closed set $A \subset \bar{\mathbf{R}}^n$ an *M-QED compactum* if (QED) holds for all disjoint continua $E, F \subset A$. For example, the boundary of any M-BQED domain is an M-QED compactum. However, unlike the Gehring-Martio situation, QED compacta may very well separate space; e.g., any sphere is a 1-QED compactum. Another difference is that every QED compactum must have empty interior (consequently the complement of a QED compactum A is a QED domain if and only if A is an NED set). Nevertheless, we shall see that QED compacta do enjoy some of the same properties that hold for the boundaries of QED domains. For example, Gehring and Martio verified that QED domains enjoy a boundary measure density condition, and an analogous property holds for QED compacta, at least when we rule out degenerate boundary components.

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Obviously any closed totally disconnected set is a QED compactum. For this reason we focus our attention on *QED continua* which are the non-degenerate connected QED compacta. While every non-degenerate component of a QED compactum is a QED continuum, it is easy to see that there are closed sets A each of whose components is a non-degenerate QED continuum, but A itself is not QED (e.g. take a union of three concentric spheres).

As we have already mentioned, QED compacta can separate space. In particular, the topology of these sets is typically more complex than for the boundaries of QED domains. For this reason, we concentrate our study on QED compacta and continua in the plane. Our main result describes the possible topology and geometry for a general QED plane continuum.

Main Theorem. *For an arbitrary QED plane continuum A, exactly one of the following conditions holds.*

(a) *A is a QED arc or QED tree and A^c is a non-QED BQED disk.*

(b) *A is a QED circle, hence a quasicircle, and A^c consists of two quasidisks.*

(c) *A is a QED theta, therefore a quasitheta, and A^c consists of three quasidisks.*

(d) *A is a 'non-elementary' QED continuum and $A^c = D \cup \bigcup_i D_i$, D_i are quasidisks, D is a distinguished component of A^c with $A = \partial D$. Furthermore, we can partition A into countably many non-degenerate (essentially disjoint) QED circles $C_i = \partial D_i$, QED arcs or trees $H \in \mathcal{H}_i$ (where $\mathcal{H}_i$ is the family of 'hairs' attached to C_i) and 'connecting' QED arcs or trees E_j, so that*

$$A = \partial D = \bigcup_i \left(C_i \cup \bigcup_{H \in \mathcal{H}_i} H \right) \cup \bigcup_j E_j \cup S,$$

where S is the (possibly uncountable) set of points w for which $\{w\}$ is a component of $A \setminus \cup C_i$. Everywhere above the quasicircle, quasidisk and quasitheta constants depend only on the QED constant.

Section 3 contains some basic properties of general QED compacta. In Section 4 we investigate the topological structure of QED plane compacta; for example, every component is locally connected. We give a detailed description for the boundaries of BQED disks and then utilize this information to explain the topology of general QED plane continua. In Section 5 we analyze the geometry of these sets. We report several results which describe the relative positions of disjoint continua in a QED plane compactum. Also, we present a proof of our main theorem. We conclude in Section 6 by exhibiting numerous examples which illustrate our results.

2. Preliminaries

Our notation is relatively standard and, for the most part, conforms with that of Väisälä's notes [Väi71]. Euclidean n-dimensional space is denoted $\mathbf{R}^n$ and $\bar{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ is its one-point compactification. We let $B(x;r) = \{y : |x-y| < r\}$ and $S(x;r) = \partial B(x;r)$ denote the open ball and sphere of radius r centered at the point x. We employ the abbreviations $B(r) = B(0;r)$, $\mathbf{B}^n = B(1)$, $S(r) = S(0;r)$ and $\mathbf{S}^{n-1} = S(1)$. The volume, surface measure of $\mathbf{B}^n$, $\mathbf{S}^{n-1}$are Ω_n, ω_{n-1}. We write $c = c(a,\dots)$ to indicate a constant c which depends only on the parameters $a,\dots$. We write $a \approx b$ to mean there exists a positive finite constant c with $a/c \le b \le a\,c$; typically c will depend on various parameters, and we try to make this as clear as possible often giving explicit values.

2.A. **Conformal Modulus.** The *conformal modulus* of a pair of disjoint, compact sets $E, F \subset \bar{D}$, relative to a domain $D \subset \mathbf{R}^n$, is

$$\operatorname{mod}(E, F; D) = \operatorname{mod} \Delta(E, F; D)$$

where $\Delta(E, F; D)$ is the family of all curves joining the sets E, F in D and $\operatorname{mod}(\Gamma)$ is the usual modulus of the curve family Γ (see [Väi71, p.16]. For the reader's convenience, we cite the following geometric estimates; these are based on the behavior of the family of all curves joining the sets E, F; see [Väi71], [Vuo88].

2.1. Fact. *Let E, F be disjoint compacta in $\bar{\mathbf{R}}^n$.*

(a) *If E, F are separated by the spherical ring $B(x; s) \setminus \bar{B}(x; t)$, then*

$$\operatorname{mod}(E, F; \bar{\mathbf{R}}^n) \leq \omega_{n-1} \left(\log \frac{s}{t} \right)^{1-n}.$$

(b) *If $E \cap S(x; r) \neq \emptyset \neq F \cap S(x; r)$ for all $t < r < s$, then*

$$\operatorname{mod}(E, F; \bar{\mathbf{R}}^n) \geq \sigma_n \log \frac{s}{t}.$$

(c) *If E, F are connected, then*

$$\operatorname{mod}(E, F; \bar{\mathbf{R}}^n) \geq \sigma_n \log \left(1 + \min\{\operatorname{diam}(E), \operatorname{diam}(F)\} / \operatorname{dist}(E, F)\right).$$

(d) *(Comparison Principle) For compacta $E, F, A, B \subset D$,*

$$\operatorname{mod}(E, F; D) \geq 3^{-n} \min\{\operatorname{mod}(E, A; D), \operatorname{mod}(F, B; D), I\},$$

where

$$I = \inf\{\operatorname{mod}(\alpha, \beta; D) : \alpha \in \Delta(E, A; D), \beta \in \Delta(F, B; D)\}.$$

Here σ_n is the spherical cap constant and ω_{n-1} is the $n-1$-measure of the unit sphere; $\sigma_2 = 2/\pi$.

2.B. **Topology.** We collect some basic topological definitions and facts. All notions are with respect to $\bar{\mathbf{R}}^n$, so for example by a *compactum* we mean a non-empty closed subset of $\bar{\mathbf{R}}^n$, and a *continuum* is then a non-degenerate connected compactum. Next, a *Jordan curve* (or *arc*) is a homeomorphic image of a circle (or an interval—so an arc may be closed or open or neither), and a Jordan curve may go through the point at infinity. Recall that a *theta* curve is the union of three closed Jordan arcs which intersect by pairs in just their endpoints [Why64, p.31].

We call x a *cutpoint* of a set A if $A \setminus \{x\}$ is not connected. We say that A is *finitely connected at x* if there are arbitrarily small neighborhoods U of x such that $U \cap A$ has finitely many components. We say that A is *m-connected at x* if m is the smallest integer with the property that there are arbitrarily small neighborhoods U of x such that $U \cap A$ has m components. See Näkki's thesis [Näk70] for more information about these concepts.

Recall that a space is *locally connected* provided each point has arbitrarily small connected neighborhoods. For later purposes we remind the reader that, for compacta in metric spaces, this is equivalent to requiring that for each $\varepsilon > 0$ there is a $\delta > 0$ such that all points within distance δ of each other can be joined, in the set, by a connected set of diameter at most ε. We refer the interested reader to the books of Newman [New51], Whyburn [Why64], and Kuratowski [Kur68]; see also [Pom92]. Here is a partial list of properties enjoyed by locally connected plane continua.

2.2. Fact. *Let $A \subset \bar{\mathbf{R}}^2$ be a locally connected continuum. Then*

(a) *A is a continuous curve (there is a continuous map from $[0,1]$ onto A) and any pair of points in A can be joined by a Jordan arc in A.*
(b) *A separates two points if and only if A contains a Jordan curve which separates these points.*
(c) *Every component D of A^c satisfies:*
  1. *D is simply connected and finitely connected at each boundary point.*
  2. *∂D and $\bar{D}$ are locally connected continua.*
  3. *When A has no cut points, ∂D is a Jordan curve.*
  4. *Each point of ∂D is accessible from D.*
  5. *Any Riemann map from the unit disk to D has a continuous extension to the closed unit disk.*

According to [Kur68, Thm1,p.172] we have the following result concerning the components of the intersection of a continuum and an open set.

2.3. Fact. *Let $K \subset \mathbf{R}^n$ be a continuum and $U \subset \mathbf{R}^n$ be open. Suppose $K \setminus U \neq \emptyset$. Then each component of $K \cap \bar{U}$ meets ∂U.*

This yields information about continua which join disjoint sets. We utilize this in the special case of plane continua which join the boundary circles of an annulus.

2.4. Lemma. *Let $U = \mathbf{R}^n \setminus (C_1 \cup C_2)$ where C_1, C_2 are disjoint closed sets in $\mathbf{R}^n$. Suppose $K \subset \mathbf{R}^n$ is a continuum which joins C_1 and C_2. Then some component of $K \cap \bar{U}$ also joins C_1 and C_2.*

Proof. Let $U_i = \mathbf{R}^n \setminus C_i$, so $U = U_1 \cap U_2$. Assume first that $K \subset \bar{U}_1$. If $K \subset \bar{U}_2$, then $K \subset \bar{U}$ and we are done; suppose $K \setminus \bar{U}_2 \neq \emptyset$. Fix a point $x_1 \in K \cap C_1$ and let K_1 be the component of $K \cap \bar{U}_2$ which contains x_1. According to Fact 2.3, K_1 meets ∂U_2, so K_1 is a component of $K \cap \bar{U}$ which meets both C_1 and C_2. A similar argument applies when $K \subset \bar{U}_2$.

Assume that $K \setminus \bar{U}_1 \neq \emptyset \neq K \setminus \bar{U}_2$. Fix a point $x_2 \in K \cap C_2$ and let K_2 be the component of $K \cap \bar{U}_1$ which contains x_2. Now apply the above reasoning to the continuum $K_2 \subset \bar{U}_2$ which (by Fact 2.3) joins C_2 and ∂U_1. $\square$

We also require the following 'obvious' result.

2.5. Lemma. *Let K_1, K_2, K_3, K_4 be disjoint continua in the closed annulus $A = \bar{B}(r_2) \setminus B(r_1)$ each of which joins the boundary circles $C_1 = S(r_1)$ and $C_2 = S(r_2)$. Then we can label these continua so that K_1 and K_3 are separated in A by $K_2 \cup K_4$ and vice versa.*

Proof. Thanks to a result of Mort Brown [Bro72] we can choose $0 < r < \delta/10$, where $\delta = \min_{i \neq j} \operatorname{dist}(K_i, K_j)$, so that each component of $S(K_i; r) = \{z : \operatorname{dist}(z, K_i) = r\}$ is either a point, a Jordan arc, or a Jordan curve. Let J_i denote the 'outer' component of $S(K_i; r)$; i.e., J_i is the boundary of the unbounded component of $\mathbf{R}^2 \setminus \bar{B}(K_i; r)$. Then J_i is a Jordan curve bounding a Jordan disk D_i which contains K_i and $\bar{D}_i \cap \bar{D}_j = \emptyset$ when $i \neq j$.

Let K be one of the continua K_i and $J = J_i$. There are exactly two components α, β of $J \cap A$ which join C_1, C_2. Indeed, if $x_i \in K \cap C_i$, then each of $z_1 = (|x_1| - r)x_1/|x_1|$, $z_2 = (|x_2| + r)x_2/|x_2|$ belongs to J, so each component of $J \setminus \{z_1, z_2\}$ joins C_1, C_2 and Lemma 2.4 gives us arcs $\alpha, \beta \subset J \cap A$ which join C_1, C_2. Let G be the component of $A \setminus (\alpha \cup \beta)$ which contains K and for $k = 1, 2$ let I_k be the component of $C_k \setminus (\alpha \cup \beta)$ which contains $K \cap C_k$. If J contained a third subarc γ joining C_1, C_2, then $\Theta = \alpha \cup I_1 \cup \gamma \cup I_2 \cup \beta$ would be a theta-curve, which in turn would mean that K was in one of the three complementary regions determined by Θ which is impossible.

Thus $\Gamma = \partial G = \alpha \cup I_1 \cup I_2 \cup \beta$ is a Jordan curve with $K \subset \bar{G}$. Let K' be one of the other K_i. Note that $K' \cap (\bar{B}(\alpha \cup \beta; r)) = \emptyset$. Suppose K' contains a point of $\bar{G}$. Then by connectedness we must get $K' \subset \bar{G}$. But then the corresponding subarcs α', β' of J' would be crosscuts of G which separate α, β, but this would mean that $\alpha' \cup \beta'$ would separate K either from α or from β. It follows that $K' \cap \bar{G} = \emptyset$.

We see now how to associate with each continua K_i a Jordan domain G_i with the properties that G_i joins C_1 and C_2, $K_i \subset \bar{G}_i$, and $G_i \cap G_j = \emptyset$ when $i \neq j$. Clearly the domains G_i can be labeled as desired, and this provides the asserted labeling for the continua K_i. □

3. General Properties of QED Compacta

In this Section we collect some properties enjoyed by all QED compacta. We begin with a few elementary thoughts, then we examine a useful measure-theoretic condition, present a simple geometric property, and last we discuss a decomposition result for BQED domains.

First, it is apparent that QED compacta are invariant with respect to Möbius transformations: Every Möbius image of an M-QED compactum is again M-QED. Thanks to this property we are free to use various normalizations, e.g. such as assuming that the point at infinity is, or is not, in the set. It is also clear that any closed subset of a QED compacta is again a QED compacta, and similarly for continua.

Gehring and Martio established a boundary measure density condition for QED domains which guarantees that the boundary of such a domain has n-measure zero [GM85, 2.13,2.16]. Such a result fails even for BQED domains because, e.g., there are totally disconnected closed sets which have positive n-measure. However, a modification of their proof does yield a similar, although restricted, result for QED compacta. We thank Pekka Koskela for pointing this out to us.

3.1. Lemma. *Let C be a non-degenerate component of an M-QED compactum $A \subset \bar{\mathbf{R}}^n$. Then for all $x \in C \setminus \{\infty\}$ and $0 < r \leq \operatorname{diam}(C)$,*

$$|B(x;r) \setminus A| \geq \frac{a}{M}|B(x;r)|,$$

where $a = a(n) > 0$ and $|E|$ is the Lebesgue n-measure of E.

Proof. Fix $x \in C$, $0 < r \leq \operatorname{diam}(C)/2$ and select a point $y \in C$ with $|x-y| \geq r$. Let E and F be the components of $C \cap \bar{B}(x;r/3)$ and $C \setminus B(x;2r/3)$ containing x and y respectively. Appealing to Fact 2.1(c) we have

$$\operatorname{mod}(E,F;\bar{\mathbf{R}}^n) \geq \mu = \mu(n) = \sigma_n \log(3/2).$$

Since each curve joining E to F in $\bar{\mathbf{R}}^n \setminus A$ has length at least $r/3$, we find that the density $\rho = 3/r$ in $B(x;r) \setminus A$ (and $\rho = 0$ elsewhere) is admissible and so

$$\mu \leq M \operatorname{mod}(E,F;\bar{\mathbf{R}}^n \setminus A) \leq \int \rho^n = M(3/r)^n |B(x;r) \setminus A|,$$

which yields the asserted conclusion with $a = \mu/3^n\Omega_n$. □

3.2. Corollary. *Let A be a QED compactum in $\bar{\mathbf{R}}^n$ and let B be the set of all points $x \in A$ with $\{x\}$ a component of A. Then $A \setminus B$ has n-measure zero.*

Proof. By Lebesgue's theorem, almost every point of A is a point of density for A. However, any point of $A \setminus B$ belongs to a non-degenerate component of A, and according to Lemma 3.1 such a point cannot be a point of density for A. $\square$

Here is a geometric condition satisfied by any QED continuum.

3.3. Lemma. *Let A be an M-QED continuum in $\bar{\mathbf{R}}^n$. Fix $0 < \lambda < 1/2$, points $x, y \in A \setminus \{\infty\}$, and let X, Y be the components of $A \cap \bar{B}(x; \lambda d)$, $A \cap \bar{B}(y; \lambda d)$ respectively where $d = |x - y|$. Then there is a constant $a = a(\lambda, M, n)$ such that*

$$\delta_A(X, Y) \le a|x - y|,$$

where

$$\delta_A(X, Y) = \inf\{\operatorname{diam}(\gamma) : \gamma \in \Delta(X, Y; A^c)\}.$$

Proof. Since both X, Y have diameter at least λd and $\operatorname{dist}(X, Y) \le d$, Fact 2.1(c) asserts that $\operatorname{mod}(X, Y; \bar{\mathbf{R}}^n) \ge \mu = \mu(\lambda, n) = \sigma_n \log(1 + \lambda)$. Suppose that each (locally rectifiable) $\gamma \in \Delta(X, Y; A^c)$ has $\operatorname{diam}(\gamma) \ge r > 2d/3$. Then each such γ leaves the ball $B(x; r/2)$, so it must join the spheres $S(x; d/3)$, $S(x; r/2)$. According to Fact 2.1(a),

$$\operatorname{mod}(X, Y; A^c) \le \omega_{n-1} \left(\log \frac{r/2}{d/3}\right)^{1-n},$$

and consequently, since A is M-QED, we obtain $r \le a\, d$ where $a = 2\lambda \exp\left((\omega_{n-1} M/\mu)^{1/(n-1)}\right)$. $\square$

We conclude this section with a 'uniform decomposition' result which is useful for generating examples (cf. Example 6.2).

3.4. Proposition. *Let $D \subset \bar{\mathbf{R}}^n$ be a domain and suppose there are uniform domains $G_j \subset D$, $0 \le j \le N$, satisfying*

$$\partial D \subset \bigcup_1^N \partial G_j \quad \textit{and} \quad \overline{\partial D \setminus \partial G_0} \subset \partial G_j \quad \textit{for all } 1 \le j \le N.$$

Then D is $2(N+1)M$-BQED where M depends only on the uniformity constants.

Proof. Since each G_j is a uniform domain, there is a constant M_j, which depends only on the uniformity constant for G_j, so that

$$\operatorname{mod}(E, F; \bar{\mathbf{R}}^n) \le M_j \operatorname{mod}(E, F; G_j)$$

for disjoint *compacta* $E, F \subset \bar{G}_j$ (see [GM85, 2.18]). Put $M = \max\{M_0, \dots, M_N\}$. Fix disjoint continua $E, F \subset \partial D$ and let

$$E_j = E \cap \partial G_j, \quad F_j = F \cap \partial G_j, \quad E' = \overline{E \setminus E_0} \quad F' = \overline{F \setminus F_0}.$$

Note that $E = \cup_1^N E_j$ and $F = \cup_1^N F_j$. According to our hypotheses, $E' \subset \partial G_j$, so $\operatorname{mod}(E', F_j; \bar{\mathbf{R}}^n) \le M_j \operatorname{mod}(E', F_j; D)$.

First,

$$\operatorname{mod}(E', F; \bar{\mathbf{R}}^n) \le \sum_1^N \operatorname{mod}(E', F_j; \bar{\mathbf{R}}^n) \le NM \operatorname{mod}(E, F; D).$$

Second,

$$\operatorname{mod}(E_0, F; \bar{\mathbf{R}}^n) \le \operatorname{mod}(E_0, F_0; \bar{\mathbf{R}}^n) + \operatorname{mod}(E_0, F'; \bar{\mathbf{R}}^n) \le$$
$$\le \sum_0^N \operatorname{mod}(E_j, F'; \bar{\mathbf{R}}^n) \le (N+1)M \operatorname{mod}(E, F; D).$$

Finally, in general we have

$$\operatorname{mod}(E, F; \bar{\mathbf{R}}^n) \le \operatorname{mod}(E_0, F; \bar{\mathbf{R}}^n) + \operatorname{mod}(E', F_0; \bar{\mathbf{R}}^n) +$$
$$+ \operatorname{mod}(E', F'; \bar{\mathbf{R}}^n) \le 2(N+1)M \operatorname{mod}(E, F; D);$$

here the first term is estimated as in case two above, the second term is dealt with as in case one above, and the last term is easily handled since E', F' belong to each of ∂G_j. □

4. Topology of QED Plane Compacta

Here we focus our attention on the topology of QED compacta in the plane. First we look at some general properties, then we give a comprehensive analysis of possible topologies for BQED disks, and finally we examine the Jordan curves and arcs in a QED plane continuum.

4.A. **General Properties.** We begin by showing that QED plane continua are locally connected. To see that this is false for QED compacta, consider disjoint circles which converge to some point. Then we verify that each complementary component of a QED plane continuum is at worst 3-connected on the boundary.

4.1. Theorem. *A QED continuum in $\bar{\mathbf{R}}^2$ is locally connected.*

Proof. Suppose a QED continuum A in $\bar{\mathbf{R}}^2$ fails to be locally connected at some point, say $a = 0$. This means that there is an $\varepsilon > 0$ such that for each $\delta > 0$ we can find a point $x \in A \cap B(\delta)$ with $x \notin A_0$, where A_0 is the component of $A \cap \bar{B}(\varepsilon)$ containing $a = 0$. We may assume that $A \setminus B(\varepsilon) \ne \emptyset$, so by Fact 2.3 we know that each component of $B = A \cap \bar{B}(\varepsilon)$ meets the circle $S(\varepsilon) = \partial B(\varepsilon)$.

Let $r_n = \varepsilon/n$. Select a point $x_1 \in A \cap B(r_1)$ with $x_1 \notin A_0$. Let C_1 be the component of B containing x_1; C_1 joins $S(r_1)$ to $S(\varepsilon)$. Since C_1 is closed, there must be a point $x_2 \in A \cap B(r_2)$ with $x_2 \notin A_0 \cup C_1$. The component C_2 of B containing x_2 satisfies $C_2 \cap C_1 = \emptyset = C_2 \cap A_0$ and C_2 joins $S(r_2)$ to $S(\varepsilon)$. Continuing this procedure we obtain components C_i of B which join $S(r_i)$ to $S(\varepsilon)$ and satisfy $C_i \cap C_j = \emptyset = C_i \cap A_0$ for all $i \ne j$.

Fix n and consider $A_0, C_n, C_{n+1}, C_{n+2}$; these four disjoint continua all join $S(r_n)$ to $S(\varepsilon)$. Appealing to Proposition 5.1 we deduce that $\varepsilon \le c\, r_n$ where c depends only on the QED constant for A. Since $r_n \to 0$, we have arrived at a contradiction; therefore we conclude that A must in fact be locally connected. □

We now have a list of nice properties for QED plane continua (see Fact 2.2) which in turn can be utilized to ascertain additional properties.

4.2. Corollary. *Let A be a QED continuum in $\bar{\mathbf{R}}^2$. Then for each $a \in A$, there are at most three components of $A \setminus \{a\}$.*

Proof. Suppose $A \setminus \{a\}$ has distinct components C_1, C_2, C_3, C_4 for some point $a \in A$. Choose points $x_i \in C_i$ with $|x_i - a| \ge r_i = \operatorname{diam}(C_i)/2 > 0$. According to Fact 2.2(a) there are Jordan arcs $\alpha_i \subset A$ joining x_i and a. We must have $\alpha_i \subset C_i \cup \{a\}$. Now for all $0 < \varepsilon < r = \min\{r_1, r_2, r_3, r_4\}$, each arc α_i joins the circles $S(a;\varepsilon)$, $S(a;r)$, so an appeal to

Proposition 5.1 yields $r \le c\varepsilon$ where $c = c(M)$. Letting $\varepsilon \to 0$ produces the contradiction $r = 0$. □

Note that the complement of a QED continuum need not be finitely connected at some boundary points. For example, if A is the union of a sequence of spheres which shrink rapidly to the origin, together with appropriate connecting line segments, then A is a QED continuum, but A^c is not finitely connected at the origin. Nevertheless, Lemma 3.3 (for $n = 2$) and Corollary 4.2 both indicate that the components of the complement of QED plane continua must satisfy some topological requirement.

4.3. Theorem. *Each component of the complement of a QED continuum in $\bar{\mathbf{R}}^2$ is at worst 3-connected at each of its boundary points.*

Proof. Let D be a component of the complement of some QED plane continuum. Let U be a neighborhood of some boundary point $w \in \partial D$. Fix $r > 0$ so that $B(w; r) \subset U$ and $\partial D \cap S(w; r) \neq \emptyset$. Note that $\partial D \cup S(w; r)$ is locally connected.

First we explain why we can write

$$D \cap B(w; r) = V_1 \cup \cdots \cup V_m \cup \bigcup U_j$$

where $w \in \partial V_i$, $w \notin \bar{U}_j$, $\operatorname{dist}(w, U_j) \ge s > 0$. By Fact 2.2(c), D is finitely connected at w, and hence according to [Näk70, Thm1.11] there is a neighborhood $W \subset B(w; r)$ of w such that $W \cap D = W_1 \cup \cdots \cup W_p$ with $w \in \partial W_k$ for $k = 1, \ldots, p$. Choose $s > 0$ so that $B(w; s) \subset W$. If V_i is any component of $D \cap B(w; r)$ which contains points of $B(w; s)$, then V_i must contain some W_k, so $w \in \partial V_i$ and there are a finite number of such V_i. Every other component U_j of $D \cap B(w; r)$ must lie outside $B(w; s)$.

Next, $V = B(w; s) \cup (V_1 \cup \cdots \cup V_m)$ is a neighborhood of w, $V \subset U$, and $V \cap D = V_1 \cup \cdots \cup V_m$. It remains to certify that $m \le 3$. Since V_i is a complementary domain of the locally connected continuum $\partial D \cup S(w; r)$, its boundary points are accessible and so there is a Jordan arc $\gamma_i \subset V_i$ which joins w to a point of $D \cap \partial V_i \cap S(w; r)$ (cf. Fact 2.2(c)). For $i \neq j$ γ_i and γ_j are separated by $\partial D \cup S(w; r)$, so by Fact 2.2(b) there are Jordan curves C_{ij} in $\partial D \cup S(w; r)$ which separate γ_i, γ_j. Some of these C_{ij} may 'overlap' or even coincide, but we must get m distinct Jordan arcs in ∂D which join w to $S(w; r)$. The fact that ∂D lies in a QED continuum now guarantees that $m \le 3$. □

4.B. **BQED Disks.** Now we present a detailed topological description for the possible boundaries of *BQED disks* (simply connected plane BQED domains). Subsection 4.C contains a similar analysis for general QED plane continua.

4.4. Lemma. *Let D be a BQED disk. Then either $\bar{D} = \bar{\mathbf{R}}^2$, or ∂D is a Jordan curve.*

Proof. If ∂D contains no Jordan curve, then it does not separate the plane [Why64, (2.3),p.33], so D is its only complementary component and thus $\bar{D} = \bar{\mathbf{R}}^2$. Suppose ∂D contains some Jordan curve C and there is some point $w \in \partial D \setminus C$. Select a Jordan arc $\alpha \subset \partial D$ which joins w to C. Let z be the first point of C encountered as α is traversed from w. It is easy to see that there are disjoint subarcs E, F of C on either side of z for which the condition (QED) is violated. □

Next we scrutinize the case where D is a BQED disk whose closure is the entire plane. Thus ∂D is a QED plane continuum which contains no Jordan curve and which has a connected complement. In particular we note that in this circumstance each pair of points on ∂D are joined by a unique Jordan arc in ∂D. Now according to Theorem 4.3 we know that D is at worst 3-connected at each of its boundary points; thus we can write $\partial D = B_1 \cup B_2 \cup B_3$ where

$$B_k = \{w \in \partial D : D \text{ is } k\text{-connected at } w\} \quad \text{for } k = 1, 2, 3.$$

We establish the following information regarding the sets B_k.

4.5. Theorem. *Let D be a BQED disk with $\bar{D} = \bar{\mathbf{R}}^2$. Then*

$$B_k = \{w \in \partial D : \partial D \setminus \{w\} \text{ has } k \text{ components}\} \quad \text{for } k = 1, 2, 3;$$

thus $B_2 \cup B_3$ is the set of cut points for ∂D. In addition:

(a-i) *B_1 contains at least two points, and exactly two points if and only if ∂D is a closed Jordan arc.*

(a-ii) *B_1 is totally disconnected, not necessarily: open, closed, discrete, nor countable.*

(a-iii) *A point w belongs to B_1 if and only if any Jordan arc $\alpha \subset \partial D$ which contains w has w as an endpoint.*

(b-i) *B_2 is dense in ∂D, may be totally disconnected, and need not be open nor closed.*

(b-ii) *Given any two points $x, y \in \partial D$ there is a point $w \in B_2$ which separates x, y in ∂D.*

(b-iii) *Each component of B_2 is either a Jordan arc in ∂D, or a singleton $\{w\}$. In the latter case, or when w is an endpoint of a component arc in B_2, w is a limit point of each of B_1, B_2, B_3.*

(c-i) *B_3 is empty if and only if ∂D is a Jordan arc, is countable and totally disconnected, but need not be: open, closed nor discrete.*

(c-ii) *A point w belongs to B_3 if and only if there are three Jordan arcs in ∂D each having w as an endpoint but no other points in common.*

Proof. We begin by verifying that $B_k = A_k$ where $A_k = \{w \in \partial D : \partial D \setminus \{w\} \text{ has } k \text{ components}\}$. Note that $\partial D = A_1 \cup A_2 \cup A_3$ by Corollary 4.2; also, A_1 consists of the non-cut points of ∂D. By [New51, 15.1,p.165], a simply connected domain is not 1-connected at cut points of its boundary, so $B_1 \subset A_1$. On the other hand, thanks to [NV91, 2.22], and the fact that D is at worst 3-connected at each of its boundary points, we find that the cut points of ∂D are precisely those points at which D fails to be 1-connected. Thus $B_2 \cup B_3 = A_2 \cup A_3$ and so $B_1 = A_1$. Clearly D cannot be 2-connected at any point of A_3, so $A_3 \subset B_3$ and $B_2 \subset A_2$.

To get to the heart of things: we confirm that $A_2 \subset B_2$. Let $w \in A_2$ and let U be a neighborhood of w. We claim that there exists a Jordan curve Γ in U which surrounds w and meets $A = \partial D$ in exactly two points; it follows that $V = \operatorname{int}(\Gamma)$ satisfies $w \in V \subset U$ and $V \cap D$ has two components, so $w \in B_2$. To find such a Jordan curve, consider a Riemann map $f : \mathbf{B} \to D$. Then as $\partial D \setminus \{w\}$ has two components, we know that $f^{-1}(w) = \{\zeta_1, \zeta_2\}$ for two distinct points $\zeta_1, \zeta_2 \in \mathbf{S}^1$ [Pom92, 2.5,p.23]. Choose $r > 0$ sufficiently small so that $\bar{B}(\zeta_1; r) \cap \bar{B}(\zeta_2; r) = \emptyset$ and $f(\bar{B}(\zeta_i; r) \cap \bar{\mathbf{B}}) \subset U$ for $i = 1, 2$. Then $\alpha_i = f(S(\zeta_i; r) \cap \mathbf{B})$ is a Jordan arc in D joining the two components of $\partial D \setminus \{w\}$, and $\alpha_1 \cap \alpha_2 = \emptyset$.

Next let β_i be a Jordan arc in ∂D joining the endpoints of α_i. Note that w must be an interior point of each arc β_i, so there is a closed Jordan arc $\beta \subset \beta_1 \cup \beta_2$ which lies in U and contains w as an interior point. Let w_1, w_2 be the endpoints of β; these belong to different components of $\partial D \setminus \{w\}$. Now choose circular arcs $\kappa_i \subset B(\zeta_i; r) \cap \mathbf{B}$ so that f maps the endpoints of κ_i to w_1, w_2. Then $\Gamma = f(\bar{\kappa}_1 \cup \bar{\kappa}_2)$ is our desired Jordan curve.

Now we establish the stated facts concerning B_1. First, appealing to [New51, Thm10.2,p.93 & Cor2,p.99] we deduce that B_1 always contains at least two points, and only two exactly when ∂D is a Jordan arc, so (a-i) holds. Next, it is easy to see that each point of B_1 has the property that it must be the endpoint of any arc in ∂D which contains it. On the other hand, for each point w of $B_2 \cup B_3$ we can find an arc in ∂D containing w as an interior point; thus (a-iii) follows. Clearly B need not be closed nor open nor discrete; it is totally disconnected by (b-ii) (proven below) and may be uncountable as illustrated by Example 6.3.

Next we demonstrate the basic fact (b-ii) concerning B_2. Let $x, y \in \partial D$. There is a unique arc $\alpha \subset \partial D$ joining x, y. Since the interior of α contains no B_1 points, and since B_3 is countable (proven below), we deduce that α must contain a point $w \in B_2$. As α is the only arc in ∂D joining x, y it follows that w separates x, y. In fact we see that α contains points of B_2 arbitrarily close to x, so B_2 is dense in ∂D. Example 6.4 elucidates the fact that B_2 may be totally disconnected, so (b-i) follows. It remains to clarify (b-iii). Certainly components of B_2 can be (open, closed or 'mixed') arcs on ∂D, or singletons. Fix a point $w \in B_2$ and let K be the component of B_2 containing w. Suppose $K \neq \{w\}$. Then any points $x, y \in K$ can be joined by an arc $\alpha \subset D$, and we find that $\alpha \subset B_2$ (because no interior point of α belongs to B_1 while if α contained a point of B_3 then x, y would be in different components of B_2), so we see that K contains arcs and in fact is equal to the maximal such arc. Now suppose that either $K = \{w\}$ or that w is an endpoint of K. Since B_2 is dense in ∂D, w is a limit point of B_2. By selecting points in different components of $\partial D \setminus \{w\}$ we obtain an arc $\alpha \subset \partial D$ with w an interior point of α. Note that α must contain points arbitrarily close to w which are not B_2 points. Since the interior points of α are not B_1 points, we see that w is a limit point of B_3 points. Finally, associated with each B_3 point z in the interior of α we have an arc γ_z which has an endpoint in B_1. As $z \to w$ along α we must have $\operatorname{diam}(\gamma_z) \to 0$ (This actually follows from Theorem 5.3!), and thus w is also a limit point of B_1.

At last we turn to the properties of B_3. When ∂D is a Jordan arc we have $B_3 = \emptyset$; the converse follows because then B_1 contains only two points. That B_3 is countable follows from the Moore Triod Theorem; e.g. see [Pom92, Cor2.19,p.37]. The rest of (c-i) is as above. Given a point $w \in B_3 = A_3$ we choose points x, y, z in distinct components of $\partial D \setminus \{w\}$ and join each of them to w by an arc. In this way we arrive at (c-ii). □

Based on the above information, we introduce the following terminology for the boundary $A = \partial D$ of a BQED disk D: we call A a *QED arc* or a *QED tree* according as to whether or not $B_3 = \emptyset$. We emphasize that this terminology includes the assumption that A is the boundary of a BQED disk whose closure is the entire plane; i.e., A is a QED plane continuum with a connected complement.

4.C. **QED Continua.** At this point we are ready to look at the topological structure of a general QED plane continua A. Notice that A contains no Jordan curves if and only if A is a QED arc or a QED tree. Thus we assume that A contains at least one Jordan curve, so A has at least two complementary components.

4.6. Lemma. *Let A be a QED plane compactum. Suppose A contains a Jordan curve C. Then either $A \cap D$ or $A \cap D^*$ is totally disconnected, where D and D^* are the components of $\bar{\mathbf{R}}^2 \setminus C$. When A is connected, either $A \cap D = \emptyset$ or $A \cap D^* = \emptyset$.*

Proof. The modulus condition (QED) cannot possibly hold for any non-degenerate continua $E \subset D$ and $F \subset D^*$. □

Next we have the following surprisingly useful result which describes when a QED continuum can contain a theta curve.

4.7. Lemma. *Let A be a QED plane compactum. Suppose A contains a theta curve Θ. Then $A\setminus\Theta$ is totally disconnected. When A is connected, $A=\Theta$.*

Proof. Suppose there is a non-degenerate continuum $E\subset A\setminus\Theta$. Then one of the three arcs making up Θ is separated from E by the other two arcs, so we can find a subarc F of this arc with $\operatorname{mod}(E,F;A^c)=0<\operatorname{mod}(E,F;\bar{\mathbf{R}}^2)$. □

We can generalize the above as follows.

4.8. Corollary. *Let A be a QED plane continuum which contains a Jordan curve C. Suppose that $A\setminus C$ separates A^c but contains no Jordan curve. Then A is a theta curve.*

Proof. According to Lemma 4.6, one of the complementary components, say D, of C satisfies $A\setminus C\subset D$. By hypothesis there are points $x,y\in D$ which are separated by $A\setminus C$ in D. Thus x,y are separated by the locally connected continuum A, so there exists a Jordan curve $\Gamma\subset A$ which separates x,y [Kur68, Thm5,p.513]. Since x,y are separated by Γ in D, there is a component γ of $D\cap\Gamma$ which separates them in D [New51, Thm7.1,p.151]. Thus A contains the theta curve $\Theta=C\cup\gamma$, and so by Lemma 4.7 $A=\Theta$. □

Now we take a closer look at the subarcs of a QED continuum which do not lie on a Jordan curve. We begin with the following information.

4.9. Lemma. *Let A be any QED continuum. Suppose $C\subset A$ is a Jordan curve and x,y are points in $A\setminus C$. Let α be a closed Jordan arc in A joining x,y. Then either $\alpha\cap C=\emptyset$, or $\alpha\cap C$ is a closed subarc of C. Also, if there exist Jordan arcs α and β joining x,y in A with $\alpha\cap C\neq\emptyset$ but $\beta\cap C=\emptyset$, then $A=\alpha\cup\beta\cup C$ is a theta curve.*

Proof. Note that $\alpha\cap C$ cannot be a single point by Proposition 5.1. Suppose $\alpha\cap C\neq\emptyset$ but $\alpha\cap C$ is not a closed subarc of C. Then α contains a closed proper subarc α' with endpoints x',y' and $\alpha'\cap C=\{x',y'\}$. According to Lemma 4.7, this means that $A=\alpha'\cup C$ is a theta curve, but neither x nor y belongs to α'.

Now suppose there are Jordan arcs α and β joining x,y in A with $\alpha\cap C\neq\emptyset$, $\beta\cap C=\emptyset$. Then we see that $\alpha\cup\beta\cup C$ contains a theta curve, so our final assertion is another consequence of Lemma 4.7. □

An immediate corollary of the above is that, when A is a QED plane continuum which contains a Jordan curve C and A is not a theta curve, then for all points $x,y\in A\setminus C$ either

$$\alpha\cap C\neq\emptyset\qquad\text{for all arcs }\alpha\text{ joining }x,y\text{ in }A$$

or

$$\alpha\cap C=\emptyset\qquad\text{for all arcs }\alpha\text{ joining }x,y\text{ in }A.$$

In the former case we say that the points x,y are separated by C in A. Obviously, points which are separated by C lie in different components of $A\setminus C$.

Next we consider the following especially simple case.

4.10. Lemma. *Let A be a QED plane continuum which contains exactly one Jordan curve C. Then either $A=C$ or, when $A\neq C$: there are countably many components of $A\setminus C$ and each has closure which is a QED arc or a QED tree.*

Proof. Let D, D^* be the components of $\bar{\mathbf{R}}^2\setminus C$ and suppose $\emptyset\neq A\setminus C\subset D^*$. Let H be a component of $A\setminus C$. Note that H itself is a locally connected generalized continuum, and therefore is arcwise connected (see [Why64, (12.1),p.15; (12.3),p.16; (4.11),p.27]). Employing this it is not difficult to certify that there is a unique point z in $\bar{H}\cap C$ and $\bar{H}=H\cup\{z\}$. Moreover, this point z is a cut point of A.

Next let f be a conformal map from the unit disk onto $G=D^*\setminus A$ (which is the component of C^c containing H). Since z is a cut point of $A=\partial G$ and $A\setminus\{z\}$ has two components, we find that $f^{-1}(z)=\{\zeta_1,\zeta_2\}$ for two distinct points of the unit circle [Pom92, Prop.2.5,p.23]. Put $\gamma_k=f[\frac{1}{2}\zeta_k,\zeta_k]$ for $k=1,2$. Next, fix some point $z_0\in D$ and let γ_0 be the 'last half' of the hyperbolic geodesic ray from z_0 to z in D. Then $\gamma_0\cup\gamma_1\cup\gamma_2$ is a topological triod with branch point z.

Thus for each such component H we have constructed a topological triod, and different components correspond to disjoint triods. According to the Moore Triod Theorem [Pom92, Prop.2.18,p.36], there are only countably many such triods.

Clearly $\bar{H}$ is a QED continuum. From above, $\bar{H}=H\cup\{z\}$, and by hypothesis $\bar{H}$ does not contain any Jordan curves; hence it is a QED arc or QED tree. □

Remarks: (1) We call such a component H described above a 'hair' attached to the Jordan curve C. (2) Notice that we have $A=C\cup\bigcup_{H\in\mathcal{H}}H$ where $\mathcal{H}$ is the family of hairs attached to C. (3) It is possible to have $\overline{A\setminus C}\supset C$ (cf. Example 6.4).

We continue our investigation by examining the next simplest case when there are two Jordan curves. We say that a closed Jordan arc α joins two Jordan curves provided each of the curves contains exactly one of the endpoints of α and no other points of α.

4.11. Lemma. *Let A be a QED plane continuum which contains two distinct Jordan curves C_0 and C_1. Then either $A=C_0\cup C_1$ is a theta curve, or $C_0\cap C_1=\emptyset$ and there exists a unique component B of $A\setminus(C_0\cup C_1)$ which contains a Jordan arc joining these curves in A with unique endpoints in C_0, C_1. When A contains exactly two Jordan curves, A is not a theta curve, the joining arc is unique, and $\bar{B}$ is a QED arc or a QED tree.*

Proof. Suppose first that $C_0\cap C_1\neq\emptyset$. According to Proposition 5.1, $C_0\cap C_1$ cannot consist of just one point. Thus we see that A contains some theta curve $\Theta\subset C_0\cup C_1$, and therefore by Lemma 4.7 we have $A=\Theta=C_0\cup C_1$. Now assume $C_0\cap C_1=\emptyset$. Fix points $w_i\in C_i$ and let α be a Jordan arc in A which joins w_0,w_1. Then there are unique points $z_i\in\alpha\cap C_i$ such that $\beta=\alpha(z_0,z_1)$ satisfies $\beta\cap(C_0\cup C_1)=\{z_0,z_1\}$. According to Lemma 4.7 any other Jordan arc in A which joins C_0 to C_1 must leave C_0 at z_0 and enter C_1 at z_1. Of course B is the component of $A\setminus(C_0\cup C_1)$ which contains the open arc $\beta\setminus\{z_0,z_1\}$. □

Remarks: (1) When A contains exactly two Jordan curves C_0, C_1 in the lemma above, we have

$$A=C_0\cup\bigcup_{H\in\mathcal{H}_0}H\cup B\cup\bigcup_{H\in\mathcal{H}_1}H\cup C_1$$

where $\mathcal{H}_i$ is the family of hairs attached to C_i. (2) In general, it is possible that B contains a Jordan curve in which case the arc from C_0 to C_1 is not unique and B is not a QED arc or tree; but $\bar{B}$ is of course a QED continuum.

We are now ready to discuss the topological structure of a general QED plane continuum. However, it seems prudent to postpone this until we have a better understanding of the geometry of these sets, which is explained below in Section 5.

5. Geometry of QED Plane Continua

Here we demonstrate that QED plane compacta cannot contain disjoint continua which are large and close together. In fact we present two different versions of this result. An easy consequence of this is that the QED condition prohibits cusps. Then we show that, at least when the topology is simple enough, a QED continuum is the QC image of a canonical curve. We conclude by establishing our main theorem, which provides a fairly complete picture of all the possibilities which may arise for a general QED plane continuum.

5.A. **Diameter vs Distance.** We begin with a basic result which describes a fundamental property of QED plane compacta. It has already been utilized in the previous section and is employed repeatedly below. Its proof does not rely on any results from Section 4.

5.1. Proposition. *Let A be an M-QED plane compactum. Fix $z \in \mathbf{R}^2$ and $0 < r < s$. Suppose K_1, K_2, K_3, K_4 are disjoint continua in A each of which joins the circles $S(z;r)$, $S(z;s)$. Then $s \le c\,r$ where $c = c(M) = \exp(2\pi\sqrt{M})$.*

Proof. First, appealing to Lemmas 2.4, 2.5 we may assume that each K_i lies in the closed annulus $\bar{B}(z;s) \setminus B(z;r)$ and that K_1 and K_3 are separated by $K_2 \cup K_4$ in this annulus. Fix $r < \rho < \sigma < s$ with $\rho/r = s/\sigma$ and put $\lambda = \sigma/\rho$. Let E and F be components of $K_1 \cap \left(\bar{B}(z;\sigma) \setminus B(z;\rho)\right)$ and $K_3 \cap \left(\bar{B}(z;\sigma) \setminus B(z;\rho)\right)$ (respectively) each of which join the circles $S(z;\rho)$, $S(z;\sigma)$. According to Fact 2.1(b), $M \operatorname{mod}(E,F;A^c) \ge \operatorname{mod}(E,F;\mathbf{R}^2) \ge (2/\pi)\log\lambda$.

Note that E, F are separated by $K_2 \cup K_4$—hence by A—in the closed annulus $\bar{B}(z;s) \setminus B(z;r)$. Thus each curve in $\Delta = \Delta(E,F;A^c)$ must meet one of the circles $S(z;r)$, $S(z;s)$. Write $\Delta = \Gamma_1 \cup \Gamma_2$ where $\Gamma_2 = \Delta \setminus \Gamma_1$ and $\Gamma_1 = \{\gamma \in \Delta : \gamma \cap \bar{B}(z;r) \ne \emptyset\}$. Then each curve in Γ_1 contains two subcurves which join the circles $S(z;r)$, $S(z;\rho)$ and thus

$$M_1 = \operatorname{mod}(\Gamma_1) \le (1/4)2\pi/\log(\rho/r).$$

Similarly,

$$M_2 = \operatorname{mod}(\Gamma_2) \le (1/4)2\pi/\log(s/\sigma).$$

Therefore,

$$(2/\pi)\log\lambda \le M(M_1 + M_2) = \pi M/\log(\rho/r).$$

Now $(\rho/r)^2 = (\rho/r)(s/\sigma) = s/\lambda r$, so we obtain

$$\log\frac{s}{\lambda r} \le \frac{\pi^2 M}{\log\lambda} \qquad \text{or} \qquad \frac{s}{r} \le \lambda \exp\left(\frac{\pi^2 M}{\log\lambda}\right).$$

The far right-hand-side is minimized by taking $\lambda = e^{\pi\sqrt{M}}$ which yields the asserted estimate $s \le \lambda^2 r$. ☐

The above gives a simple proof that two Jordan curves in a QED plane compactum must be 'relatively isolated' from each other; see also [Her87], [HK91].

5.2. Corollary. *Suppose C_1, C_2 are disjoint Jordan curves in an M-QED plane compactum A. Then*

$$\min_{i=1,2} \operatorname{diam}(C_i) \le c\operatorname{dist}(C_1, C_2),$$

where $c = c(M)$ is the constant from Proposition 5.1.

Proof. Choose points $x_i \in C_i$ so that $d = |x_1 - x_2| = \operatorname{dist}(C_1, C_2)$ and put $z = (x_1 + x_2)/2$. Let $r > d/2$ be the smallest radius with $C_i \subset \bar{B}(z; r)$ for some $i = 1, 2$. According to Proposition 5.1, $r \leq c\, d/2$ and thus $\min_{i=1,2} \operatorname{diam}(C_i) \leq 2r \leq cd$ as claimed. □

Notice that $A = \mathbf{S}^1 \cup [r, s]$ is a 4-QED compactum for any $1 < r < s$, so no such result as above holds unless both of the continua are Jordan curves. However, we do have an analog of the above for general continua in a QED plane compactum, which is based on the following substantial generalization of Proposition 5.1.

5.3. Theorem. *Let H, K be disjoint continua in an M-QED plane compactum A. Suppose that $H \subset B(K; r)$. Then* $\operatorname{diam}(H) \leq c\, r$ *where* $c = c(M)$.

Proof. We assume that $\operatorname{diam}(H) \geq 32c_0^2\, r$ where $c_0 = c_0(M)$ is the constant from Lemma 5.1. We begin by replacing H with an appropriate Jordan arc, and by establishing some preliminary notation. Select points $\zeta_1, \zeta_2 \in H$ with $|\zeta_1 - \zeta_2| = \operatorname{diam}(H)$. Since H itself is a QED continuum, it is locally connected and so there is a Jordan arc $\alpha \subset H$ which joins ζ_1, ζ_2. Let z_0 be a point of α on the perpendicular bisector of $[\zeta_1, \zeta_2]$. Assume $z_0 = 0$ and let α_1, α_2 be the subarcs of α which lie in $B(s)$ and join $z_0 = 0$ to the circle $S(s)$ where $s = \operatorname{diam}(H)/2$. For $i = 1, 2$: Let $\tilde{\alpha}_i$ be the (first) subarc of α_i which lies inside the closed annulus $\bar{B}(3\rho) \setminus B(2\rho)$ and joins the boundary circles and let x_i, y_i be its endpoints with $|x_i| = 2\rho$, $|y_i| = 3\rho$. Here $\rho = \sqrt{rs}$. Also, choose points $\xi_i, \eta_i \in K$ with $|x_i - \xi_i| < r$ and $|y_i - \eta_i| < r$.

We claim that $\operatorname{dist}(\tilde{\alpha}_1, \tilde{\alpha}_2) \geq 2\rho/c_0$. For suppose that $|z_1 - z_2| = d = \operatorname{dist}(\tilde{\alpha}_1, \tilde{\alpha}_2)$ with $z_i \in \tilde{\alpha}_i$. Since $|z_i| \leq 3\rho$, $x = (z_1 + z_2)/2 \in \bar{B}(3\rho)$. If $|x| \leq \rho$, then $d/2 = |z_i - x| \geq |z_i| - |x| \geq \rho$ so $d \geq 2\rho$. Assume $|x| > \rho$. Then each arc α_i contains two arcs which join the circles $S(x; d/2)$, $S(x; \rho)$ and our assertion is then a consequence of Lemma 5.1.

Choose $w_0 \in K$ with $|w_0| = |w_0 - z_0| < r$ and let D be the component of $B(s) \setminus (\alpha_1 \cup \alpha_2)$ which contains w_0. As K is also locally connected, and there are points of K within distance r of the points ζ_i in H, we deduce that there is a Jordan arc β in K which joins w_0 to the circle $S(t)$ where $t = s - r$. Notice that β 'separates' D into two 'pieces', loosely speaking, and any curve which joins these two 'pieces' in A^c must cross either the circle $S(t)$ or $S(r)$. The proof now proceeds by carefully examining a number of possible situations; in each case we demonstrate that $s \leq c\, r$ by a judicious choice of continua E, F in A and utilizing the modulus estimates

$$\operatorname{mod}(E, F, \mathbf{R}^2) \geq b > 0 \quad \text{and} \quad \operatorname{mod}(E, F; A^c) \leq c/\log(s/r)$$

which are obtained via Facts 2.1(a,c).

First, suppose there is a point $w \in K \cap \bar{B}(3\rho)$ with $w \notin D$. Let E and F be the components of $K \cap \bar{B}(w; \rho)$ and $\beta \cap \bar{B}(w_0; \rho)$ containing w and w_0 respectively. Then each of E, F has diameter at least ρ and $\operatorname{dist}(E, F) \leq 4\rho$, so $\operatorname{mod}(E, F; \mathbf{R}^2) \geq b_1 = (2\pi)\log(5/4)$. On the other hand, $\alpha_1 \cup \alpha_2$ separates E, F in $B(s)$, so each $\gamma \in \Delta(E, F; A^c)$ meets both $S(s)$ and $S(2\rho)$ (since $r < \rho$) whence

$$\operatorname{mod}(E, F; A^c) \leq 2\pi/\log(s/2\rho) = 4\pi/\log(s/4r)$$

and therefore $s/r \leq c_1 = 4\exp(4\pi M/b_1)$. Thus we may henceforth assume that $K \cap \bar{B}(3\rho) \subset D$.

Next we explain why we can assume that $r \leq \operatorname{dist}(\beta, \tilde{\alpha}_1 \cup \tilde{\alpha}_2) \leq \rho/2c_0$. Suppose there are points $z \in \tilde{\alpha}_1 \cup \tilde{\alpha}_2$ and $w \in \beta$ with $|w - z| < r$. Then each of α and β contains two arcs which join the circles $S(z; r)$, $S(z; \rho)$ and so Lemma 5.1 furnishes $s \leq c_0^2 r$. Suppose instead that $\operatorname{dist}(\beta,\tilde\alpha_1\cup\tilde\alpha_2) > \rho/2c_0$. Let F be the x_2-component of $\alpha_2\cap\bar B(x_2;\rho)$ and let E be the w-component of $K\cap\bar B(w;\rho/4c_0)$ where $w\in K$ satisfies $|w-x_1|<r$. Then $\operatorname{diam}(F)\ge\operatorname{diam}(E)\ge\rho/4c_0$ and $\operatorname{dist}(E,F)\le 4\rho$, so $\operatorname{mod}(E,F;\mathbf{R}^2)\ge b_2=(2/\pi)\log(1+1/16c_0)$. As $\beta\cap\bar B(w;\rho/4c_0)=\emptyset$, curves joining E,F in A^c must 'go around' β or α and so writing $\Delta=\Delta(E,F;A^c)=\Gamma_1\cup\Gamma_2$, where $\Gamma_2=\Delta\setminus\Gamma_1$ and $\Gamma_1=\{\gamma\in\Delta:\gamma\cap\bar B(r)\ne\emptyset\}$, we obtain

$$\operatorname{mod}(\Gamma_1)\le 2\pi/\log(\rho/r)=4\pi/\log(s/r)$$

and

$$\operatorname{mod}(\Gamma_2)\le 2\pi/\log((t-2\rho)/\rho)\le 4\pi/\log(0.81s/r).$$

We conclude that $s/r\le c_2=2\exp(8\pi M/b_2)$.

We now assume $r\le\operatorname{dist}(\beta,\tilde\alpha_1\cup\tilde\alpha_2)\le\rho/2c_0$. Let w_1 be the first point w of β, as we go from w_0, with $\operatorname{dist}(w,\tilde\alpha_1\cup\tilde\alpha_2)=\rho/2c_0$. Suppose that w_1 is closest to $\tilde\alpha_1$ and pick a point $z_1\in\tilde\alpha_1$ with $|w_1-z_1|=\operatorname{dist}(w_1,\tilde\alpha_1)=\rho/2c_0$. Next, let δ be a Jordan arc in K which joins ξ_2 to w_0. We consider two cases depending on whether or not the ξ_2-component of $\delta\cap\bar B(\xi_2;\rho/4c_0)$ meets β.

Suppose that $F\cap\beta=\emptyset$ where F is the ξ_2-component of $\delta\cap\bar B(\xi_2;\rho/4c_0)$. Let E be the x_1-component of $\alpha_1\cap\bar B(x_1;\rho)$. Then $\operatorname{diam}(E)\ge\operatorname{diam}(F)\ge\rho/4c_0$ and $\operatorname{dist}(E,F)\le 5\rho$, so $\operatorname{mod}(E,F;\mathbf{R}^2)\ge b_3=(2/\pi)\log(1+1/20c_0)$. Next, the curves in $\Delta(E,F;A^c)$ either meet $\bar B(r)$ or 'go around' β or α, so

$$\operatorname{mod}(E,F;A^c)\le\frac{2\pi}{\log(\rho/r)}+\frac{2\pi}{\log((t-2\rho)/\rho)}\le\frac{8\pi}{\log(0.81s/r)}$$

and thus $s/r\le c_3=2\exp(8\pi M/b_3)$.

Finally, we assume δ meets β. Let ζ be the first point of β encountered as we traverse δ from ξ_2 to w_0. By the above we may suppose that $|\zeta-\xi_2|\le\rho/4c_0$. Notice that $\zeta\notin\bar B(w_1;\rho/2c_0)$ because $\operatorname{dist}(\tilde\alpha_1,\tilde\alpha_2)\ge 2\rho/c_0$. Let E be the w_1-component of $\beta\cap\bar B(w_1;\rho/2c_0)$. Then $\zeta\notin E$, so ζ lies either before or after E on β. However, $\operatorname{dist}(\zeta,\tilde\alpha_1\cup\tilde\alpha_2)\le|\zeta-\xi_2|+|\xi_2-x_2|\le\rho/2c_0$, so according to our choice of w_1, ζ must come after E as β is traversed from w_0. Now let F be the y_2-component of $\alpha_2\cap\bar B(y_2;\rho)$. Then $\operatorname{diam}(F)\ge\operatorname{diam}(E)\ge\rho/2c_0$ and $\operatorname{dist}(E,F)\le 7\rho$, so $\operatorname{mod}(E,F;\mathbf{R}^2)\ge b_4=(2/\pi)\log(1+1/14c_0)$. Write $\Delta=\Delta(E,F;A^c)=\Gamma_1\cup\Gamma_2\cup\Gamma_3$ where $\Gamma_3=\Delta\setminus(\Gamma_1\cup\Gamma_2)$, $\Gamma_1=\{\gamma\in\Delta:\gamma\cap\bar B(r)\ne\emptyset\}$ and $\Gamma_2=\{\gamma\in\Delta:\gamma\cap S(t)\ne\emptyset\}$. As before,

$$\operatorname{mod}(\Gamma_1)\le 2\pi/\log(\rho/r)=4\pi/\log(s/r)$$

and

$$\operatorname{mod}(\Gamma_2)\le 2\pi/\log((t-3\rho)/\rho)\le 4\pi/\log(0.81s/r).$$

We claim that each $\gamma\in\Gamma_3$ satisfies $\gamma\cap\bar B(x_2;r)\ne\emptyset\ne\gamma\cap S(x_2;\rho/c_0)$, thus

$$\operatorname{mod}(\Gamma_3)\le 2\pi/\log(\rho/c_0r)=4\pi/\log(s/c_0^2r)$$

and therefore $\operatorname{mod}(E,F;A^c)\le 12\pi/\log(s/c_0^2r)$ and so we obtain $s/r\le c_4=c_0^2\exp(12\pi M/b_4)$.

Having inspected all possibilities we conclude that $\operatorname{diam}(H)\le c\,r$ where $c=c(M)=2\max\{c_1,c_2,c_3,c_4\}$. The interested reader might ponder the size of c, e.g., when $M=1$. □

We can now give an analog of Corollary 5.2. In particular, this asserts that a QED plane compactum cannot contain any 'cusps'.

5.4. Corollary. *The Hausdorff distance between two disjoint continua K_1, K_2 in an M-QED plane compactum satisfies*

$$\min_{i=1,2} \operatorname{diam}(K_i) \le c \operatorname{dist}_{\mathcal{H}}(K_1, K_2),$$

where $c = c(M)$ is the constant in Theorem 5.3.

5.B. **Quasiconformal Sets.** We recall that a (K-)*quasicircle*, *quasiarc*, or *quasidisk* is the image of a circle, an interval, or a disk under a (K-)quasiconformal self-homeomorphism of the closed plane. Thanks to Ahlfors [Ahl63] (see also [LV73, II.8.7]) we know that the quasicircles are precisely the *bounded turning* Jordan curves C which satisfy the requirement

$$\operatorname{diam} C(x, y) \le a\,|x - y| \qquad \text{for all } x, y \in C$$

for some constant $a \ge 1$, where $C(x, y)$ is one of the subarcs of C joining x, y. We abbreviate this by saying that C is a-BT. Here the constants a, K depend only on each other. Rickman [Ric66] established the analagous geometric characterization for *closed* quasiarcs: these are the closed Jordan arcs which have *bounded distortion*; see [Ric66, Theorem 1] and its proof. Again the constants depend only on each other. While it is true that a BT arc always has bounded distortion (and hence is a quasiarc when closed), in general a K-quasiarc need not be a-BT with $a = a(K)$. However, in the special case when Γ is a closed Jordan arc one of whose endpoints is at infinity, we again find that Γ is a K-quasiarc if and only if it is a-BT, and the constants K, a depend only on each other.

Rickman also introduced the notion of *bounded local distortion* and used this to establish a metric characterization for (K-)*quasiconformal arcs* Γ which are defined by the requirement that there exist a domain $G \supset \Gamma$ and a (K-)quasiconformal homeomorphism of G which maps Γ onto a line segment. He proved [Ric66, Theorem 4] that a Jordan arc is quasiconformal exactly when it has finite maximal local distortion. Obviously every quasiarc is a quasiconformal arc, but a quasiconformal arc need not be a quasiarc.

Here we examine the Jordan curves and arcs in a QED plane continuum to determine when they will be quasiconformal. First we point out the following result, which could also be proved using ideas similar to those given for [HK90, Cor3.7].

5.5. Theorem. *Let A be an M-QED compactum in the plane. If A contains a Jordan curve C, then C is a K-quasicircle, $K = K(M)$.*

Proof. Let C be a Jordan curve in A. We verify that C must be bounded turning, which guarantees that it is a quasicircle. Fix points $x, y \in C$ and let γ be the smaller subarc of C joining x, y. Assume that $\operatorname{diam}(\gamma) > 2|x - y|$. Choose points $z \in \gamma$ and $w \in C \setminus \gamma$ with $|w - x| \ge |z - x| \ge \operatorname{diam}(\gamma)/2$. Then the two arcs of C which join z, w each must contain two subarcs which meet the circles $S(x; |x - y|)$, $S(x; \operatorname{diam}(\gamma)/2)$. Appealing to Proposition 5.1 we may assert that $\operatorname{diam}(\gamma) \le 2c|x - y|$ where $c = c(M)$. □

5.6. Corollary. *Let D be a BQED disk in the plane. Then either D is a quasidisk or $\bar{D} = \bar{\mathbf{R}}^2$.*

Proof. According to Lemma 4.4, if $\bar{D} \ne \bar{\mathbf{R}}^2$, then ∂D is a Jordan curve which, by Theorem 5.5, is a quasicircle. □

We define a K-*quasitheta* to be the image of $\mathbf{S}^1 \cup [-1, 1]$ under a K-quasiconformal self-homeomorphism of $\bar{\mathbf{R}}^2$, and an M-*QED theta* is a theta curve which is an M-QED continuum.

5.7. Theorem. *Let A be a plane theta curve. Then A is an M-QED theta if and only if A is a K-quasitheta; here $K = K(M)$ and $M = 4K^2$.*

Proof. It is easy to see that a K-quasitheta is $4K^2$-QED. Let A be an M-QED theta. Throughout the proof we write K to denote a constant which depends only on M. According to Theorem 5.5, each of the Jordan curves making up A is a K-quasicircle. Thus by using a preliminary K-QC map, if necessary, we may assume that $A = \mathbf{R} \cup \gamma$ where γ is a Jordan arc which joins the origin to ∞ in the upper half-plane $U = \{\Im(z) > 0\}$. Let $\Gamma = \gamma \cup \gamma^*$ where γ^* is the reflection of γ across the real line. We claim that Γ is a K-quasicircle. For let $z, w \in \Gamma$. Assume $w \in \gamma^*$, $z = x + iy \in \gamma$ with $x \geq 0$, and $\operatorname{diam}\Gamma(z,0) \geq \operatorname{diam}\Gamma(w,0)$. Note that $\Gamma_+ = \gamma \cup [0, +\infty]$ is a K-quasicircle, since it is a Jordan curve in A. Thus

$$\begin{aligned}\operatorname{diam}\Gamma(z,w) &\leq \operatorname{diam}\Gamma(z,0) + \operatorname{diam}\Gamma(w,0) \leq 2\operatorname{diam}\Gamma(z,0)\\ &\leq 2\operatorname{diam}\Gamma_+(z,x) \leq 2c|z-x| = 2cy \leq 2c|z-w|\end{aligned}$$

where $c = c(M)$.

Now let G be the quasidisk 'to the left' of Γ and let $f : G \to H$ map G conformally onto the left half-plane $H = \{\Re(w) < 0\}$. We may stipulate that f is symmetric with respect to the real line. Since Γ is a quasicircle, we can extend f to a global K-QC self-homeomorphism F of the plane. A glance at the Beurling-Ahlfors extension formula (cf. [BA54, (14),p.135]) certifies that we can construct such an F with the added property that F too is symmetric with respect to the real line. Now we examine $F|_U$ and observe that A is mapped to $\mathbf{R} \cup I$ where I is the positive imaginary axis. Postcomposing with a Möbius transformation now furnishes the desired map sending A to $\mathbf{S}^1 \cup [-1, 1]$. □

Finally, we turn our attention to QED arcs. It is easy to see that a K-quasiarc is K^2-QED. It was surprising to us that there are QED arcs which fail to be quasiarcs; see Example 6.1. However, we believe that the following idea may prove useful in determining a geometric characterization for QED arcs. We say that a Jordan arc A is *weakly bounded turning* provided there exists a constant $b \geq 1$ such that for all points x, y in the interior of A,

$$\min_{i=0,1,2} \operatorname{diam}(\alpha_i) \leq b\,|x-y|,$$

where α_i are the subarcs of $A \setminus \{x, y\}$. We abbreviate this by the phrase 'A is b-WBT'.

Example 6.1 furnishes an arc which is WBT but not BT. On the other hand, it is true that the 'middle' of a WBT arc is BT.

5.8. Proposition. *Let $A = \beta_1 \cup \gamma \cup \beta_2$ be a Jordan arc where γ, β_1, β_2 are Jordan arcs with γ separating β_1, β_2. Suppose A is b-WBT. Then γ is a-BT where $a = b\max\{1, \operatorname{diam}\gamma / \min_{k=1,2}\operatorname{diam}\beta_k\}$.*

Proof. Let $x, y \in \gamma$. Then $A = \alpha_0 \cup \alpha_1 \cup \alpha_2$ where $\alpha_0 = \gamma(x,y)$, $\alpha_1 = \beta_1 \cup \gamma_1$, $\alpha_2 = \beta_2 \cup \gamma_2$ and γ_1, γ_2 are the subarcs of γ joining its endpoints to x, y (so $\gamma = \gamma_1 \cup \alpha_0 \cup \gamma_2$). Since A is b-WBT, we know that $\min_{i=0,1,2}\operatorname{diam}(\alpha_i) \leq b\,|x-y|$. Thus either

$$\operatorname{diam}\gamma(x,y) = \operatorname{diam}(\alpha_0) \leq b\,|x-y|$$

or

$$b\,|x-y| \geq \min_{i=1,2}\operatorname{diam}(\alpha_i) \geq \min_{k=1,2}\operatorname{diam}\beta_k \frac{\operatorname{diam}\gamma(x,y)}{\operatorname{diam}\gamma},$$

which corroborates our assertion. □

Example 6.1 shows that the above is essentially best possible.

5.9. Corollary. *If $A = \beta_1 \cup \gamma \cup \beta_2$ as above is b-WBT and* $\operatorname{diam}\gamma \le \min_{k=1,2} \operatorname{diam}\beta_k$*, then γ is b-BT.*

Now we verify that QED plane arcs are weakly bounded turning.

5.10. Proposition. *An M-QED plane arc is c-WBT with $c = c(M)$ the constant from Proposition 5.1.*

Proof. Let A be an M-QED plane arc. Fix $x, y \in A$ and let $\gamma_0, \gamma_1, \gamma_2$ be the components of $A \setminus \{x, y\}$ with γ_0 the subarc joining x, y. Put $z = (x + y)/2$, $r = |x - y|/2$ and let $s > 0$ be the smallest radius with some $\gamma_i \subset \bar{B}(z; s)$. According to Proposition 5.1, $s \le cr$ and thus $\min_{i=0,1,2} \operatorname{diam}(\gamma_i) \le 2s \le c\,|x - y|$ as claimed. $\square$

While it is true that a QED arc need not be a quasiarc, we do have the following information. Recall that α is a *K-quasiconformal arc* provided there is a domain $G \supset \alpha$ and a K-quasiconformal homeomorphism $f : G \to f(G)$ which maps α onto an interval.

5.11. Lemma. *Any* open *Jordan arc in an M-QED plane continuum is a K-quasiconformal arc where $K = K(M)$.*

Proof. Let α be such an arc. Then $A = \bar{\alpha}$ is an M-QED continuum. It is not difficult to see that A is either a Jordan curve or a closed Jordan arc. (If one of the 'ends' of α had two distinct limits points, then A would not be locally connected.) In the first case A is a K-quasicircle by Theorem 5.5, hence in this case α is actually an open K-quasiarc.

Suppose that A is a closed Jordan arc. Then by Proposition 5.10, A is c-WBT with $c = c(M)$. We assume that $\infty \notin \alpha$. According to a result of Rickman, it suffices to show that α has maximal local distortion $c_0(\alpha) = \sup_{w\in\alpha} r(\alpha, w) < \infty$ where

$$r(\alpha, w) = \inf_{\gamma} R(\gamma) \quad \text{and} \quad R(\gamma) = \sup_{x,y,z} \frac{|x - y| + |y - z|}{|x - z|};$$

here the infimum is taken over all arcs $\gamma \subset \alpha$ which are open in α and contain w, and the supremum is over all successive points $x, y, z \in \gamma$.

So, fix a point $w \in \alpha$. Write $A = \beta_1 \cup \gamma \cup \beta_2$ as a union of subarcs where $w \in \gamma$, γ is open in α, and $\operatorname{diam}\gamma \le \min_{k=1,2} \operatorname{diam}\beta_k$. According to Corollary 5.9, γ is c-BT, which easily gives $R(\gamma) \le 2c$. We conclude that $r(\alpha, w) \le 2c$, and as this holds for all $w \in \alpha$, it follows that α has maximal local distortion $c_0(\alpha) \le 2c$ and hence by [Ric66, Theorem 4] α is a K-quasiconformal arc with $K = K(M)$. $\square$

Using the above ideas we can verify that many subarcs of QED continua are in fact quasiarcs. We believe the following warrants attention. We say that a closed Jordan arc α joins two Jordan curves provided each of the curves contains exactly one of the endpoints of α and no other points of α.

5.12. Proposition. *Let C_0, C_1 be disjoint Jordan curves in an M-QED plane continuum A. Suppose α is any closed Jordan arc in A which joins C_0, C_1. Then α is a K-quasiarc with $K = K(M, \rho)$ where $\rho = \operatorname{dist}(C_0, C_1)/\min_{k=0,1} \operatorname{diam}(C_k)$.*

Proof. Select points $z_k \in C_k$ with $d = |z_0 - z_1| = \operatorname{dist}(C_0, C_1)$. Initially, assume that $\infty \in C_0$. Thus $\rho = d/\operatorname{diam} C_1$. We verify that α is a-BT with $a = a(M, \rho)$. This means that α has bounded distortion, and hence by Rickman's characterization (see [Ric66, Theorem 1] and its proof) we conclude that α is a K-quasiarc with $K = K(a)$. Let w_0, w_1 be the endpoints of α and suppose $x, y \in \alpha$ are such that w_0, x, y, w_1 are successive points of α. Assume

$x+y=0$, put $r=|x-y|/2$, and let $s>0$ be the smallest radius with $\alpha(x,y)\subset \bar{B}(0;s)$. We demonstrate that $s\le c\,r$ which shows that α is c-BT.

Let $t=\max\{|z|:z\in\alpha(y,w_1)\cup C_1\}$. Since C_0 is unbounded, we see that A contains four disjoint arcs joining the circle $|z|=r$ to the circle $|z|=\min\{s,t\}$, so $\min\{s,t\}\le c_0 r$ where $c_0=c_0(M)$ is the constant from Proposition 5.1. If $s\le t$, we are done with $c=c_0$. Assume $s>t$. Let $u=\min\{|z|:z\in C_0\}=\operatorname{dist}(0,C_0)$. Another application of Proposition 5.1 now yields $s\le c_0\max\{r,u\}$. If $u\le r$, we are again done with $c=c_0$. Assume $u>r$. We now have: $t<s$, $r<u$, $t\le c_0 r$, $s\le c_0 u$. We claim that $u\le(1+2\rho)t$ and therefore

$$s\le c_0(1+2\rho)t\le c_0^2(1+2\rho)r$$

which gives the desired conclusion with $c=c_0^2(1+2\rho)$. To see that our claim is valid, notice that

$$u\le|z_0|\le|z_0-z_1|+|z_1|\le d+t,$$

so

$$\rho=d/\operatorname{diam}C_1\ge(u-t)/2t.$$

We have thus established that, in the case when some C_k is unbounded, α is a-BT with $a=c(M)(1+2\rho)$, and from this our assertion follows. It remains to consider the case when both C_0, C_1 are bounded. By affine invariance, we may assume that $z_k=k$ for $k=0,1$. Further, we may assume that $\operatorname{diam}C_1\le\operatorname{diam}C_0$. Let A',C_0',C_1',α' be the images of A,C_0,C_1,α under the inversion $z\mapsto 1/z$. Then A' is an M-QED continuum containing the disjoint Jordan curves C_0',C_1' which are joined in A' by the Jordan arc α'. Since $\infty\in C_0'$, the first part of our proof shows that α' (hence α) is a K-quasiarc with $K=K(M,\rho')$ where $\rho'=\operatorname{dist}(C_0',C_1')/\operatorname{diam}(C_1')$. It is not difficult to check that, e.g., $\rho'\le 4\rho\max\{3,4\rho\}$, so in fact $K=K(M,\rho)$ as asserted. $\square$

5.C. **Proof of Main Theorem.** First we have the *elementary* curves which are the QED arcs, QED trees, QED circles and QED thetas. A careful review of our previous results reveals a fairly complete description for these curves and their complements. We call A a *non-elementary* QED continuum if it is not one of these.

Let A be an QED continuum. When A^c is connected, it must be a BQED disk, and since A would then contain no Jordan curve, A^c cannot be QED and thus (a) follows (see 4.B). Assume A^c is not connected; then A contains some Jordan curve. Items (b), (c) follow immediately from Theorems 5.5, 5.7. For the remainder of this subsection, we assume that A is a non-elementary QED continuum. Lemmas 4.10, 4.11, and their subsequent remarks, describe A and A^c in some special cases, and it is not hard to generalize to the case when A contains finitely many Jordan curves. We wish to extend these ideas for the case when A has an arbitrary number of Jordan curves (of course countable).

Suppose that $\infty\in A^c$. We are assuming A contains one or more Jordan curves, say C_i, $(1\le i\le n\le\infty)$ with $C_i\cap C_j=\emptyset$ for $i\ne j$ (because A is not a theta curve). Let D_i and D_i^* be the components of $\bar{\mathbf{R}}^2\setminus C_i$; according to Lemma 4.6 we can label these so that $A\cap D_i=\emptyset$ (i.e., $D_i^*\supset A\setminus C_i$). Note that according to Theorem 5.5, each D_i is a quasidisk. We can further assume that D_i is the interior of C_i. Finally, let D be the component of A^c containing ∞. Since A is a locally connected plane continuum, each component of A^c is simply connected with locally connected boundary and closure, and each bounded component is in fact a Jordan domain; see Fact 2.2. It therefore follows that

$$A^c=D\cup\bigcup_i D_i.$$

Next we show that A is a *boundary curve*; in fact we have $A = \partial D$. Clearly $\partial D \subset A$. We must show that $A \subset \partial D$. Suppose there is some point $x \in A$ with $x \notin \partial D$. Then there is an $r > 0$ so that $B(x;r) \cap \bar{D} = \emptyset$. Thus ∂D separates x, ∞, so by [Why42, (2.51),p.107], there is a bounded Jordan curve $C \subset \partial D$ with x in the interior of C and D lying in the exterior of C. Since $\partial D \neq C$, there is a non-degenerate continuum $E \subset \partial D$ also lying in the exterior of C. (Take a point of $\partial D \setminus C \subset A$ and join it to C with an arc in A and use a piece of this.) Next, let F be a small subarc of A which contains x and lies inside $B(x;r)$. Then $\Delta(E, F; A^c) = \emptyset$, so $\operatorname{mod}(E, F; A^c) = 0$ which violates the QED condition.

Now we take a closer look at the components H of $A \setminus \bigcup C_i$. It is possible that H is a single point w. This only happens if w is a limit point of $\cup C_i$ with the property that for any other point z of A, there is some C_i which separates w, z in A. We let S be the set of all such points w; this may be an uncountable set! (Modify Example 6.3 to see this.) We assume that H is non-degenerate. Note that H still may contain limit points of $\cup C_i$.

Let us explain why H is arcwise connected. Recall Lemma 4.9 and the discussion immediately thereafter. Consider points $x, y \in A \setminus \bigcup C_i$. If there is some Jordan curve C_i which separates x, y in A, then x, y lie in different components of $A \setminus \bigcup C_i$. Therefore, when x, y lie in the same component, say H, they are not separated by any C_i; we conclude that in this case there is a (unique) closed Jordan arc $\alpha \subset A$ which joins x, y and lies in H. In particular, H is arcwise connected.

Armed with the crucial information that H is arcwise connected, we can reason as in the proof of Lemma 4.10 to corroborate that: $E = \bar{H}$ meets no C_i in more than one point, $E = H \cup \bigcup\{z_i\}$ where $\{z_i\} = E \cap C_i$, and E is a QED arc or a QED tree. When E meets just one of the Jordan curves C_i, then E is one of the hairs attached to C_i as described in the comments following Lemma 4.10. However, it is possible that E may meet many of the curves (which is somewhat described by Lemma 4.11)in which case we call it a 'connecting' arc or 'connecting' tree.

It remains to verify that there are a countable number of such non-degenerate components H. Fix one of the Jordan curves C_i. Consider all of the H for which $\bar{H}$ meets C_i: Arguing as in the proof of Lemma 4.10, only now using a Riemann map onto D, we see that each such H corresponds to a unique triod with two subarcs in D and one in D_i. Since different H yield disjoint triods, there are countably many such H 'attached' to C_i. $\square$

We have the following application of our main result.

5.13. Corollary. *Let A be an M-QED plane continuum. Then any pair of points in A can be joined by an arc in A^c. Also, either A^c is an M-BQED disk, or, except for possibly one exception, every component of A^c is a K-quasidisk where $K = K(M)$.*

6. Examples

Here we exhibit several non-trivial examples of QED arcs and QED trees. We begin with a QED arc which fails to be a quasiarc; it is also a weakly bounded turning arc which is not bounded turning. We call this our 'square example'.

6.1. Example. *There is a closed Jordan arc in the plane which is a QED arc, hence WBT, but not BT and not a quasiarc.*

Proof. We construct a QED arc A which can be written as

$$A = [0, +\infty] \cup \bigcup_{n=0}^{\infty} \alpha_n \cup \{w_0\},$$

where each α_n is a closed subarc of the boundary of a square Q_n to be determined below and w_0 is the limit point of α_n as $n \to \infty$ (or $w_0 = \cap_0^\infty Q_n$). Intuitively speaking, the arc A comes from infinity along the positive real axis, then spires inward clockwise along squares, and finally reaches the limit end point w_0.

For $n = 0, 1, 2, \ldots$, let $s_0 = 1$ and $s_n = 2^{-n} s_{n-1}$. We define closed squares Q_n of side length s_n and arcs $\alpha_n \subset \partial Q_n$ inductively as follows. The sides of Q_n are required to be parallel to the coordinate axes. Let Q_0 be the closed unit square with vertices at $0, i, 1+i, 1$ in the plane and let $\alpha_0 = [x_0, y_0]$, with $x_0 = 0$ and $y_0 = 1 + s_1 i$, where (and throughout this example) the notation $\alpha_n = [x_n, y_n]$ means that α_n is the subarc on ∂Q_n from x_n to y_n traveling clockwise. Once Q_n and $\alpha_n = [x_n, y_n]$ are defined, we let Q_{n+1} be the square of side length s_{n+1} with one vertex at y_n and such that $Q_{n+1} \subset Q_n$ and Q_{n+1}, Q_n do not share any vertices. Note that such a Q_{n+1} is uniquely determined and one edge of Q_{n+1} lies in an edge of Q_n. Next, let $\alpha_{n+1} = [x_{n+1}, y_{n+1}]$, where $x_{n+1} = y_n$ and, when traveling on ∂Q_{n+1} clockwise starting at x_{n+1}, y_{n+1} is the second point of ∂Q_{n+1} satisfying $\operatorname{dist}(y_{n+1}, \partial Q_n) = s_{n+2}$. Thus α_{n+1} consists of two full edges of ∂Q_{n+1} together with most of a third edge, and none of these three edges meet ∂Q_n except at the point y_n. Finally, set $w_0 = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. This completes the construction of the Jordan arc A.

It is easy to verify that A is weakly bounded turning, but not bounded turning and hence not a quasiarc. However, we claim that A is a QED arc (and so the arc $A \setminus \{w_0\}$ is a quasiconformal arc). As our proof of this claim is tedious and technical, we only sketch the main ideas.

To show that A is an M-QED arc, fix disjoint continua E and F on A. Using the orientation of A described above, we may assume that E is "closer" to the end point w_0 than F in the sense that in order to reach w_0 from F one has to pass through E. Let n_E be the smallest integer such that $E \cap \partial Q_{n_E} \neq \emptyset$ and let m_F be the largest integer such that $F \cap \partial Q_{m_F} \neq \emptyset$. Then $n_E \geq m_F$. We examine two cases separately.

Case I. E and F are close together, i.e. $m_F \geq n_E - 1$. In this case, the key fact we use about A is that the domain G_n constructed below is a QED domain with an absolute constant (independent of n). Let

$$G_n = (Q_n \cup Q'_{n+1}) \setminus (Q_{n+1} \setminus Q^*_{n+2}),$$

where Q'_{n+1} is the symmetric image of Q_{n+1} about the vertex x_{n+1}, and Q^*_{n+2} is the symmetric image of Q_{n+2} about the (unique) side of Q_{n+2} that has x_{n+2} as an end point and is not contained in ∂Q_{n+1}. Estimates for the moduli $\operatorname{mod}(E, F; A^c)$ and $\operatorname{mod}(E, F; \bar{\mathbf{R}}^2)$ can be obtained from estimates for moduli of certain subfamilies related to the QED domains G_n.

Case II. E and F are apart, i.e. $m_F < n_E - 1$. In this case, E and F are separated in A^c by some narrow "gates". Through certain reductions using the QED domains G_n constructed above, we may further assume that $\operatorname{dist}(E, F) \geq (3/2) \min\{\operatorname{diam}(E), \operatorname{diam}(F)\}$, so by Fact 2.1(a) we deduce that $\operatorname{mod}(E, F; \bar{\mathbf{R}}^2) \leq 2\pi / \log(3/2)$.

To estimate $\operatorname{mod}(E, F; A^c)$, we construct suitable closed balls $B_E \subset G_n$ and $B_F \subset G_m$, with $n = n_E - 1$ and $m = m_F + 1$, such that

$$\operatorname{dist}(B_E, E) \approx \min\{\operatorname{diam}(B_E), \operatorname{diam}(E)\}, \quad \operatorname{dist}(B_F, F) \approx \min\{\operatorname{diam}(B_F), \operatorname{diam}(F)\}.$$

By the Comparison Principle (Fact 2.1(d)),

$$\operatorname{mod}(E, F; A^c) \geq \frac{1}{9} \min\{\operatorname{mod}(E, B_E; A^c), \operatorname{mod}(F, B_F; A^c), I\},$$

where I can be controlled by $\operatorname{mod}(B_E, B_F; A^c)$. Lower bounds for $\operatorname{mod}(E, B_E; A^c)$ and $\operatorname{mod}(F, B_F; A^c)$ can be obtained by using similar techniques as in Case I, while quasihyperbolic distance estimates can be utilized to estimate $\operatorname{mod}(B_E, B_F; A^c)$ from below. Finally, we conclude that A is a QED arc. □

The remaining examples are all QED trees. First we describe two infinite QED 'combs'; these are due to Pekka Koskela and we thank him for allowing us to include them. Verification of the following is left to the interested reader. (It is easy to see that A_2 is not QED. Proposition 3.4 is useful for demonstrating that $D = A_1^c$ is a BQED domain; simply let G_0 be the lower half-plane and let G_1 (resp., G_2) be the upper half-plane with the following squares *removed*: $\{x + iy : 0 \le y \le 1, 2n - 1 \le x \le 2n, n = 1, 2, \dots\}$ (resp., $\{x + iy : 0 \le y \le 1, 2n \le x \le 2n + 1, n = 0, 1, 2, \dots\}$).)

6.2. Example. *The continuum*

$$A_1 = [0, +\infty) \cup [1, 1 + i] \cup [2, 2 + i] \cup [3, 3 + i] \cup \dots$$

is QED. However the 'two-sided comb' obtained by reflecting every other 'tooth', i.e.,

$$A_2 = [0, +\infty) \cup [1, 1 + i] \cup [2, 2 - i] \cup [3, 3 + i] \cup [4, 4 - i] \cup \dots$$

fails to be QED. On the other hand, if the 'teeth' of a 'two-sided comb' become smaller and smaller quickly enough, then it will be QED. E.g.,

$$A_3 = [0, +\infty) \cup [1, 1 + i] \cup [2, 2 - i/2] \cup [3, 3 + i/3] \cup [4, 4 - i/4] \cup \dots$$

is QED.

Based on the above and other examples we expect that a 'one-sided comb' will be QED if and only if two 'big teeth' cannot be close together. Such a characterization would provide 'Cantor type' examples, but more importantly would point the way for a better understanding of general QED trees; a proof of this should generalize to provide techniques for determining when more general continua are QED.

Here is another tree example which illustrates that the boundary set B_1 of 'endpoints' may be uncountable.

6.3. Example. *There is a QED tree A such that B_1 is uncountable.*

Proof. We construct A as an infinite binary tree whose edges get small fast enough so that there is a limiting continuum which is QED. Since we use two edges from each node, we find that the 'limit end points' are indeed uncountable, and these form the set B_1. We start with $A_0 = [0, 1] \cup [0, i]$. At each of the points $1 - \varepsilon_1, i(1 - \varepsilon_1)$, where $0 < \varepsilon_1 << 1$, we place a vertical, horizontal (resp.) segment of length ε_1 to obtain A_1. Now iterate this idea taking $0 < \varepsilon_2 << \varepsilon_1$ and adding an appropriate horizontal or vertical segment of length ε_2 at the end of each segment of A_1 to get A_2. Continue this. Then there is a limit continua A, its B_1 points are uncountable, and provided we choose $\varepsilon_{n+1} << \varepsilon_n$, A will be QED. □

We conclude with a 'hairy' tree example which illustrates how the boundary set B_2 may actually be totally disconnected.

6.4. Example. *There is a QED tree A such that B_2 is totally disconnected and B_3 is dense in A.*

Proof. We start with the unit interval $[0, 1]$. Then we add a vertical segment at the midpoint $1/2$, say $[1/2, 1/2 + ih_1]$ for some $h_1 > 0$. Next we add vertical segments each of some length $h_2 << h_1$ at the 'new' midpoints $1/4$ and $3/4$ placing these below the original segment $[0, 1]$. Continue this placing smaller and smaller vertical segments at the dyadic rationals alternating above and below the segment $[0, 1]$. In this way we obtain a QED tree and the B_3 points are dense in $[0, 1]$. Now we apply this process to each of the vertical segments, and then to each of the new horizontal segments, and so forth. In the end we obtain a 'hairy' QED tree and B_2 will be totally disconnected. We can be sure of the QED condition provided the new segments introduced are much smaller than the ones already present. □

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